# Tilings with tiles in infinitely many orientations

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International conference
"Geometry, Topology and Applications"

*in honour of Nikolai Dolbilins 70th birthday*

Yaroslavl
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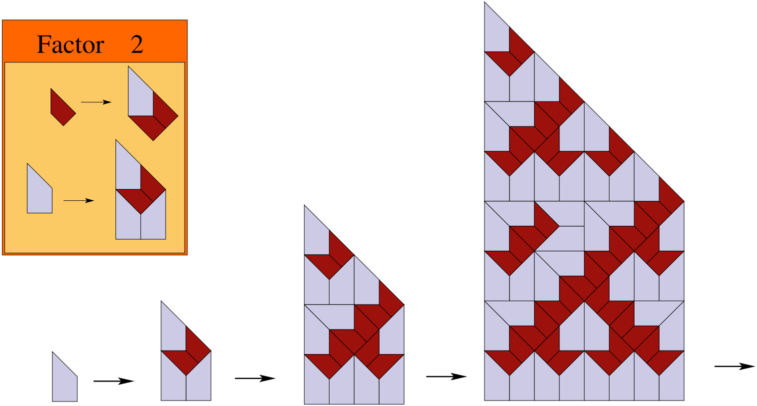
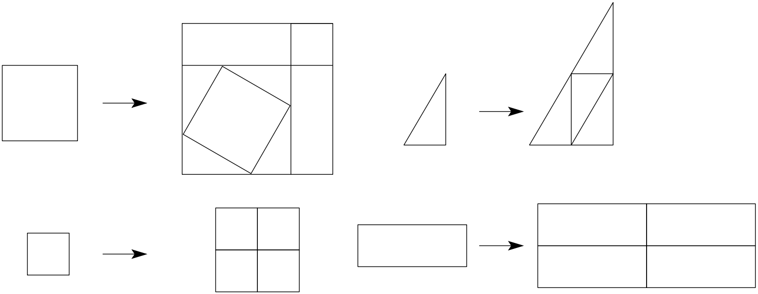
1. Substitution tilings with tiles in infinitely many orientations
2. Dense tile orientations (DTO)
3. Tile shapes forbidding DTO
4. Tilings with rotational symmetry and DTO

Substitution tilings:



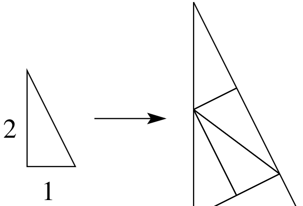

Usually, tiles occur in finitely many different orientations only.

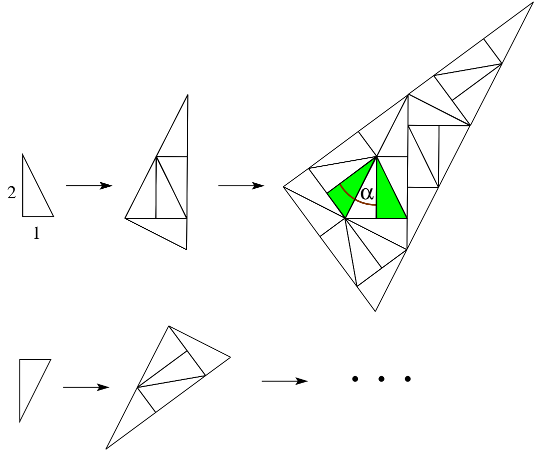
Not always. Cesi's example (1990):



