

# SUBSTITUTION TILINGS WITH DENSE TILE ORIENTATIONS AND $n$ -FOLD ROTATIONAL SYMMETRY

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ABSTRACT. It is shown that there are primitive substitution tilings with dense tile orientations invariant under  $n$ -fold rotation for  $n \in \{2, 3, 4, 5, 6, 8\}$ . The proof for dense tile orientations uses a general result about irrationality of angles in certain parallelograms.

## 1. INTRODUCTION

From the discovery of Penrose tilings in the 70s [13] and of quasicrystals in the 80s [19] evolved a theory of aperiodic order. One main method to produce interesting patterns showing aperiodic order is a tile substitution. For a more precise description see below. The idea is illustrated in Figure 1: a tile substitution is a rule of how to enlarge a given *prototile* (or a set of several prototiles) and dissect it into congruent copies of the prototiles. The rule can be iterated to fill larger and larger regions of the plane. Formally one considers a fixed point of the substitution: an

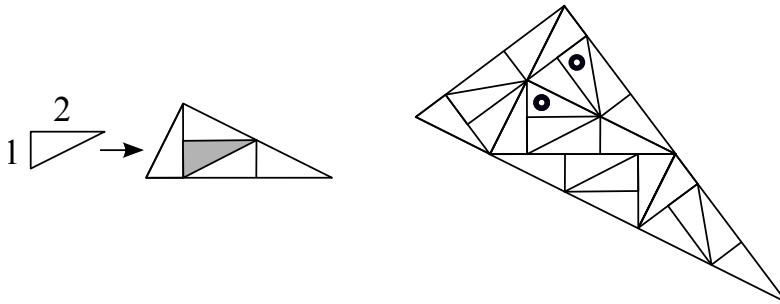


FIGURE 1. Two iterations of the substitution for the pinwheel tiling.

infinite tiling  $\mathcal{T}$  of the plane invariant under the substitution rule. This fixed point  $\mathcal{T}$  yields the *hull* of the tiling: the closure of the image  $G\mathcal{T}$  of  $\mathcal{T}$ , where  $G$  is a group acting on  $\mathbb{R}^2$  (usually all translations in  $\mathbb{R}^2$ , or all rigid motions), and closure is taken with respect to the local topology. For details see below, for more details see for instance [3].

Several mathematical fields interact in the theory of aperiodic order. A lot of literature is dedicated to studying the topology of the hull of an aperiodic tiling. One way to do this is to compute its cohomology groups. For substitution tilings this can be done by the methods introduced in [2]. For the pinwheel tiling this was done in [4] and [9]. One problem for the pinwheel tiling is that the tiles occur in infinitely many different orientations. More precisely: the pinwheel tiling has *dense tile orientations* (DTO), i.e. the orientations of the tiles are dense on the circle. For the treatment of hulls of tilings with DTO in the context of dynamical systems see [8]. A further problem is that the hull of the pinwheel tiling contains six different tilings invariant under 2-fold rotation. These tilings correspond to cone singularities of the quotient of the hull by the circle. These give rise to a torsion part in the second cohomology group  $H^2$  of the hull. In particular,  $m$  tilings in the hull that are invariant under  $n$ -fold rotation contribute a  $\mathbb{Z}_n^{m-1}$  subgroup to  $H^2$ , where  $\mathbb{Z}_n$  denotes the cyclic group of order  $n$  [4, Theorem 12]. In view of this problem Jean Savinien [16] asked in 2013 for which values of  $n$  primitive substitution tilings with

DTO can be invariant under  $n$ -fold rotation. (For the definition of primitivity see below.) This question motivated this paper. Our main result is the following.

**Theorem 1.** *There are primitive substitution tilings with DTO that are invariant under rotation by  $\frac{2\pi}{n}$  for  $n \in \{2, 3, 4, 5, 6, 8\}$ . These tilings are not mirror symmetric, hence they occur in pairs for each such  $n$ .*

Hence we have  $m \geq 2$  in the discussion above, and the contributions  $\mathbb{Z}_n^{m-1}$  are not trivial. It is likely that the idea carries over to any  $n \in \mathbb{N}$ , but since the proof is constructive (and the constructions become tedious for large  $n$ ) we can deal only with the small cases here. The cases  $n \in \{3, 4, 6\}$  are considered in more detail in [7]. The case  $n = 7$  is treated in [17]. The case  $n = 2$  is known already to occur in pinwheel tilings.

This paper is organised as follows. Section 2 contains some basic definitions and facts on substitution tilings. Readers familiar with this topic may skip this section. In order to show that all tilings have DTO we need a result on the irrationality of certain angles. This is provided in Theorem 5 in Section 3. The construction of the substitution rules is given in Section 4. Theorem 1 is then a consequence of Propositions 6, 7, 8, and 9 in Section 4.

## 2. BASICS

For the purpose of this paper a *tile* is a nonempty compact set  $T \subset \mathbb{R}^2$  which is the closure of its interior. A *tiling* of  $\mathbb{R}^2$  is a collection of tiles  $\mathcal{T} = \{T_i \mid i \in \mathbb{N}\}$  that is a covering (i.e.  $\bigcup_{i \in \mathbb{N}} T_i = \mathbb{R}^2$ ) as well as a packing (i.e. the intersection of the interiors of any two distinct tiles  $T_i$  and  $T_j$  is empty). A finite subset of  $\mathcal{T}$  is called a *patch* of  $\mathcal{T}$ . A tiling  $\mathcal{T}$  has *finite local complexity* with respect to rigid motions (FLC for short) if for any  $r > 0$  there are only finitely many pairwise non-congruent patches in  $\mathcal{T}$  fitting into a ball of radius  $r$ . (In many other contexts one may replace “non-congruent” by “not translates of each other”, but here the first option is the appropriate one.)

