Self–Duality of Galois–dual tilings

Dirk Frettlöh

University of Bielefeld

Bielefeld, Germany

Generalized substitutions, tilings and numeration

March 6–10, 2006

Marseille

Geometric substitution:

- $T_1, T_2, \dots T_m$ *prototiles* in $\mathbb{R}^d$,
- $\lambda > 1$ an algebraic integer (the *inflation factor*),
- $\mathcal{D}_{ij}$ $(1 \le i, j \le m)$ *digit sets* (set of translation vectors)

such that

$$\lambda T_j = \bigcup_{i=1}^{m} T_i + \mathcal{D}_{ij}$$

(non-overlapping). This yields a (geometric) *substitution*

$$\sigma(T_j) := \{ T_i + \mathcal{D}_{ij} \mid i = 1 \dots m \}.$$

σ extends to all sets $\{T_{i(k)} + x_k \mid T_{i(k)} \text{ prototile}, k \in I\}$.

The equation system

$$\lambda T_j = \bigcup_{i=1}^{m} T_i + \mathcal{D}_{ij}$$

gives rise to the corresponding IFS

$$T_j = \bigcup_{i=1}^{m} \lambda^{-1}(T_i + \mathcal{D}_{ij})$$

and vice versa.

σ extends to all sets $\{ T_{i(k)} + x_k \mid T_{i(k)} \text{ prototile}, k \in I \}$.

The equation system

$$\lambda T_j = \bigcup_{i=1}^{m} T_i + \mathcal{D}_{ij}$$

gives rise to the corresponding IFS

$$T_j = \bigcup_{i=1}^{m} \lambda^{-1} T_i + \mathcal{D}_{ij}$$

and vice versa.

The prototiles are the unique compact nonempty solution of the corresponding IFS.

Def.: $M = (|\mathcal{D}_{ij}|)_{1 \le i,j \le m}$ is the *substitution matrix*.

From M we obtain the inflation factor, volumes and frequencies of tiles by Perron–Frobenius theory.

Ex.: $a \to ab, b \to aab$ (symbolic substitution)

$M = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $\lambda = 1 + \sqrt{2}$ (Perron–Frobenius eigenvalue of M).

The prototiles $a = [0,1]$, $b = [0, \sqrt{2}]$ are the solution of the IFS

$$a = \lambda^{-1} a \cup \lambda^{-1} b + 1, \quad b = \lambda^{-1} a \cup \lambda^{-1} a + 1 \cup \lambda^{-1} b + 2$$

resp. a — non unique — solution of the expanding counterpart

$$\lambda a = a \cup b + 1, \quad \lambda b = a \cup a + 1 \cup b + 2$$

Thus $\mathcal{D} = \begin{pmatrix} \{0\} & \{0,1\} \\ \{1\} & \{2\} \end{pmatrix}$

Dual substitution

Let σ be a substitution, such that the inflation factor λ is a *PV* number (Pisot Vijayaraghavan number) of algebraic degree m.

In this talk:

- λ unimodular, real PV.
- Tilings in dimensions $d = 1$ or $d = 2$ only.
- All vertices, maps... can be expressed in $\mathbb{Z}[\lambda]$ $(d = 1)$ respectively $\mathbb{Z}[i, \lambda]$ $(d = 2)$.

We know: (from Thurston, Gelbrich, Pleasants,...)

Every point $x \in \mathbb{Z}[\lambda]$ can be lifted to a lattice Γ in $\mathbb{R}^m$. The canonical projection π_1 is a bijection $\Gamma \to \mathbb{Z}[\lambda]$.

E.g., if $d = 1$, then the lattice

$$\langle (1,1,\ldots,1),(\lambda,\lambda_2,\ldots,\lambda_m)\ldots(\lambda^{m-1},\lambda_2^{m-1},\ldots,\lambda_m^{m-1})\rangle_{\mathbb{Z}}$$

will do.

Multiplication by λ in $\mathbb{Z}[\lambda]$ corresponds to a lattice automorphism g of Γ.

Let π_2 be the projection to the orthogonal complement of $\pi_1(\mathbb{R}^m)$.

The star–map:

If $x = \sum_{i=0}^{m-1} \alpha_i \lambda^i$, then let

$$x^* := \pi_2(\pi_1^{-1}(x)).$$

In the present setting, that is

$$x^* := \sum_{i=0}^{m-1} \alpha_i (\lambda_2^i, \ldots, \lambda_m^i),$$

where $\lambda_2, \ldots, \lambda_m$ are the algebraic conjugates of λ.