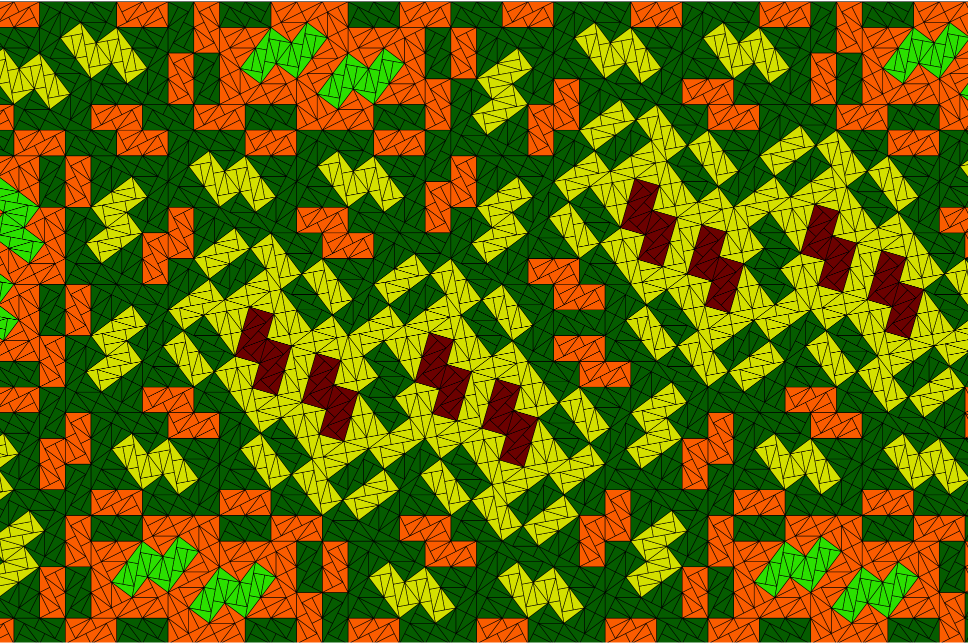
A substitution $\sigma$ is *primitive*, if for any tile $T$ there is $k \geq 1$ such that $\sigma^k(T)$ contains all tile types.

Conway's Pinwheel substitution (1991):

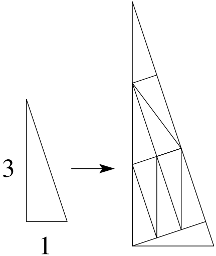




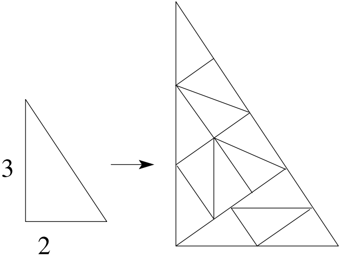

The angle $\alpha$ is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.



Obvious generalizations: Pinwheel $(n, k)$



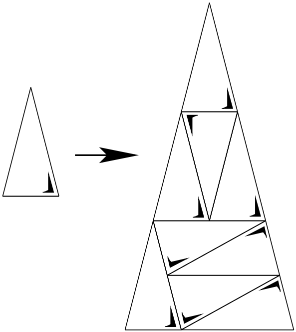

$n = 3, k = 1$

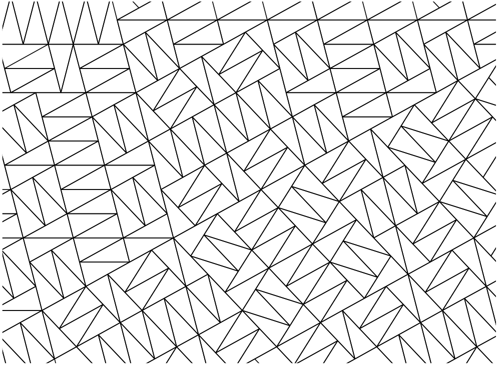



$n = 3, k = 2$

etc.