A tiling  $\mathcal{T}$  is *nonperiodic*, if  $\mathcal{T} + t = \mathcal{T}$  ( $t \in \mathbb{R}^2$ ) implies  $t = 0$ . In addition,  $\mathcal{T}$  is called *aperiodic* if each tiling in the hull of  $\mathcal{T}$  is nonperiodic. The *hull* of the tiling  $\mathcal{T}$  in  $\mathbb{R}^2$  is the closure of the set  $\{x\mathcal{T} \mid x \in G\}$  in the local topology. Usually one takes  $G = \mathbb{R}^2$  regarded as translations acting on  $\mathcal{T}$ , or  $G$  the group of all rigid motions in  $\mathbb{R}^2$ . In our case it does not matter which one of the two we choose, see [8]. The local topology can be defined via a metric. In this metric two tilings are  $\varepsilon$ -close if they agree on a large ball of radius  $\frac{1}{\varepsilon}$  around the origin, possibly after a small motion (e.g. a translation by less than  $\varepsilon$ ). If  $\mathcal{T}$  arises from a primitive substitution  $\sigma$  one may as well speak of the *hull of  $\sigma$* , since all tilings generated by  $\sigma$  define the same hull. See for instance [20, 15, 3, 12, 8] for more details.

A *substitution rule*  $\sigma$  is a simple method to generate nonperiodic tilings. A substitution rule consists of several *prototiles*  $T_1, \dots, T_m$ , an *inflation factor*  $\lambda > 1$  and for each  $i = 1, \dots, m$  a dissection of  $\lambda T_i$  into congruent copies of some of the prototiles  $T_1, \dots, T_m$ . The patch resulting from the dissection is denoted by  $\sigma(T_i)$ . A substitution  $\sigma$  can be iterated on the resulting patch, by inflating the patch by  $\lambda$  and dissecting all tiles according to  $\sigma$ . Hence it makes sense to write  $\sigma^2(T_i)$  or  $\sigma^k(T_i)$ . A simple example is the substitution for the pinwheel tiling shown in Figure 1. This substitution uses just one prototile. The inflation factor is  $\lambda = \sqrt{5}$ . One may as well formulate the pinwheel substitution for two prototiles: if we distinguish a tile and its mirror image then the pinwheel substitution  $\sigma_P$  has two prototiles  $T_1$  and  $T_2$  (where  $T_2$  is the mirror image of  $T_1$ ), the substitution  $\sigma_P(T_2)$  is the mirror image of  $\sigma_P(T_1)$ .

In certain instances we want to consider congruent tiles in  $\mathcal{T}$  as different. This is achieved by markings or colours. Two tiles are *equivalent* if they are congruent and have the same marking or colour. See Subsection 4.4 below for an example where we consider congruent prototiles as different (e.g.  $T_3^{(5)}, T_4^{(5)}, T_5^{(5)}$ ), with a different substitution for each prototile.

Given a substitution  $\sigma$  with prototiles  $T_1, \dots, T_m$  a patch of the form  $\sigma(T_i)$  is called a *supertile*. More generally, a patch of the form  $\sigma^k(T_i)$  is called a  *$k$ -th order supertile*. A substitution rule is called *primitive* if there is  $k \in \mathbb{N}$  such that each  $k$ -th order supertile contains congruent copies of all prototiles.

Equivalently one may define primitivity of a substitution by an associated matrix. The *substitution matrix* of a substitution  $\sigma$  with prototiles  $T_1, T_2, \dots, T_m$  is  $M_\sigma := (a_{ij})_{1 \leq i, j \leq m}$ , where  $a_{ij}$  is the number of tiles equivalent to  $T_i$  in  $\sigma(T_j)$ ,  $i, j \in \{1, 2, \dots, m\}$ . The substitution is primitive if there is some power of the substitution matrix containing positive entries only. Primitivity is an important property for substitutions. One reason is the following result, the Perron-Frobenius theorem.

**Theorem 2** ([14]). *Let  $M$  be a primitive non-negative square matrix. Then  $M$  has a real eigenvalue  $\lambda > 0$  which is simple. Moreover,  $\lambda > |\lambda'|$  for any eigenvalue  $\lambda' \neq \lambda$ . This eigenvalue  $\lambda$  is called Perron-Frobenius-eigenvalue or PF-eigenvalue for short. Furthermore, the associated left and right eigenvectors of  $\lambda$  can be chosen to have positive entries. Such eigenvectors are called the left PF-eigenvector and right PF-eigenvector of  $M$ .*

Applied to a substitution tiling  $\mathcal{T}$  the Perron Frobenius theorem has the following consequences, see for instance [15, 3].

**Theorem 3.** *Let  $\sigma$  be a primitive substitution in  $\mathbb{R}^2$  with inflation factor  $\lambda$  and prototiles  $T_1, T_2, \dots, T_m$ ; let  $M_\sigma$  be the substitution matrix of  $\sigma$ . Then the PF-eigenvalue of  $M_\sigma$  is  $\lambda^2$ . The left PF-eigenvector contains the areas of the different prototiles, up to scaling. The normalised right PF-eigenvector  $\mathbf{v} = (v_1, v_2, \dots, v_m)^T$  of  $M_\sigma$  contains the relative frequencies of the prototiles of the tiling in the following sense: The entry  $v_i$  is the relative frequency of  $T_i$  in  $\mathcal{T}$ .*

