Symmetries of monocoronal tilings

Dirk Frettlöh

Technische Fakultät
Universität Bielefeld

Geometry and Symmetry
29 June - 3 July 2015, Veszprém

Dedicated to Károly Bezdek and Egon Schulte on the occasion of their 60th birthdays

Mathematics Genealogy Project

Home
Search
Extrema
About MGP
Links
FAQs
Posters
Submit Data
Contact
Mirrors

A service of the NDSU Department of Mathematics, in association with the American Mathematical Society.

Ludwig Danzer

MathSciNet

Dr. rer. nat. Technische Universität München 1960

Dissertation: *Über zwei Lagerungsprobleme*

Mathematics Subject Classification: 52—Convex and discrete geometry

Advisor 1: Hanfried Lenz
Advisor 2: Frank Löbell
Advisor 3: Robert Sauer

Students:
Click here to see the students listed in chronological order.

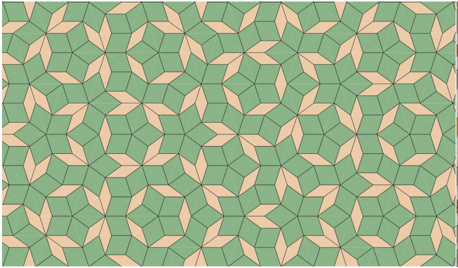
Name	School	Year	Descendants
Ulrich Bolle	Universität Dortmund	1976	
Jürgen Eckhoff	Georg-August-Universität Göttingen	1969	
Dirk Frettloeh	Universität Dortmund	2002	
Dietrich Kramer	Universität Dortmund	1974	
Hanno Schecker	Universität Dortmund	1972	
Egon Schulte	Universität Dortmund	1980	12
Petra Sonneborn	Universität Dortmund	1994	
Thomas Stehling	Universität Dortmund	1989	
Rolf Stein	Universität Dortmund	1982	
Gerd Wegner	Georg-August-Universität Göttingen	1967	1

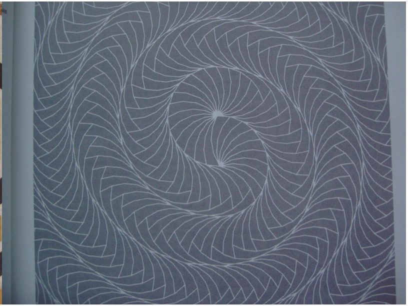
1. Monohedral and isohedral tilings
2. Monogonal and isogonal tilings
3. Monocoronal (and isocoronal) tilings

Joint work with Alexey Garber.

Tiling (=tessellation) covering of $\mathbb{R}^2$ which is also a packing.

Pieces (**tiles**): nice compact sets (squares, triangles...).



A central question:

Which shapes do tile?

- Euclidean plane $\mathbb{R}^2$
- Euclidean space $\mathbb{R}^d$ ($d \geq 2$)
- hyperbolic space $\mathbb{H}^2$
- finite regions like $\square$, $\triangle$, ...

A central question:

Which shapes do tile?

- Euclidean plane $\mathbb{R}^2$
- Euclidean space $\mathbb{R}^d$ $(d \geq 2)$
- hyperbolic space $\mathbb{H}^2$
- finite regions like $\square$, $\triangle$, ...

Here: tilings of $\mathbb{R}^2$.

Def.: The **symmetry group** of a tiling $\mathcal{T}$ of $\mathbb{R}^2$:

$$Sym(\mathcal{T}) = \{\varphi \text{ isometry in } \mathbb{R}^2 \,|\, \varphi(\mathcal{T}) = \mathcal{T}\}$$

Monohedral and isohedral tilings

A tiling is called **monohedral** if all tiles are congruent.

A tiling $\mathcal{T}$ is called **isohedral** if its symmetry group acts transitively on the tiles.

A tiling is called **monohedral** if all tiles are congruent.

A tiling $\mathcal{T}$ is called **isohedral** if its symmetry group acts transitively on the tiles.