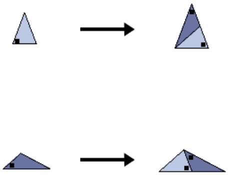
Unknown (< 1996, communicated to me by Danzer):

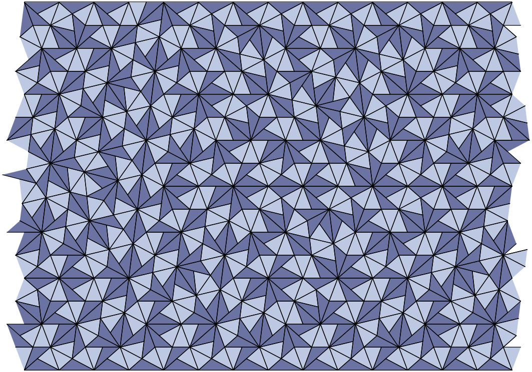




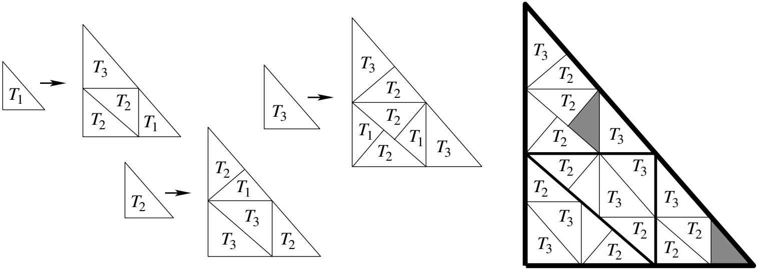
(+ obvious generalizations)

C. Goodman-Strauss, L. Danzer (ca. 1996):





Pythia $(m, j)$, here: $m = 3$, $j = 1$.



# Dense Tile Orientations (DTO)

For all examples: the orientations are dense in $[0, 2\pi[$.

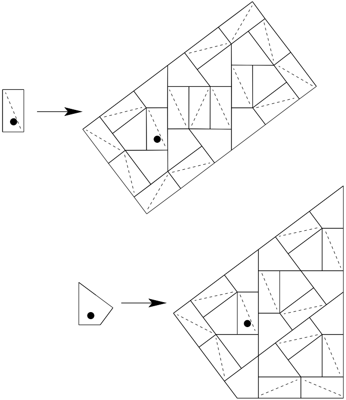
Even more: The orientations are equidistributed in $[0, 2\pi[$.

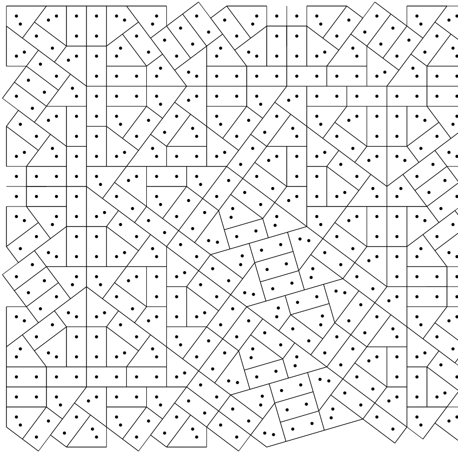
**Theorem (F. '08)**

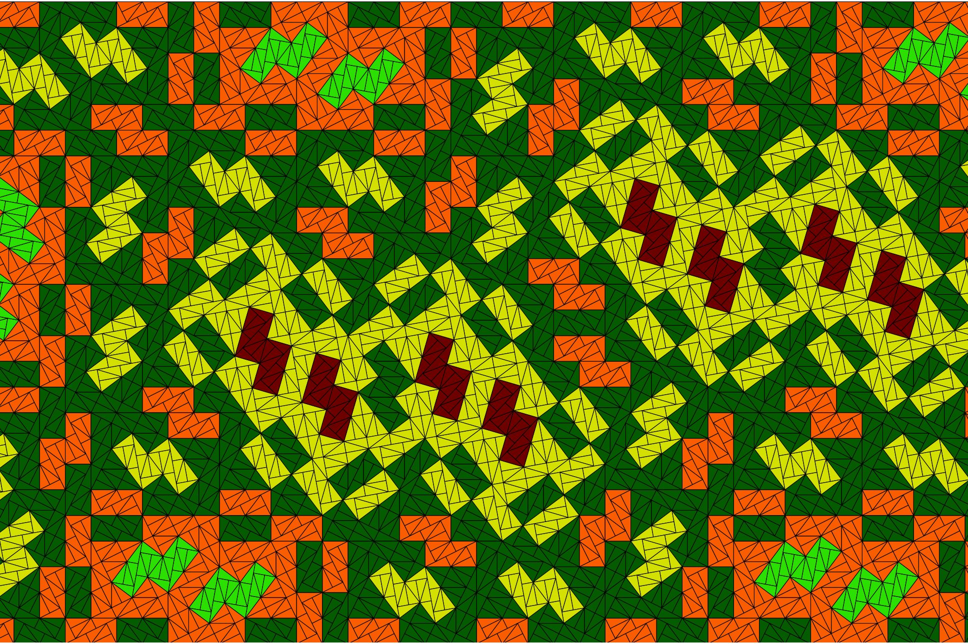
*In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are equidistributed in* $[0, 2\pi[$.