### 3. AN IRRATIONALITY RESULT

An angle  $\theta \in [0, 2\pi[$  is called *irrational* if  $\theta \notin \pi\mathbb{Q}$ . The pinwheel tilings have indeed DTO because of the fact that the second order supertile contains two congruent tiles which are rotated against each other by an irrational angle (see Figure 1 right, the two tiles are marked by dots). The angle here is  $2 \arctan(1/2)$ . It is known that  $\arccos(\frac{1}{\sqrt{n}}) \notin \pi\mathbb{Q}$  for  $n \geq 3$  odd [1]. Hence  $2 \arctan(\frac{1}{2}) = 2(\frac{\pi}{2} - \arccos(\frac{1}{\sqrt{5}}))$  is irrational. By induction the entire tiling contains tiles that are rotated against each other by  $n \cdot 2 \arctan(\frac{1}{2}) \pmod{2\pi}$  for all  $n \in \mathbb{N}$ . Since  $2 \arctan(\frac{1}{2})$  is irrational these values are dense on a circle. More generally we have the following result:

**Theorem 4** ([5, Proposition 3.4]). *Let  $\sigma$  be a primitive substitution in  $\mathbb{R}^2$  with prototiles  $T_1, T_2, \dots, T_m$ . Any substitution tiling in the hull of  $\sigma$  has DTO if and only if there are  $k, i$  such that  $\sigma^k(T_i)$  contains two equivalent tiles  $T$  and  $T'$  that are rotated against each other by some irrational angle.*

Hence the desired substitutions need to involve some irrational angles. The following result provides such angles. The authors believe that this result must be known already, but we are not aware of any reference.

**Theorem 5.** *Let  $P$  be a parallelogram with edge lengths 1 and 2 and interior angles  $\frac{2\pi}{n}$  and  $\frac{(n-2)\pi}{n}$ ,  $n \geq 3$ . Then the angles between the longer diagonal of  $P$  and the edges of  $P$  are irrational.*

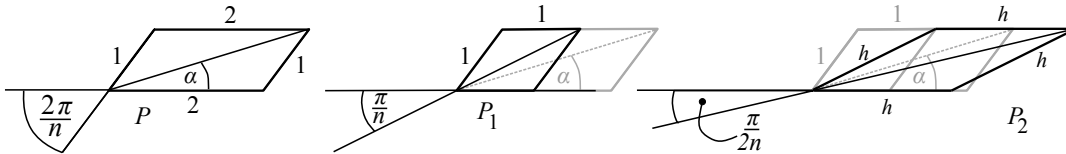


FIGURE 2. The angle  $\alpha$  (left);  $\alpha$  is smaller than  $\frac{\pi}{n}$  (middle);  $\alpha$  is bigger than  $\frac{\pi}{2n}$ .

*Proof.* Embed  $P$  in the complex plane such that the lower left corner of  $P$  coincides with the origin and the lower base lies along the real axis as shown in Figure 2. So the upper left corner of  $P$  coincides with the point  $\xi_n := e^{2\pi i/n}$  and the upper right corner coincides with the point  $z = \xi_n + 2$ . Let  $\alpha$  be the angle between the long diagonal of  $P$  and the  $x$ -axis, see Figure 2 left. We will show that  $\alpha$  is irrational.

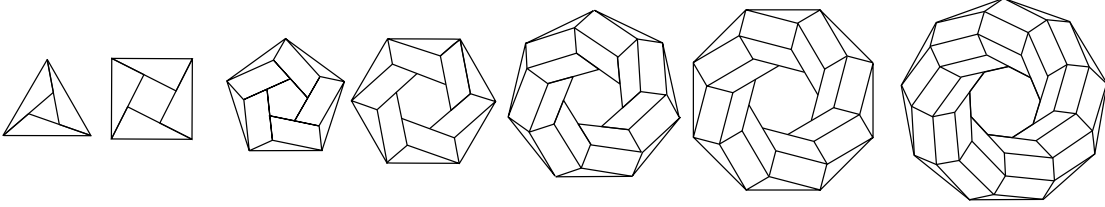


FIGURE 3. Dissection of regular  $n$ -gons into a small regular  $n$ -gon  $T_1^{(n)}$ ,  $n$  triangles  $T_2^{(n)}$  and possibly several parallelograms.

Suppose  $\alpha$  is rational. Then there is  $m \geq 1$  such that  $z^m \in \mathbb{R}$ , which yields

$$\left(\frac{z}{|z|}\right)^m = \pm 1 \Rightarrow \left(\frac{z}{|z|}\right)^{2m} = 1 \Leftrightarrow \left(\frac{z^2}{zz}\right)^m = 1 \Leftrightarrow \left(\frac{z}{z}\right)^m = 1,$$

hence  $\frac{z}{|z|}$  is some  $m$ -th root of unity.

Because  $z = \xi_n + 2 \in \mathbb{Q}(\xi_n)$  we have  $\frac{z}{|z|} \in \mathbb{Q}(\xi_n)$ . It is known (compare [22, Exercise 2.3]) that roots of unity in  $\mathbb{Q}(\xi_n)$  are of the form  $\pm \xi_n^k$ ,  $0 \leq k \leq n-1$ . Hence  $\frac{z}{|z|} = \pm \xi_n^k$  for some  $k \in \mathbb{N}$ . Then  $2\alpha = \arg\left(\frac{z}{|z|}\right) = \arg(\pm \xi_n^k)$ , and so  $\alpha = \frac{k\pi}{n}$  if  $n$  is even, and  $\alpha = \frac{k\pi}{2n}$  if  $n$  is odd.

For  $n$  even consider a second parallelogram  $P_1$  with vertices  $0, 1, 1 + \xi_n, \xi_n$  (compare Figure 2 middle). Since  $\frac{\pi}{n}$  is the angle between 1 and the diagonal of  $P_1$  we get  $0 < \alpha < \frac{\pi}{n}$ , yielding a contradiction for  $n$  even.