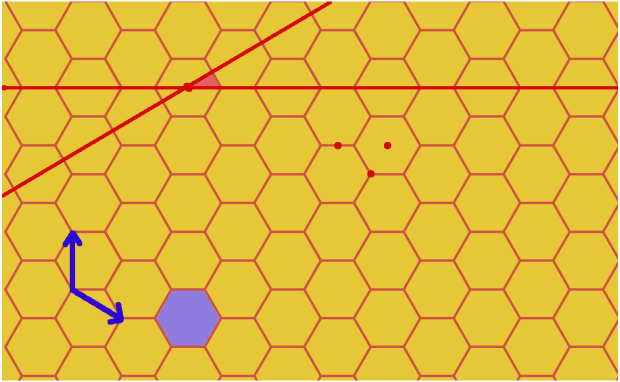
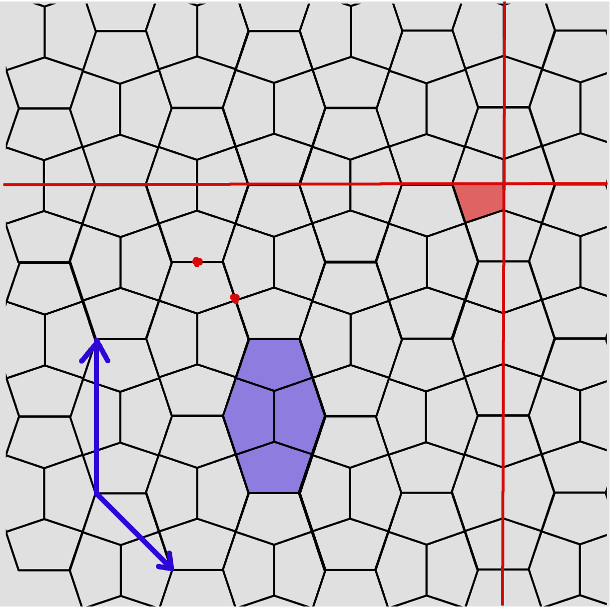
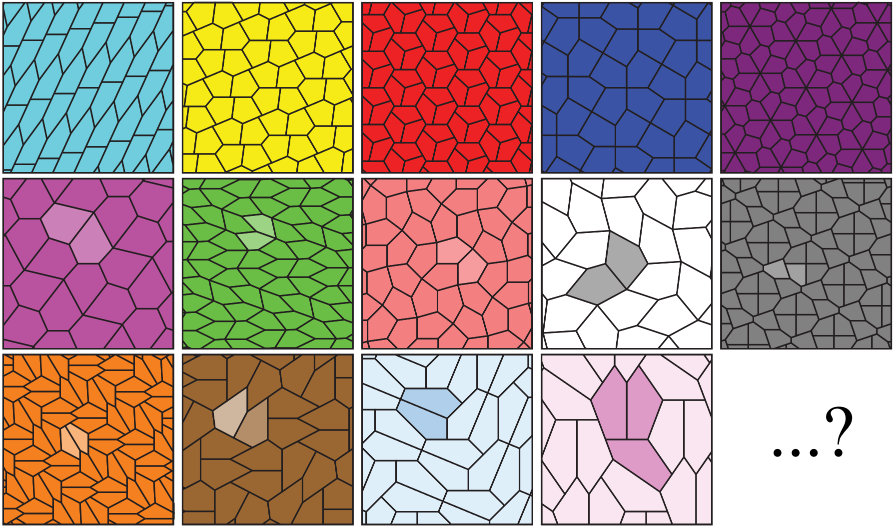


Image: isohedral (hence monohedral) tiling.

Image: another isohedral and monohedral tiling.



Some isohedral and non-isohedral tilings:



The list of all isohedral tilings of $\mathbb{R}^2$ by convex polygons is known (Reinhardt 1918, see also Grünbaum and Shephard 1987)

The list of all convex polygons allowing monohedral tilings of $\mathbb{R}^2$ is maybe incomplete.

- All triangles can tile $\mathbb{R}^2$ by congruent copies.
- All quadrangles (convex or non-convex) can tile $\mathbb{R}^2$.
- There are three kinds of convex hexagons that can tile $\mathbb{R}^2$.