So far: tiles are always triangles. One exception:

Kite Domino (equivalent with Pinwheel):







# Tile shapes forbidding DTO

*Qusstion:* Can we find examples with rhombic tiles for instance?

*Answer:* No.

**Theorem (F.-Harriss, 2013)**

*Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many prototiles (i.e., finitely many different tile shapes). Let all prototiles be centrally symmetric convex polygons. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.*

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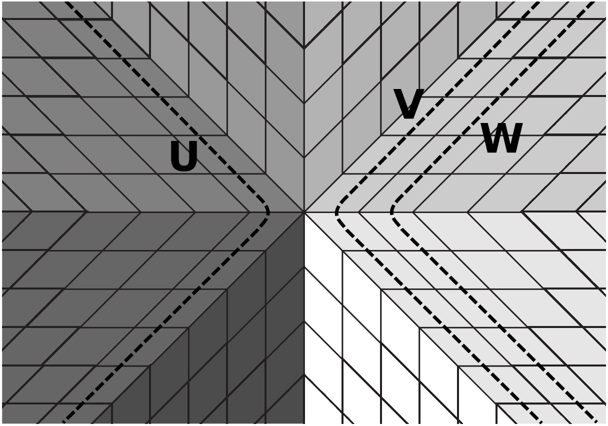
**Theorem (F.-Harriss, 2013)**

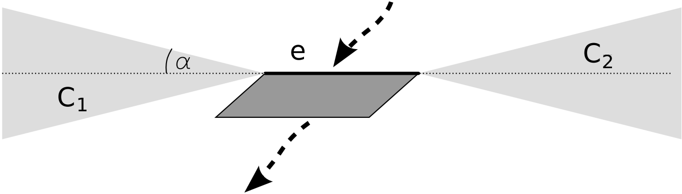
*Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many parallelograms as prototiles. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.*

Assume all tiles are vertex-to-vertex.



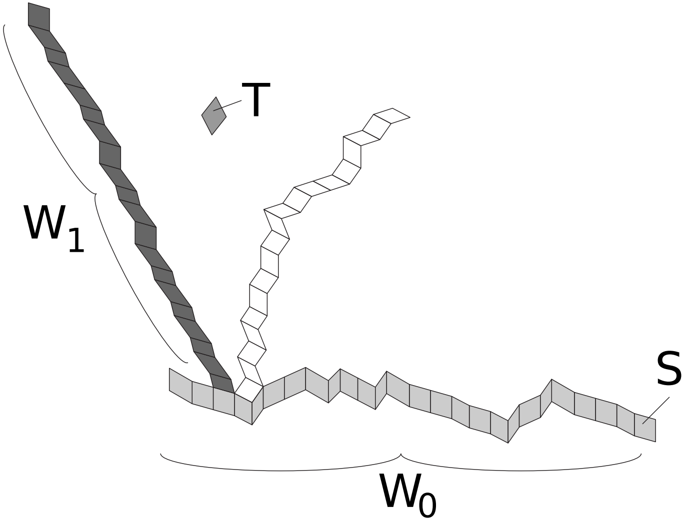

A *worm* is a sequence of tiles $\ldots, T_{-1}, T_0, T_1, T_2, \ldots$ where $T_k$ and $T_{k+1}$ share a common edge, and all shared edges are parallel.