For  $n$  odd consider a third parallelogram  $P_2$  with vertices  $0, 1 + \xi_n, 1 + \xi_n + h, h$ , where  $h$  is the length of the long diagonal of  $P_1$  (compare Figure 2 right). The angle between the long diagonal of  $P_2$  and the real axis is  $\frac{\pi}{2n}$ . Since  $h > 1$  we obtain  $\alpha > \frac{\pi}{2n}$ . Together with the reasoning above we get  $\frac{\pi}{n} > \alpha > \frac{\pi}{2n}$ , yielding a contradiction for  $n$  odd. Therefore  $\alpha$  is irrational.  $\square$

#### 4. CONSTRUCTION OF THE SUBSTITUTION TILINGS

The general idea for the substitutions is to choose one prototile as a bisected parallelogram from Theorem 5. To be more precise, for  $n \geq 3$  odd the prototile  $T_2^{(n)}$  is the triangle with interior angle  $\frac{n-1}{n}\pi$  where the two edges forming this angle have length one and two, respectively. For  $n \geq 4$  even the prototile  $T_2^{(n)}$  is the triangle with interior angle  $\frac{n-2}{n}\pi$  where the two edges forming this angle have length one and two, respectively. By Theorem 5 the other two angles of this triangle are irrational for any  $n \geq 3$ . A short computation yields the length  $\lambda_n$  of the longest edge as follows:

$$\lambda_n = \begin{cases} \sqrt{5 + 4 \cos\left(\frac{\pi}{n}\right)} & \text{if } n \text{ is odd} \\ \sqrt{5 + 4 \cos\left(\frac{2\pi}{n}\right)} & \text{if } n \text{ is even.} \end{cases}$$

Let  $\lambda_n$  be the inflation factor for the desired substitutions  $\sigma_n$  in the sequel. A regular  $n$ -gon of side length  $\lambda_n$  can be dissected into copies of  $T_2^{(n)}$  (along its edges), one regular  $n$ -gon with unit edge length (in its centre), and, if  $n \geq 5$ , into several parallelograms. This dissection is illustrated in Figure 3 for the cases  $3 \leq n \leq 9$ .

In order to construct the desired substitution tilings with DTO being invariant under  $n$ -fold rotation one chooses a first prototile  $T_1^{(n)}$  to be a regular  $n$ -gon of unit edge length. The substitution of  $T_1^{(n)}$  is then given by the dissection in Figure 3. Therefore the inflation factor equals  $\lambda_n$ . If one can find a substitution for all further prototiles arising in this dissection these substitutions are good candidates for DTO tilings since by Theorem 5 the central  $n$ -gon of edge length 1 is rotated against the big  $n$ -gon by an irrational angle. Furthermore—given a substitution exists for some  $n$ —the dissection of  $\lambda_n T_1^{(n)}$  already provides a tiling invariant under  $n$ -fold rotation (given a substitution for all tiles exists at all) since it may serve as a seed for a fixed point of  $\sigma_n$  with  $T_1^{(n)}$  in the centre. (To be precise, one needs to define  $\sigma_n$  including a rotational part in order to take care of the different orientations of the large and the small regular  $n$ -gons.)

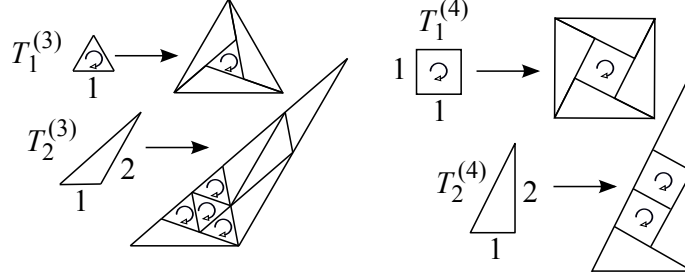


FIGURE 4. The substitutions  $\sigma_4$  (right) and  $\sigma_3$  (left). For tiles with non-trivial symmetry the arrows indicate the chirality of the tiles. By choice, all symmetric tiles in the images are right-handed copies.

**4.1. The 3-fold and 4-fold tilings.** The substitutions for  $n \in \{3, 4\}$  need only two prototiles. Two possible substitutions  $\sigma_3$  and  $\sigma_4$  are shown in Figure 4.

**Proposition 6.** *For  $n \in \{3, 4\}$  holds: The substitution  $\sigma_n$  is a primitive substitution with DTO. The hull of  $\sigma_n$  contains two aperiodic tilings invariant under  $n$ -fold rotation. Any tiling in the hull of  $\sigma_n$  is FLC with respect to rigid motions.*

*Proof.* Obviously the substitutions are primitive, the substitution matrices are  $M_{\sigma_3} = \begin{pmatrix} 1 & 4 \\ 4 & 3 \end{pmatrix}$  and  $M_{\sigma_4} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ .

Theorems 5 and 4 imply that the tilings have DTO as follows. Let  $\alpha$  denote the smallest interior angle of  $T_2^{(3)}$ . Figure 5 shows the situation for  $\sigma_3$ , the grey shaded tile on the boundary of  $\sigma_3^3(T_2^{(3)})$  is rotated by  $2\alpha$  against the grey shaded tile in the centre. Since  $2\alpha$  is irrational by Theorem 5, DTO of the tiling in the hull of  $\sigma_3$  follows by Theorem 4. This phenomenon appears in all substitutions considered here and in the sequel: copies of  $T_2^{(n)}$  are lined up along the boundary of  $\sigma_n(T_1^{(n)})$ . Mirror images of  $T_2^{(n)}$  are lined up along the boundary of  $\sigma_n^2(T_1^{(n)})$  (since they are mirror images Theorem 4 does not apply here already), and rotated copies of  $T_2^{(n)}$  are lined up along the boundary of  $\sigma_n^3(T_1^{(n)})$ . The boundaries of  $\sigma_n(T_1^{(n)})$  and  $\sigma_n^3(T_1^{(n)})$  are rotated against each other by  $2\alpha$ , thus the triangles  $T_2^{(n)}$  are rotated against each other by  $2\alpha$ .