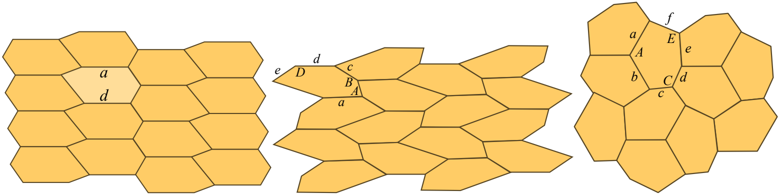
The list of all isohedral tilings of $\mathbb{R}^2$ by convex polygons is known (Reinhardt 1918, see also Grünbaum and Shephard 1987)

The list of all convex polygons allowing monohedral tilings of $\mathbb{R}^2$ is maybe incomplete.

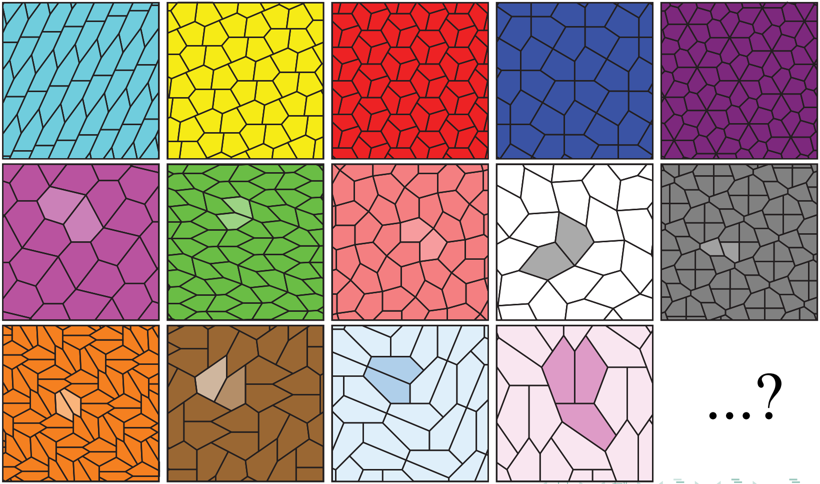
- All triangles can tile $\mathbb{R}^2$ by congruent copies.
- All quadrangles (convex or non-convex) can tile $\mathbb{R}^2$.
- There are three kinds of convex hexagons that can tile $\mathbb{R}^2$.

The three species hexagons allowing monohedral tilings:

Type 1	Type 2	Type 3
$a \parallel d$ $a = d$	$A + B + D = 2\pi$ $a = d,\ c = e$	$A = C = E = 2\pi/3$ $a = b,\ c = d,\ e = f$

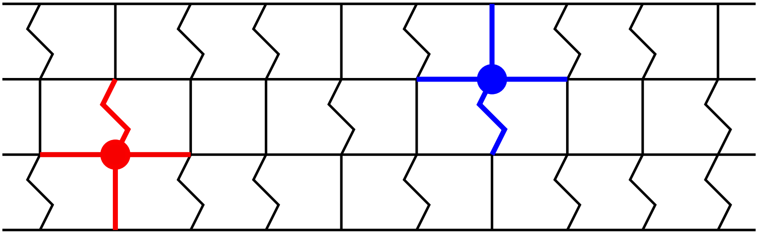


A convex n-gon cannot tile $\mathbb{R}^2$ by congruent copies for $n \geq 7$. Only open case: pentagons. Below a list of 14 species of convex pentagons that can tile $\mathbb{R}^2$ by congruent copies. It is unknown whether this list is complete.



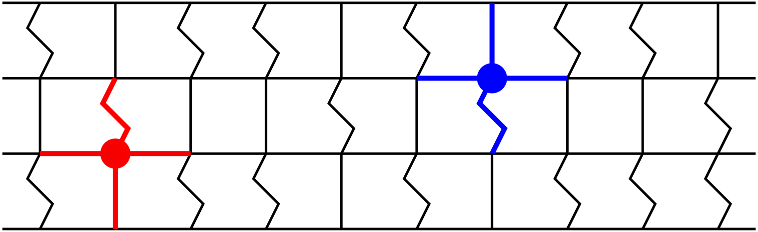
Monogonal and isogonal tilings

A tiling is called **monogonal** if all vertex stars (=vertex plus adjacent edges) are congruent.



