

Aperiodic substitution tilings with infinitely many prototiles

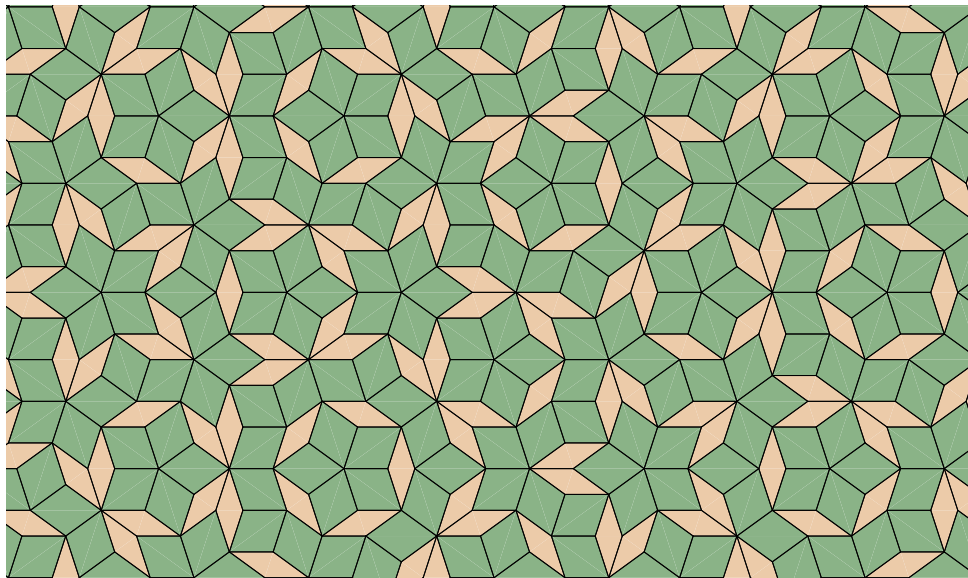
Dirk Frettlöh

Joint work with Alexey Garber, Neil Mañibo, Jan Mazáč

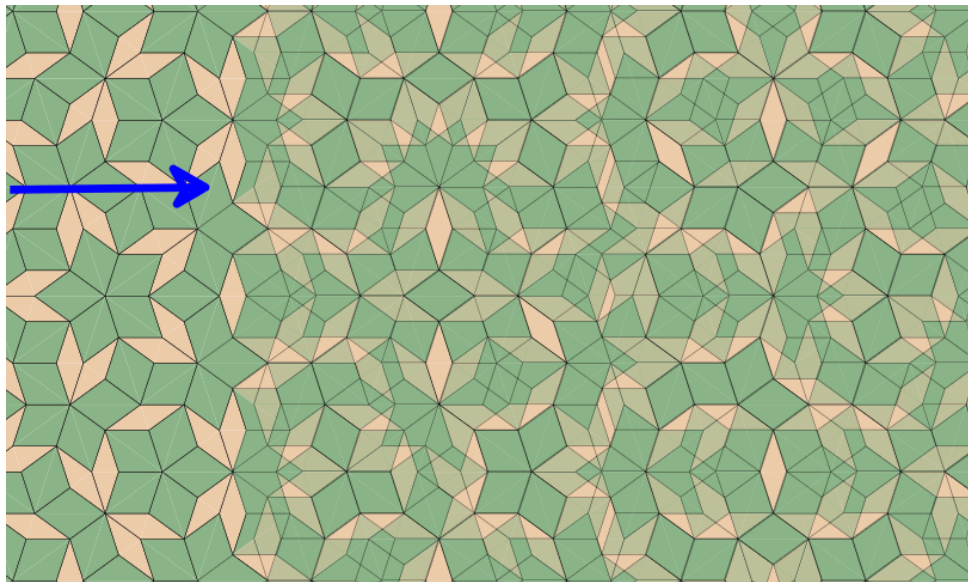
Technische Fakultät
Universität Bielefeld