In order to show that the tilings are aperiodic it suffices to show that the substitution  $\sigma_n$  has a unique inverse on the hull [10, Theorem 10.1.1], see also [21, 3]. In the cases  $n \in \{3, 4\}$  this is particularly simple: For  $n = 3$  note that each isolated regular triangle  $T_1^{(3)}$  is contained in a supertile  $\sigma_3(T_1^{(3)})$ , hence the supertiles  $\sigma_3(T_1^{(3)})$  can all be identified uniquely. The remaining part of the tiling consists of supertiles  $\sigma_3(T_2^{(3)})$ , and the patches of four connected  $T_1^{(3)}$  determine the exact location and orientation of these supertiles. A similar reasoning works for  $n = 4$ .

Let  $R_\alpha$  denote the rotation about the origin through  $\alpha$ . Let  $T_1^{(n)}$  be centred in the origin. Then  $R_\alpha \sigma_n(T_1^{(n)})$  contains  $T_1^{(n)}$  in its interior. Consequently,  $(R_\alpha \sigma_n)^k(T_1^{(n)})$  contains  $(R_\alpha \sigma_n)^{k-1}(T_1^{(n)})$  in its interior. (Figure 5 shows  $(R_\alpha \sigma_3)^k(T_1^{(3)})$  for  $k = 0, 1, 2, 3$ .) Hence  $((R_\alpha \sigma_n)^k(T_1^{(n)}))_{k \in \mathbb{N}}$  is a nested sequence that converges in the local topology. The limit is a tiling  $\mathcal{T}$  that is fixed under  $R_\alpha \sigma_n$ . Since the tilings in the hull have DTO the hull is invariant under rotations. Thus the patches  $(R_\alpha \sigma_n)^k(T_1^{(n)})$  are legal in the sense that they are contained in tilings in the hull. Hence  $\mathcal{T}$  is indeed contained in the hull. Since mirror images of all tiles occur also in all tilings in the hull, the mirror image of  $\mathcal{T}$  is also contained in the hull, yielding a second tiling invariant under  $n$ -fold rotation.

We sketch why the tilings have FLC with respect to rigid motions. The simplest way to see this is to introduce an additional (pseudo-)vertex at the midpoint of the edge of length 2 in  $T_2^{(n)}$ . Taking into account this pseudo-vertex the tilings are vertex-to-vertex. A standard argument implies that the tilings are FLC. (There are finitely many ways how two tiles can touch each other, hence there are only finitely many possible patches fitting into a ball of radius  $r$ . A complete proof of FLC

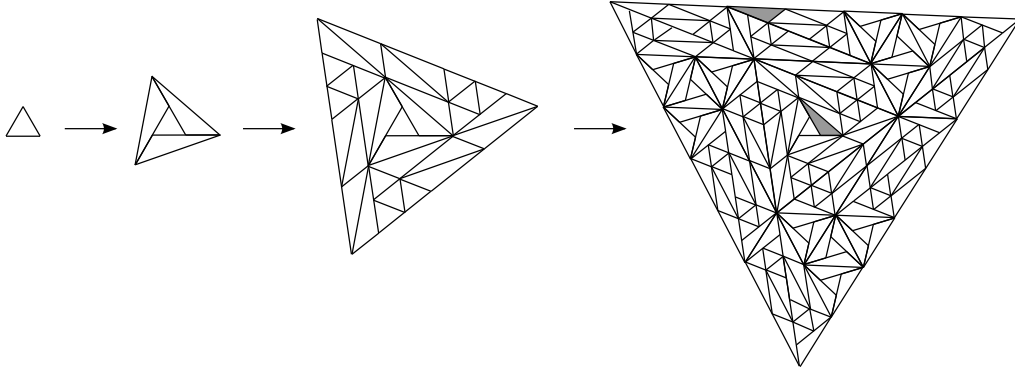


FIGURE 5. Three iterations of  $R_\alpha \sigma_3$  on  $T_1^{(3)}$ . The third order supertile  $\sigma_3^3(T_1^{(3)})$  contains two copies of  $T_2^{(3)}$  that are rotated against each other by an irrational angle.

would need a list of all possibilities how two tiles can touch each other, e.g. a list of all vertex stars. Such a list is contained in [7] for  $n = 3, 4$ . More details in [17].)  $\square$

**4.2. The 6-fold tiling.** For  $n = 6$  we get the inflation factor  $\lambda_6 = \sqrt{5 + 4 \cos(\frac{2\pi}{6})} = \sqrt{7}$ . The substitution  $\sigma_6$  is shown in Figure 6. We need to introduce an additional tile  $T_4^{(6)}$  in order to ensure primitivity:  $\lambda_6 T_3^{(6)}$  can be dissected into congruent copies of  $T_2^{(6)}$  and  $T_3^{(6)}$ , but then  $T_1^{(6)}$  would not occur in any of the supertiles of  $T_2^{(6)}$  and  $T_3^{(6)}$ , hence the substitution would not be primitive.