A tiling is called **isogonal** if the symmetry group acts transitively on the vertex stars.

A tiling is called **monogonal** if all vertex stars (=vertex plus adjacent edges) are congruent.



A tiling is called **isogonal** if the symmetry group acts transitively on the vertex stars.

The classification of all isogonal tilings is known, see

Grünbaum & Shephard: Tilings and Patterns

A classification of all monogonal tilings seems out of reach.

Monocoronal (and isocoronal) tilings

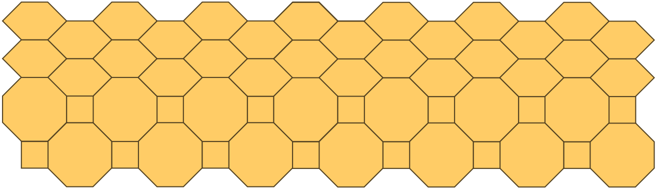
The **vertex corona** of a vertex is this vertex together with its incident tiles.
A tiling is called **monocoronal** if all vertex coronae in the tiling are congruent.

Example: All Archimedean tilings are monocoronal.

The **vertex corona** of a vertex is this vertex together with its incident tiles.
A tiling is called **monocoronal** if all vertex coronae in the tiling are congruent.

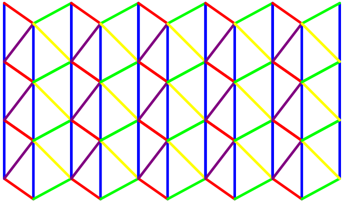
Example: All Archimedean tilings are monocoronal.

More restrictively, a tiling is called **monocoronal wrt direct idometries** if all vertex coronae in the tiling are congruent wrt direct isometries (i.e., no reflections allowed).

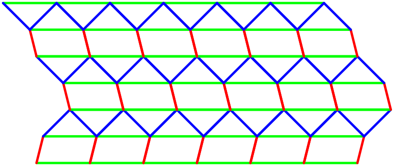


Monogonal, but not monocoronal.

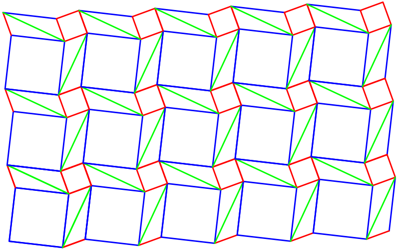
A classification of all monocoronal tilings was obtained in F-Garber 2015. It is known (see Grünbaum & Shephard) that any monocoronal tiling is of one of 11 combinatorial types:



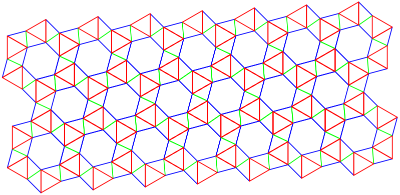
3.3.3.3.3.3



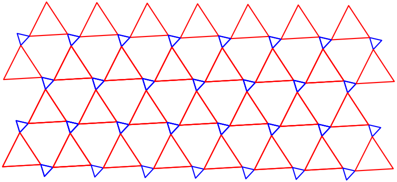
3.3.3.4.4



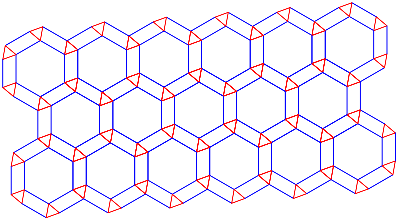
3.3.4.3.4



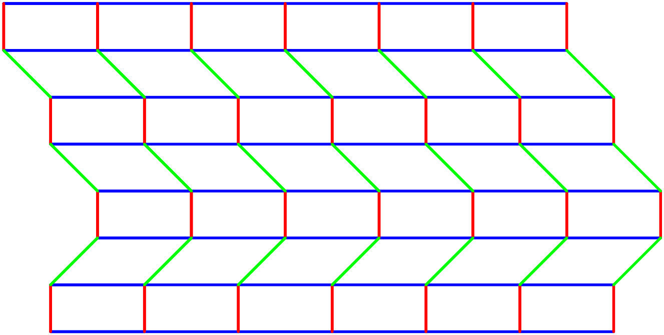
3.3.3.3.6



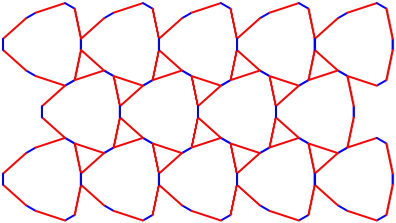
3.6.3.6



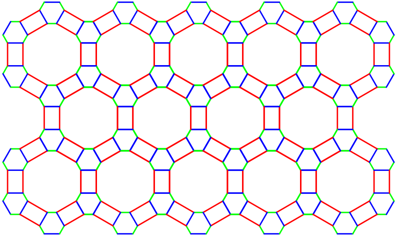
3.4.6.4



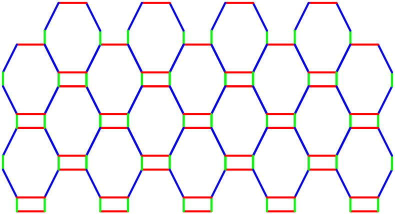
4.4.4.4



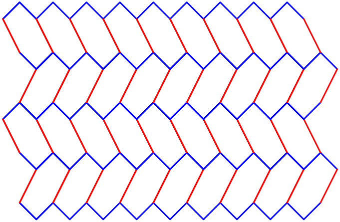
3.12.12



4.6.12



4.8.8



6.6.6

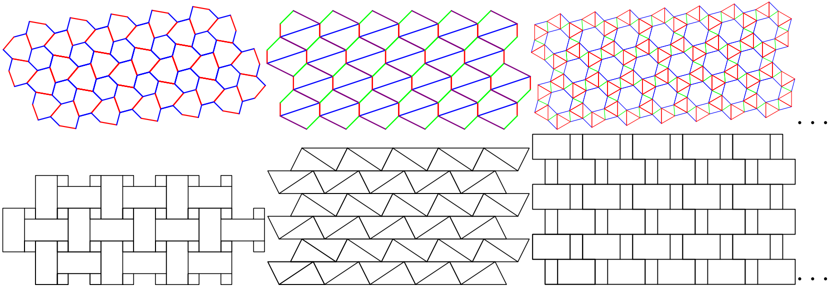
Taking into account metrical properties one gets that:

There are 34 species of monocoronal face-to-face tilings by convex polygons (wrt different vs equal edge length)

Taking into account metrical properties one gets that:

There are 34 species of monocoronal face-to-face tilings by convex polygons (wrt different vs equal edge length)

In fact one may drop the requirements "convex" and "face-to-face". This yields only 15 additional cases to consider. Hence the results below hold for all monocoronal tilings of $\mathbb{R}^2$.



Theorem (F-Garber 2015)

Every tiling that is monocoronal wrt direct isometries (no reflections allowed) has one of the following 12 symmetry groups:

$*632, *442, *333, *2222, \quad 632, 442, 333, 2222, \quad 4*2, 3*3, 2*22, 22*.$

Theorem (F-Garber 2015)

Every tiling that is monocoronal wrt direct isometries (no reflections allowed) has one of the following 12 symmetry groups:

$$*632, *442, *333, *2222, \quad 632, 442, 333, 2222, \quad 4*2, 3*3, 2*22, 22*.$$

In particular: If $\mathcal{T}$ is a monocoronal tiling wrt direct isometries, then

- $\mathcal{T}$ is crystallographic (2-periodic)
- $\mathcal{T}$ is vertex transitive (**isocoronal**)
- $\mathcal{T}$ has a center of rotational symmetry of order at least 2

(Here we use orbifold notation to denote the 17 wallpaper groups. E.g., $*442$ denotes the symmetry group of the regular square tiling; 442 denotes its rotation group)

If we allow reflected copies of vertex coronae the situation becomes more diverse.