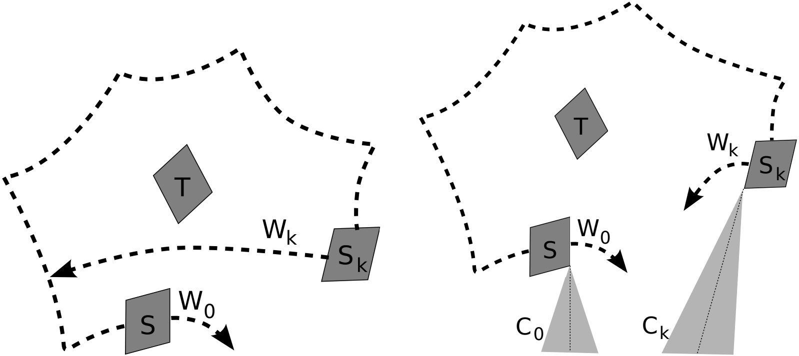
*Cone Lemma:* A worm defined by edge $e$ cannot enter $C_1$ or $C_2$.
($\alpha$ the minimal interior angle in the prototiles)




*Loop Lemma:* A worm has no loop.

*Travel Lemma:* Any two tiles can be connected by a finite sequence of finite worm pieces. (At most $k = \lceil \frac{2\pi}{\alpha} \rceil$ many.)




T
$W_k$
$S_k$
S
$W_0$
T
$W_k$
$S_k$
S
$W_0$
$C_0$
$C_k$

*Proof of theorem (parallelogram version):* Fix some tile $S$. Every tile $T$ can be connected to $S$ by at most $\lceil \frac{2\pi}{\alpha} \rceil$ worm pieces. That is, with $\lceil \frac{2\pi}{\alpha} \rceil$ turns. □

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*Proof of theorem (general)* Any centrally symmetric convex polygon can be dissected into parallelograms.
(see e.g. Kannan-Soroker 1992)

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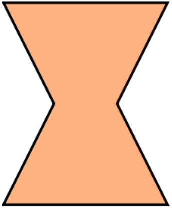
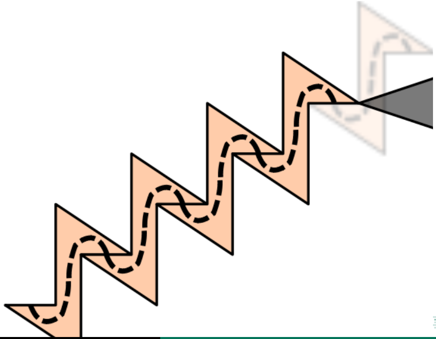
Can we drop "convex"? Hmm...