Osnabrück, 3rd December 2025

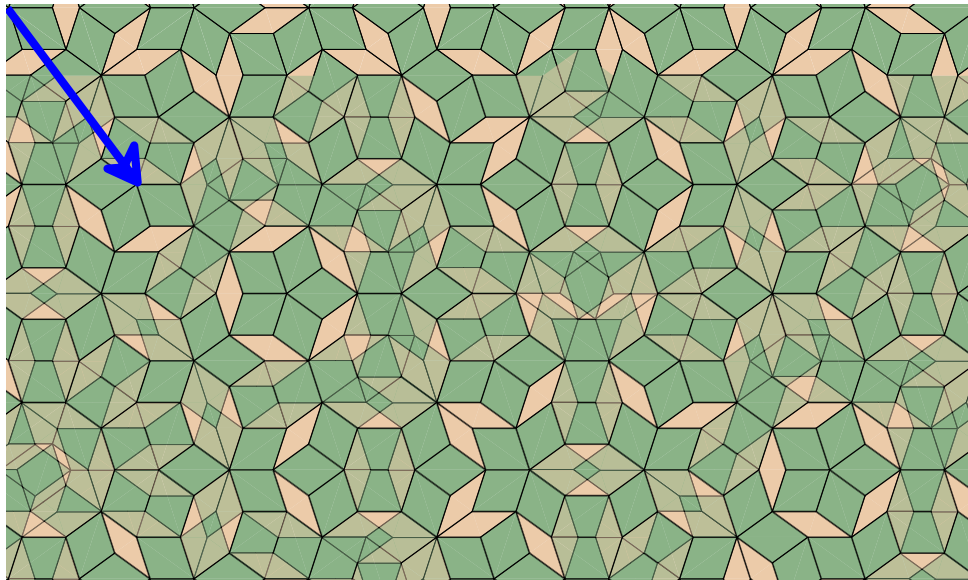
A famous aperiodic tiling: the Penrose tiling



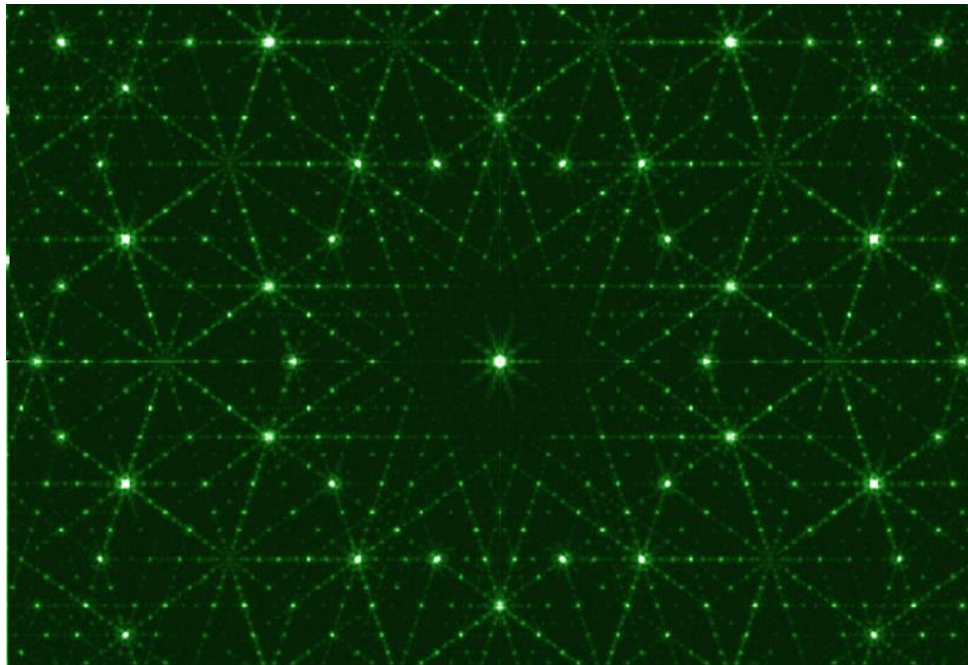
'Aperiodic': not fixed by any non-zero translation...



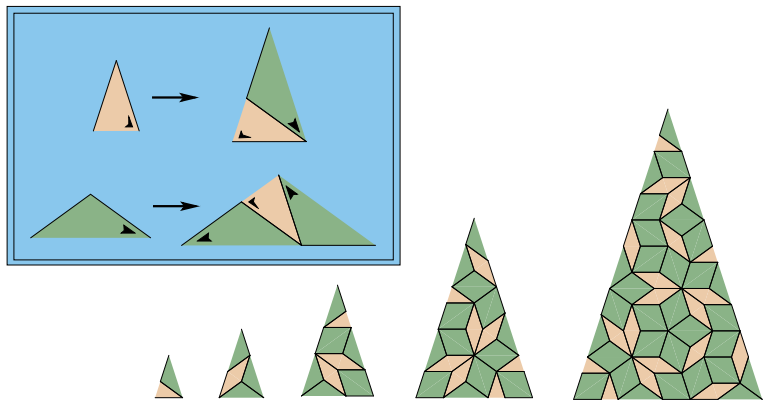
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Diffraction image of a Penrose tiling:

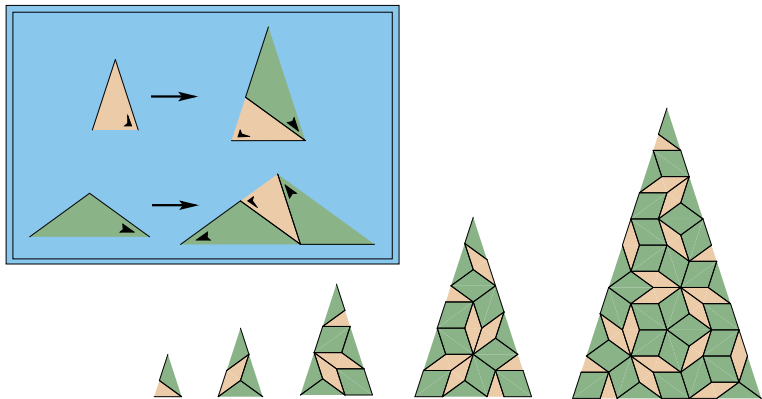


One way to generate nice aperiodic tilings: a tile substitution



Substitution matrix: $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, inflation factor $\frac{1}{2}(1 + \sqrt{5})$.

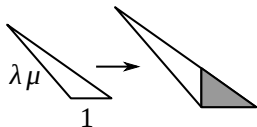
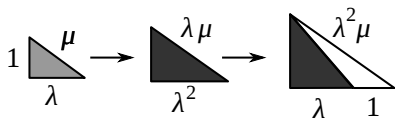
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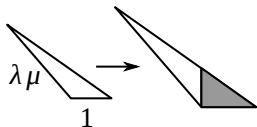
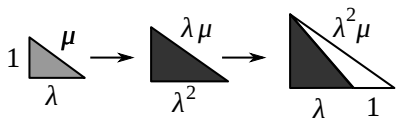
2 *prototiles*.

In dimension $d = 2$:



- ▶ substitution matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$,
- ▶ inflation factor $\lambda = 1.3247\dots$ (the *plastic number*),
- ▶ minimal polynomial $x^3 - x - 1$.

In dimension $d = 2$:



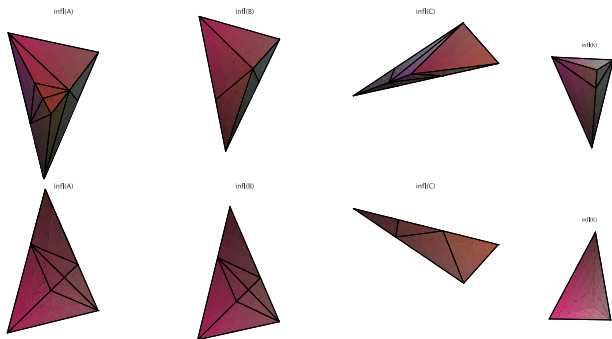
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▶ minimal polynomial $x^3 - x - 1$.

3 prototiles.

In dimension $d = 3$:

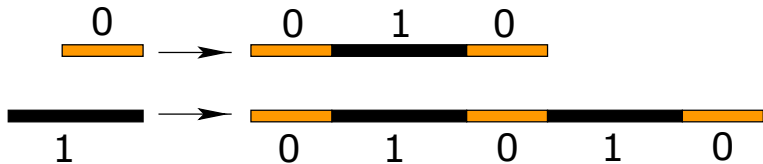


▶ substitution matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 6 & 4 & 2 & 1 \end{pmatrix}$,

▶ inflation factor $\lambda = \frac{1}{2}(\sqrt{5} + 1)$ (the *golden mean* again),

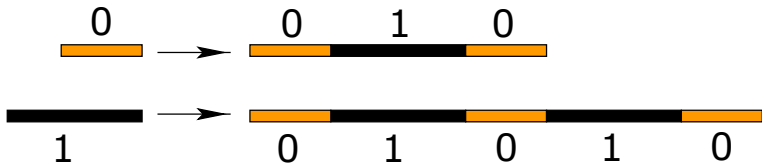
▶ minimal polynomial $x^2 - x - 1$.

Substitution tiling in dimension $d = 1$:



- ▶ substitution matrix $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$,
- ▶ inflation factor $\lambda = 2 + \sqrt{3}$,
- ▶ minimal polynomial $x^2 - 4x + 1$.

Substitution tiling in dimension $d = 1$:



- ▶ substitution matrix $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$,
- ▶ inflation factor $\lambda = 2 + \sqrt{3}$,
- ▶ minimal polynomial $x^2 - 4x + 1$.

For many more examples: visit

<https://tilings.math.uni-bielefeld.de>