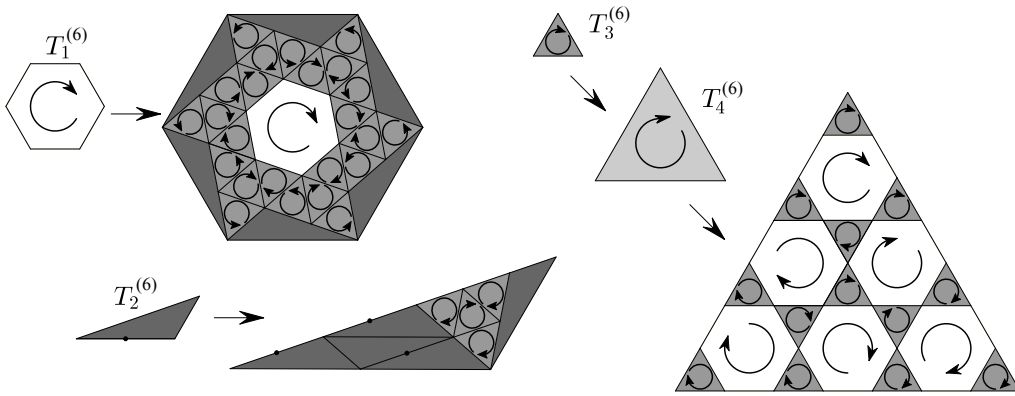


FIGURE 6. The substitution  $\sigma_6$ .

The substitution  $\sigma_6$  has the substitution matrix

$$M_{\sigma_6} = \begin{pmatrix} 1 & 0 & 0 & 6 \\ 6 & 5 & 0 & 0 \\ 24 & 4 & 0 & 13 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which is primitive because  $M_{\sigma_6}^k$  contains only positive entries for all  $k \geq 3$ . The corresponding PF-eigenvalue of  $M_{\sigma_6}$  is  $\lambda_6^2 = 7$  with left PF-eigenvector  $(6, 2, 1, 7)$  and normalised right PF-eigenvector  $(\frac{1}{12}, \frac{1}{4}, \frac{7}{12}, \frac{1}{12})^T$ . By Theorem 3 the left PF eigenvector contains the areas of the tiles up to scaling, the normalised right PF-eigenvector contains the relative frequencies of the tiles. The latter can serve as a starting point for computing the frequency module of the tilings. See [7] for the computation of the frequency module of  $\sigma_4$  by these means.

**Proposition 7.** *The substitution  $\sigma_6$  is a primitive substitution with DTO. The hull contains two aperiodic tilings invariant under 6-fold rotation. Any tiling in the hull of  $\sigma_6$  is FLC with respect to rigid motions.*

*Proof.* The proofs of DTO, FLC and the existence of a tiling invariant under 6-fold rotation are very much along the lines in the proof of Proposition 6. Aperiodicity of the tilings can be proven similarly by identifying the first order supertiles uniquely: the tile  $T_4^{(6)}$  occurs only as the supertile  $\sigma_6(T_3^{(6)})$ . A patch of six connected hexagons  $T_1^{(6)}$  occurs only in the supertile  $\sigma_6(T_4^{(6)})$ . All remaining hexagons  $T_1^{(6)}$  determine the supertiles  $\sigma(T_1^{(6)})$ . Everything that remains are supertiles  $\sigma_6(T_2^{(6)})$ . For more thorough proofs see [17].  $\square$

**4.3. The 8-fold tiling.** For  $n = 8$  we get the inflation factor  $\lambda_8 = \sqrt{5 + 4 \cos(\frac{2\pi}{8})} = \sqrt{5 + 2\sqrt{2}}$ . The substitution rule is shown in Figure 7. Since  $\lambda_8 T_3^{(8)}$  and  $\lambda_8 T_4^{(8)}$  cannot be dissected into copies of the prototiles  $T_1^{(8)}, T_2^{(8)}, T_3^{(8)}, T_4^{(8)}$  we need to introduce intermediate tiles  $T_5^{(8)} := \lambda_8 T_3^{(8)}$  and  $T_6^{(8)} := \lambda_8 T_4^{(8)}$  in order to define a substitution rule. Note that we need to define an orientation on the tiles in order to distinguish a tile from its mirror image. This is not indicated in the figure. There are several possibilities to do so. One possibility is letting all tiles in figure have the same orientation. The only point where this really matters is that the tile  $T_1^{(8)}$  in  $\sigma_8(T_1^{(8)})$  has the same orientation as the prototile  $T_1^{(8)}$  in order to ensure DTO.

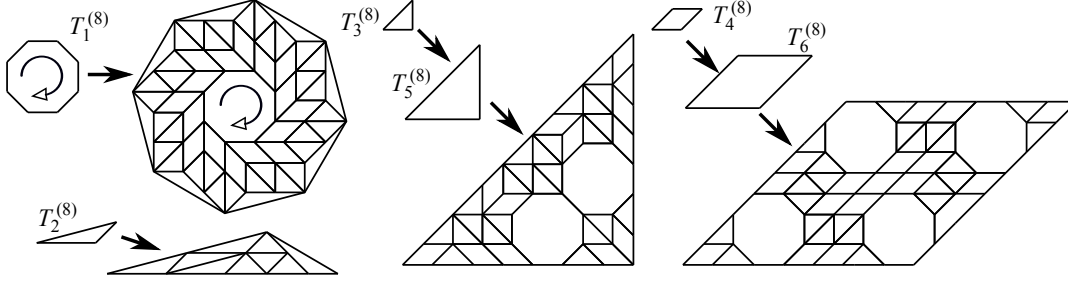


FIGURE 7. The substitution  $\sigma_8$ . Orientations and chiralities of symmetric tiles are arbitrary if not shown in the image.