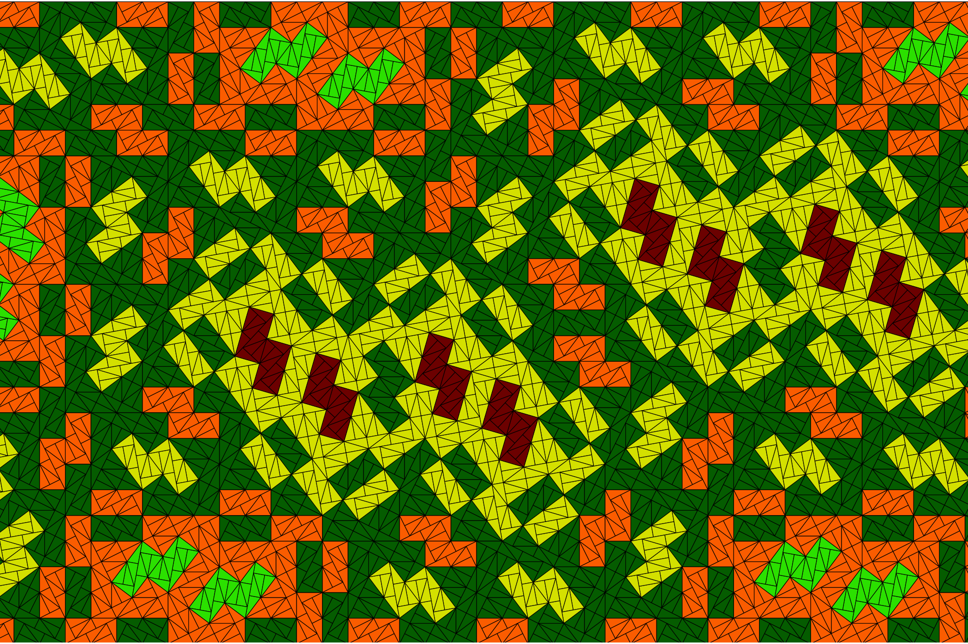




# Tilings with rotational symmetry and DTO

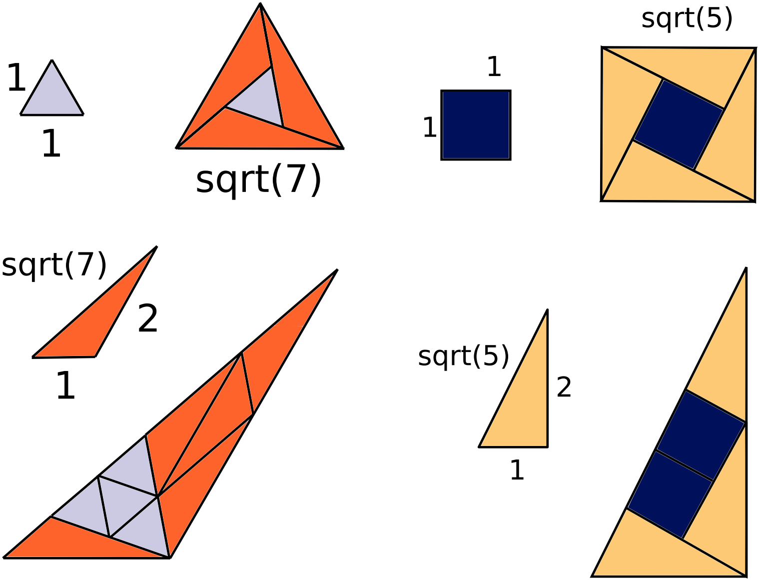
Some tilings with DTO show rotational symmetry.

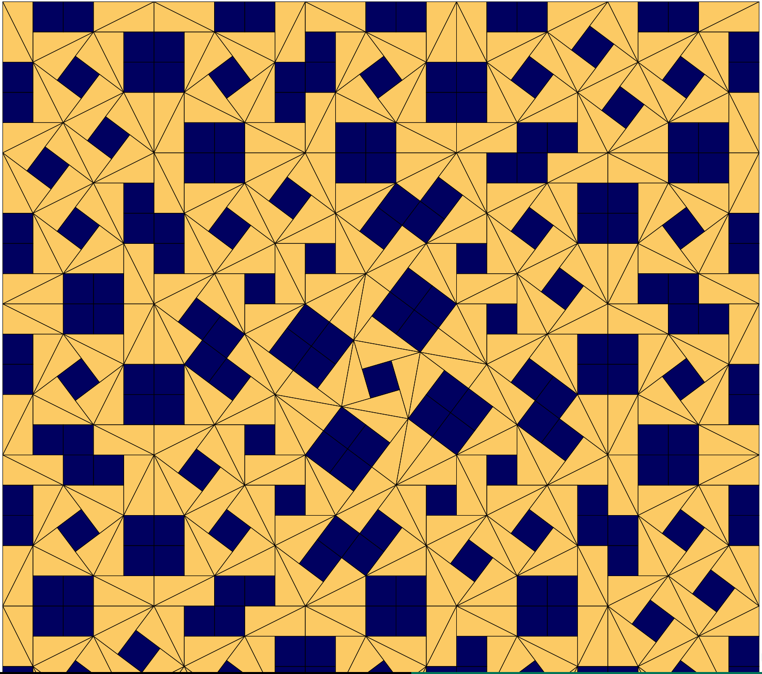
Rotational symmetry causes problems in computing cohomology of tiling spaces (...whatever this means).