Then the *dual substitution* is

$$\sigma^*(T_i^*) := \{ T_j^* + \mathcal{D}_{ij}^* \,|\, j = 1 \dots m \}.$$

The *dual prototiles* arise from the *dual IFS*:

$$T_i^* = \bigcup_{j=1}^{m} QT_i^* + \mathcal{D}_{ji}^*,$$

$(Q = \mathrm{diag}(\lambda_2, \dots, \lambda_m).)$

Ex.:(cont.) $a \to ab, b \to aab$

$$\mathcal{D} = \begin{pmatrix} \{0\} & \{0,1\} \\ \{1\} & \{2\} \end{pmatrix} \qquad \text{Conjugate } \lambda_2 = 1 - \sqrt{2}.$$

$$\mathcal{D}^* = \begin{pmatrix} \{0\} & \{1\} \\ \{0,1\} & \{2\} \end{pmatrix}$$

Dual IFS: $a^* = \lambda_2 a^* \cup \lambda_2 b^* \cup \lambda_2 b^* + 1, b^* = \lambda_2 a^* + 1 \cup \lambda_2 b^* + 2.$

Prototiles $a^* = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$, $b^* = [-\frac{\sqrt{2}}{2} - 1, -\frac{\sqrt{2}}{2}]$.

Dual substitution: $a^* \to b^* b^* a^*$, $b^* \to b^* a^*$.

With $a^* \to b$, $b^* \to a$: the same tilings as before.

$$\boxed{\sigma: \quad S \to ML, \quad M \to SML, \quad L \to LML}$$

Substitution matrix: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

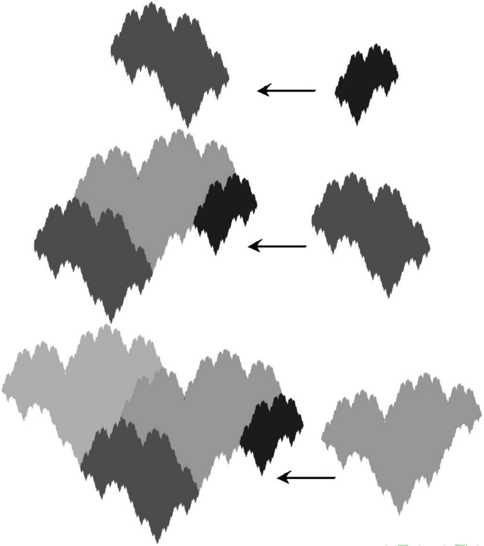
Characteristic Polynomial: $x^3 - 3x^2 + 1$

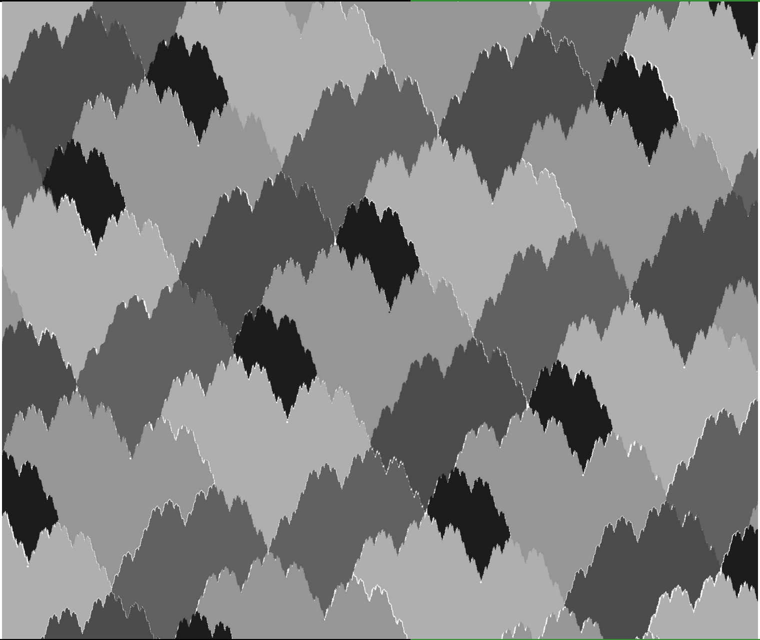
Eigenvalues: $\lambda = \lambda_1, \qquad \lambda_2, \qquad \lambda_3$

$$\frac{\sin(\frac{4\pi}{9})}{\sin(\frac{\pi}{9})}, \; -\frac{\sin(\frac{\pi}{9})}{\sin(\frac{2\pi}{9})}, \; \frac{\sin(\frac{2\pi}{9})}{\sin(\frac{4\pi}{9})}$$

$$\approx 2.879, \quad -0.532, \quad 0.6527$$

Dual of $S \to ML, M \to SML, L \to LML$:





Questions

- What are the duals of Ammann-Beenker, Penrose,...?
- Which tilings are self–dual?
- What is the correct definition of self–dual?