**Proposition 8.** *The substitution  $\sigma_8$  is a primitive substitution with DTO. The hull contains two aperiodic tilings invariant under 8-fold rotation.*

*Proof.* Again the proofs of DTO and the existence of a tiling invariant under 8-fold rotation are very much along the lines in the proof of Proposition 6. Aperiodicity of the tilings can be proven similarly as above by identifying the first order supertiles uniquely. More details in [17]. The primitivity can be checked via the substitution matrix

$$M_{\sigma_8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 4 \\ 8 & 5 & 0 & 0 & 0 & 0 \\ 32 & 4 & 0 & 0 & 25 & 24 \\ 16 & 0 & 0 & 0 & 12 & 17 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

$M_{\sigma_8}^4$  has only positive entries.  $\square$

We suppose that the tilings in the hull of  $\sigma_8$  have also FLC. But since the tilings—with or without pseudo-vertices—are not vertex-to-vertex (this can be seen in  $\sigma_8^3(T_1^{(8)})$  for instance) a rigorous proof will be rather lengthy. For details we refer to [17].

**4.4. The 5-fold tiling.** It is possible to define the desired substitution for  $n = 5$  using just six prototiles. However the tilings may not have FLC. In order to ensure FLC we define a substitution  $\sigma_5$  with 12 prototiles. The inflation factor is  $\lambda_5 = \sqrt{6 + \sqrt{5}}$ . The substitution rule is shown in Figure 8.

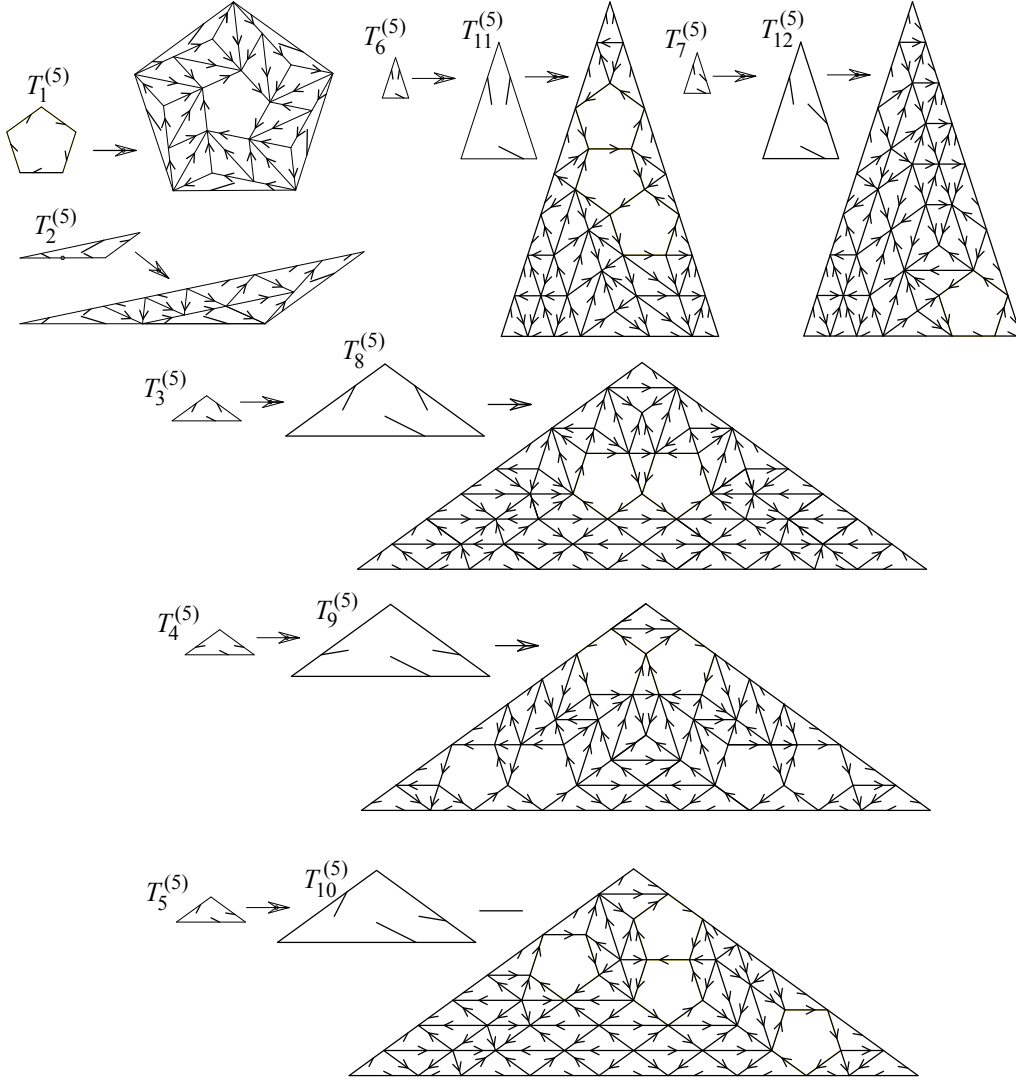


FIGURE 8. The substitution  $\sigma_5$ . Half arrows indicate the orientation of edges and tiles. All coincident edges have the same orientation.