Theorem (F-Garber 2015)

Every monocoronal tiling (reflections allowed) is either 1-periodic, or its symmetry is one out of 16 wallpaper groups: any except $*\times$. *If such a tiling is 1-periodic then its symmetry group is one of four frieze groups:*

$$\infty\infty, \infty\times, \infty*, \textit{ or } \quad 22\infty.$$

In particular, every monocoronal tiling is crystallographic or 1-periodic.

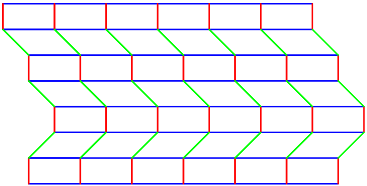
If we allow reflected copies of vertex coronae the situation becomes more diverse.

Theorem (F-Garber 2015)

Every monocoronal tiling (reflections allowed) is either 1-periodic, or its symmetry is one out of 16 wallpaper groups: any except $*\times$. *If such a tiling is 1-periodic then its symmetry group is one of four frieze groups:*

$$\infty\infty, \infty\times, \infty*, \text{ or } \quad 22\infty.$$

In particular, every monocoronal tiling is crystallographic or 1-periodic.



$\leftarrow$ A monocoronal tiling that is not 2-periodic. In the vertical direction one may stack layers like $\cdots LRLLRLLLLRLLLLLLLLRLL \cdots$ (L: layer slanted to the left, R: layer slanted to the right)

For dimensions $d > 2$:

Situation even more diverse. Depending on the exact assumptions (face-to-face or not, reflections allowed or not) the symmetry groups range from trivial to crystallographic.

For dimensions $d > 2$:

Situation even more diverse. Depending on the exact assumptions (face-to-face or not, reflections allowed or not) the symmetry groups range from trivial to crystallographic.

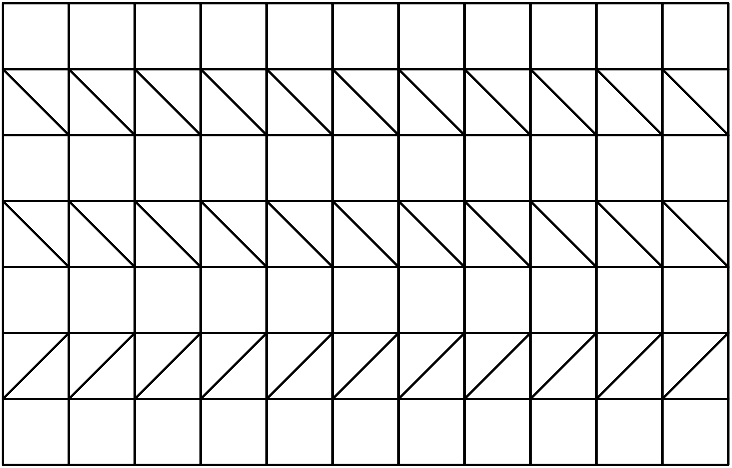
Theorem (F-Garber 2015)

For any $d \geq 3$ there are non-periodic non face-to-face tilings of $\mathbb{R}^d$ that are monocoronal (reflections allowed).

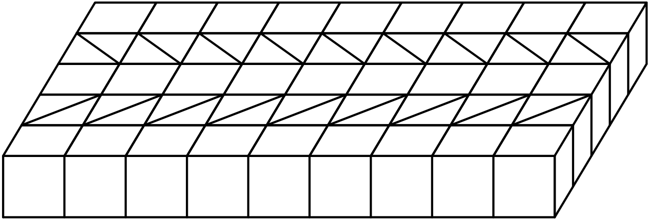
Theorem (F-Garber 2015)

For any $d \geq 4$ there are non-periodic non face-to-face tilings of $\mathbb{R}^d$ that are monocoronal (reflections forbidden).