Ex.: Penrose tiling, version with Robinson triangles:

- 40 prototiles up to translations
- 2 prototiles up to isometries

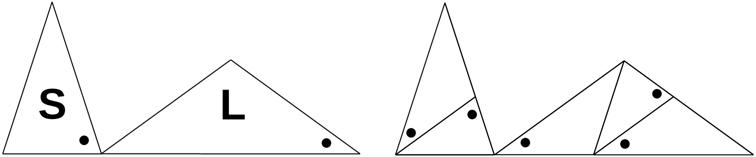
So it is better to work with isometries instead of Digit sets.

(Allow reflections and rotations.)

For the Penrose and Ammann–Beenker tilings we can use cyclotomic number fields $\mathbb{Z}[\xi]$, $\xi = e^{2\pi i/n}$.

Express all maps by elements of $\mathbb{Z}[\xi]$. Then replace each ξ by an appropriate Galois conjugate.

Ex.: Penrose (factor $\tau = \frac{\sqrt{5}+1}{2}$.) in $\mathbb{Z}[e^{2\pi/5}]$.



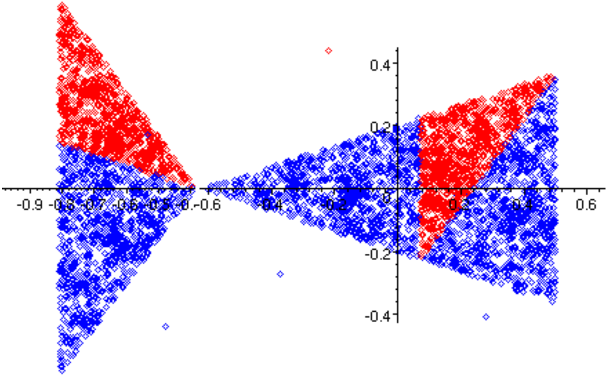
$$S = f_1(S) \cup f_2(L), \quad L = f_3(L) \cup f_4(S) \cup f_5(L)$$

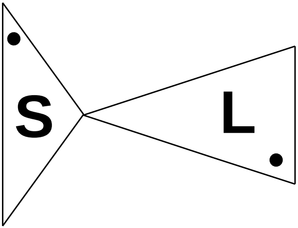
E.g. $f_5(z) = \tau^{-1} R_v z + 1 + \tau$ $\quad$ (R_v: reflection in the vertical axis)

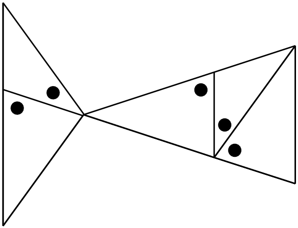
$$= (\xi + \xi^4)\xi^5 \overline{z} + 1 - \xi^2 - \xi^3. \quad (\xi = e^{\frac{2\pi i}{5}})$$

In f_i^{-1}, replace each ξ by ξ^3.

$$S^* = f_1^*(S^*) \cup f_4^*(L^*), \quad L^* = f_2^*(S^*) \cup f_3^*(L^*) \cup f_5^*(L^*)$$

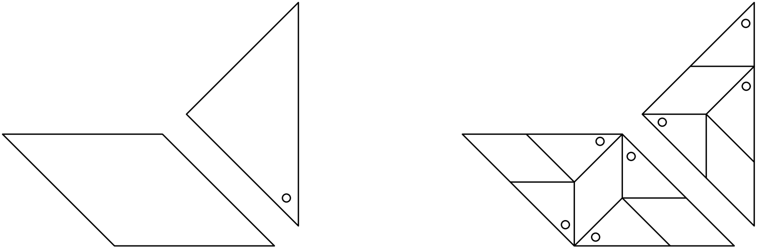

0.4
0.2
-0.2
-0.4
-0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.6 -0.4 -0.2 0 0.2 0.4 0.6



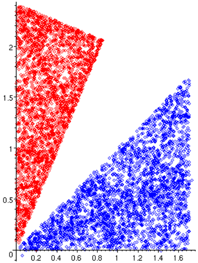


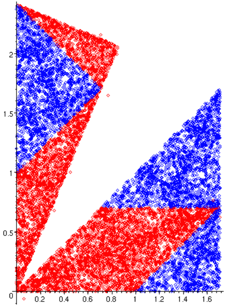
The dual of the Penrose tiling: the Tübingen triangle tiling.