**Proposition 9.** *The substitution  $\sigma_5$  is a primitive substitution with DTO. The hull contains two aperiodic tilings invariant under 5-fold rotation. Any tiling in the hull of  $\sigma_5$  is FLC with respect to rigid motions.*

*Proof.* As before the proofs of DTO and the existence of a tiling invariant under 5-fold rotation are along the lines of the proof of Proposition 6. FLC follows from the fact that coincident edges have the same orientation and that edges of the same length are dissected in the same manner under  $\sigma_5$ . Aperiodicity of the tilings can be proven similarly by identifying the first order supertiles uniquely. More details in [17]. Primitivity of  $\sigma_5$  can be checked by considering the substitution matrix  $M_{\sigma_5}$  below. Since  $M_{\sigma_5}^5$  contains only positive entries the substitution  $\sigma_5$  is primitive.  $\square$



The substitution matrix for  $\sigma_5$  looks as follows.

$$M_{\sigma_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 4 & 3 & 1 \\ 5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 & 9 & 10 & 5 & 11 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 8 & 6 & 1 & 2 \\ 15 & 4 & 0 & 0 & 0 & 0 & 0 & 8 & 18 & 25 & 9 & 8 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 12 & 6 & 7 & 9 & 11 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 10 & 12 & 13 & 17 & 17 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The powers of  $M_{\sigma_5}$ , the PF-eigenvalue and the PF-eigenvectors have been computed with the computer algebra software (CAS) *Scientific Workplace 5.5* [18]. The PF-eigenvalue is  $\sqrt{5} + 6$  (equals  $\lambda_5^2$ , as it ought to be) and its corresponding left PF-eigenvector and normalised right PF-eigenvector are respectively given by

$$\mathbf{w} = \begin{pmatrix} \frac{15}{62} + \frac{13}{62}\sqrt{5} \\ \frac{12}{31} - \frac{2}{31}\sqrt{5} \\ \frac{1}{62} + \frac{5}{62}\sqrt{5} \\ \frac{1}{62} + \frac{5}{62}\sqrt{5} \\ \frac{1}{62} + \frac{5}{62}\sqrt{5} \\ \frac{1}{6+\sqrt{5}} \\ \frac{1}{6+\sqrt{5}} \\ \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} \\ 1 \\ 1 \end{pmatrix}^T \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} \frac{2640247257}{109180718845}\sqrt{5} - \frac{233289537}{21836143769} \\ \frac{2271797364}{21836143769}\sqrt{5} - \frac{4175144835}{21836143769} \\ \frac{-69639647193}{349378300304}\sqrt{5} + \frac{200370426489}{349378300304} \\ \frac{-281379644707}{1746891501520}\sqrt{5} + \frac{164659760407}{349378300304} \\ \frac{16679139843}{21836143769}\sqrt{5} - \frac{31075028997}{21836143769} \\ \frac{-208504378761}{873445750760}\sqrt{5} + \frac{113488825221}{174689150152} \\ \frac{-214190232831}{873445750760}\sqrt{5} + \frac{126605432787}{174689150152} \\ \frac{-19942203537}{349378300304}\sqrt{5} + \frac{50013574029}{349378300304} \\ \frac{-81018602267}{1746891501520}\sqrt{5} + \frac{40946393779}{349378300304} \\ \frac{4230640905}{21836143769}\sqrt{5} - \frac{8704705587}{21836143769} \\ \frac{-58660335441}{873445750760}\sqrt{5} + \frac{28691526777}{174689150152} \\ \frac{-61876405191}{873445750760}\sqrt{5} + \frac{31413639663}{174689150152} \end{pmatrix}$$

Again the left PF eigenvector  $\mathbf{w}$  contains the areas of the tiles up to scaling, the right PF-eigenvector  $\mathbf{v}$  contains the relative frequencies of the tiles.

## 5. CONCLUSION

We were not able to define a general substitution rule  $\sigma_n$  for all  $n$ , or at least for all even  $n$ . Anyway, there is a general pattern for dissecting  $\lambda_n T_2^{(n)}$  ( $n \neq 4$ ) into five copies of  $T_2^{(n)}$  and four further triangles, and for dissecting  $\lambda_n T_1^{(n)}$  into one copy of  $T_1^{(n)}$ ,  $n$  copies of  $T_2^{(n)}$  and several parallelograms. Hence it is likely that there are substitution tilings with DTO invariant under  $n$ -fold rotation for all  $n \geq 3$ . This might be also of interest with respect to a comment in [11]: “...there is a lack of known examples of aperiodic planar tiling families with higher orders of rotational symmetry.”. That paper contains “the first substitution tiling with elevenfold symmetry appearing in the literature”. Our method might yield further primitive substitution tilings with 11-fold and also 12-fold rotational symmetry (though the number of prototiles might be huge).

The proof of Theorem 5 on irrational angles in cyclotomic parallelograms uses the fact that the considered irrational angle  $\alpha$  is smaller than  $\frac{\pi}{n}$ . Hence the result generalises immediately to the long diagonals of parallelograms with interior angle  $\frac{\pi}{n}$  and with edge lengths  $a \neq b \in \mathbb{Q}$  rather than 2 and 1. Using other arguments it might be possible to show the irrationality of other angles as well, e.g. the angle of the short diagonal.

For the sake of brevity we did not mention further implications of our constructions in the context of dynamical properties of the hull. Just to mention a few: the fact that a tiling has FLC ensures the minimality of the hull of  $\sigma_n$ . Due to primitivity of  $\sigma_n$  all tilings in the hull of  $\sigma_n$  are repetitive, hence have uniform patch frequencies. As a consequence we obtain: If we denote the

hull of  $\sigma_n$  by  $X_{\sigma_n}$  then the dynamical systems  $(X_{\sigma_n}, \mathbb{R}^2)$  and  $(X_{\sigma_n}, E(2))$  (where  $E(2)$  denotes the rigid motions in  $\mathbb{R}^2$ ) are both uniquely ergodic. For more details on these concepts see [20, 3, 8].

#### ACKNOWLEDGEMENTS

The authors express their gratitude to Franz Gähler, Lorenzo Sadun and Mike Whittaker for helpful discussions. M. De Las Peñas acknowledges the support of the German Academic Exchange Service (DAAD) (research stay programme). A. Say-awen gives thanks to the Philippine Department of Science and Technology (DOST) (DOSTASTHRDP scholarship).

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