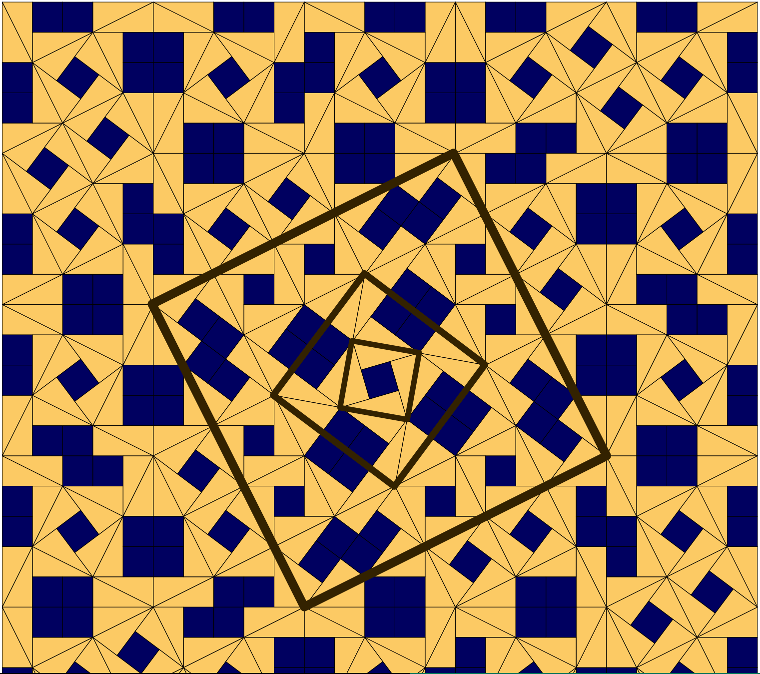


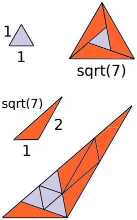
*Question:* Are there tilings with DTO and $n$-fold rotational symmetry for $n \geq 3$?

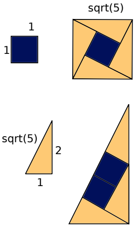
*Answer:* Yes. At least for $n \in \{3, 4, 5, 6, 8\}$.

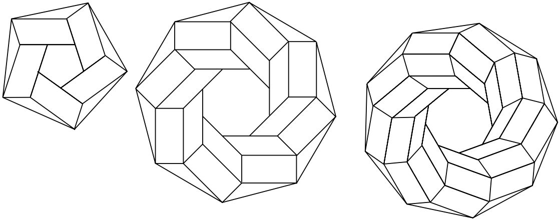

1
1
sqrt(7)
1
1
sqrt(5)
sqrt(7)
2
1
sqrt(5)
2
1






1
1
sqrt(7)
sqrt(7)
2
1


1
1
sqrt(5)
sqrt(5)
2
1



*Conjecture or Fact (?):*

In a parallelogram with edge lengths 1 and 2, and interior angle $\beta$:
If $\beta = \frac{2\pi}{n}$ $(n \geq 4)$ then $\alpha \notin \pi\mathbb{Q}$.