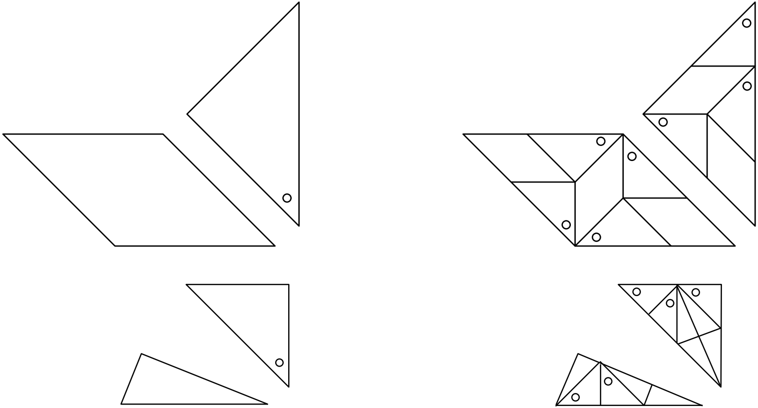
Ex.: Ammann–Beenker (factor $\sqrt{2}+1$.) in $\mathbb{Z}[e^{2\pi/8}]$.



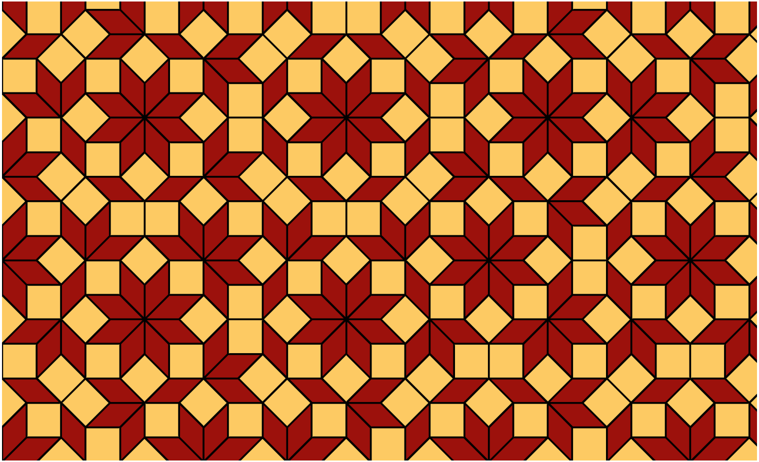
Again, in each f_i^{-1}, replace each ξ by ξ^3 to obtain the dual substitution, resp. the dual IFS.

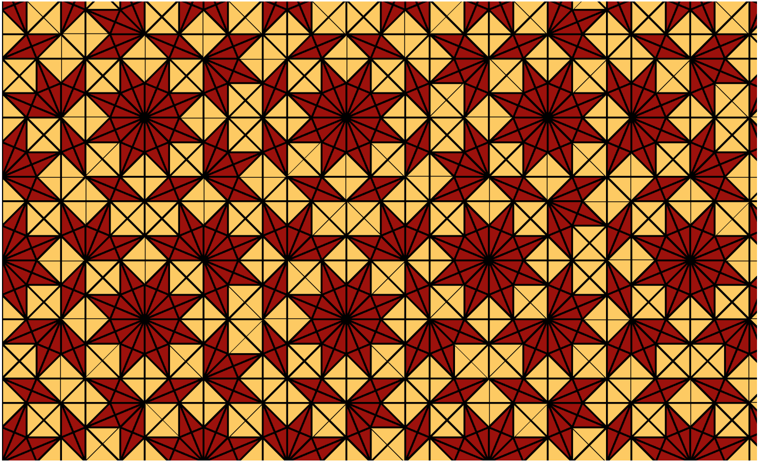

0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0.5
1
1.5
2


0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0.5
1
1.5
2

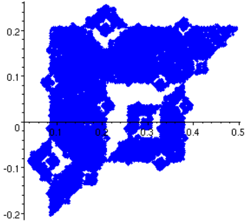


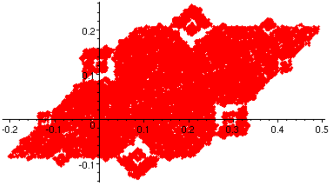
The dual of the Ammann Beenker: a slightly different 8fold tiling.

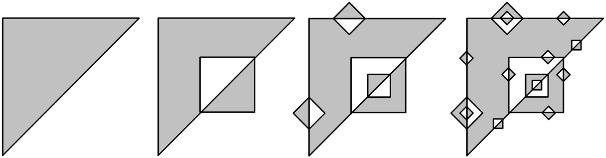




Variations:







Necessary:

- Factor λ is of algebraic degree 2.
- Substitution matrix $M^T = PMP^{-1}$.

Sufficient:

- Digit set matrix $\mathcal{D} = P(\pm\mathcal{D}^* + t)P^{-1}$.

  (P a permutatation matrix)

TODO:

- Extend the definition of duality to cover p–adic spaces
- Good definition of self-dual.
- Classify all self–dual tilings.