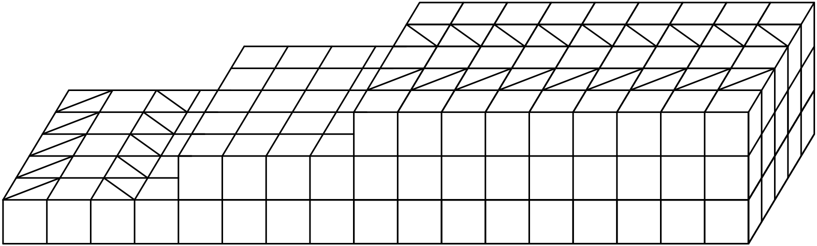
Both results can be obtained by the following construction and its analogues in higher dimensions. Start with a 1-periodic plane tiling:



...thicken it into a 3-dim layer:



...and stack these layers (some of them rotated) alternating with unit cube layers:



Open Problems

- Generalise Theorems 3 and 4 to face-to-face tilings.
- Consider the same problem for the hyperbolic plane $\mathbb{H}^2$.
- Consider the same problem for bicoronal tilings. In particular, is there a bicoronal nonperiodic tiling?

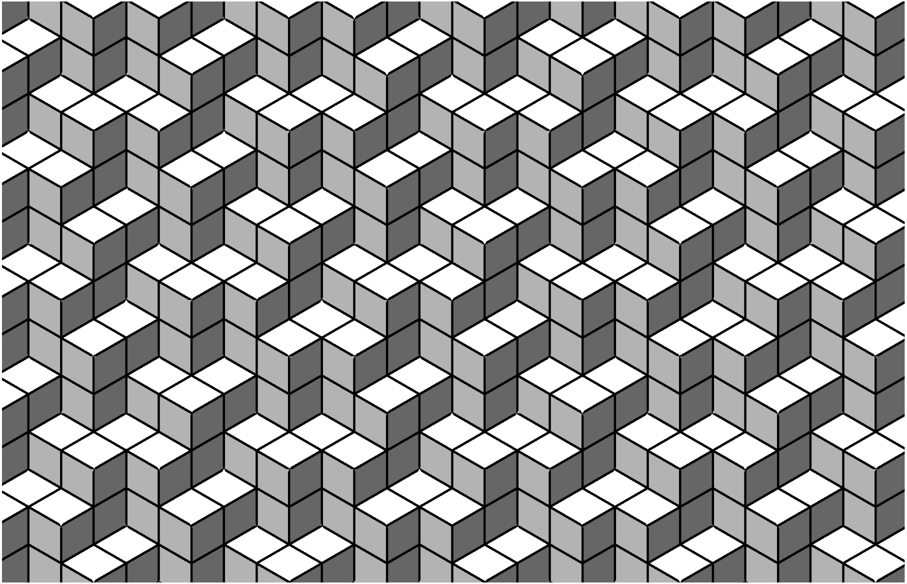
Partial answers to the second problem can be found in F-Garber 2015, using Böröczky tilings.
In particular, there are monocoronal tilings in $\mathbb{H}^2$ with trivial symmetry group.

Open Problems

- Generalise Theorems 3 and 4 to face-to-face tilings.
- Consider the same problem for the hyperbolic plane $\mathbb{H}^2$.
- Consider the same problem for bicoronal tilings. In particular, is there a bicoronal nonperiodic tiling?

Partial answers to the second problem can be found in F-Garber 2015, using Böröczky tilings.
In particular, there are monocoronal tilings in $\mathbb{H}^2$ with trivial symmetry group.

Regarding the third problem: there are non-periodic tricoronal tilings.



Open Problems

- Generalise Theorems 3 and 4 to face-to-face tilings.
- Consider the same problem for the hyperbolic plane $\mathbb{H}^2$.
- Consider the same problem for bicoronal tilings. In particular, is there a bicoronal nonperiodic tiling?

Partial answers to the second problem can be found in F-Garber 2015, using Böröczky tilings.
In particular, there are monocoronal tilings in $\mathbb{H}^2$ with trivial symmetry group.

Open Problems

- Generalise Theorems 3 and 4 to face-to-face tilings.
- Consider the same problem for the hyperbolic plane $\mathbb{H}^2$.
- Consider the same problem for bicoronal tilings. In particular, is there a bicoronal nonperiodic tiling?

Partial answers to the second problem can be found in F-Garber 2015, using Böröczky tilings.
In particular, there are monocoronal tilings in $\mathbb{H}^2$ with trivial symmetry group.

* *
*

Thank you!