About duality of cut-and-project tilings.

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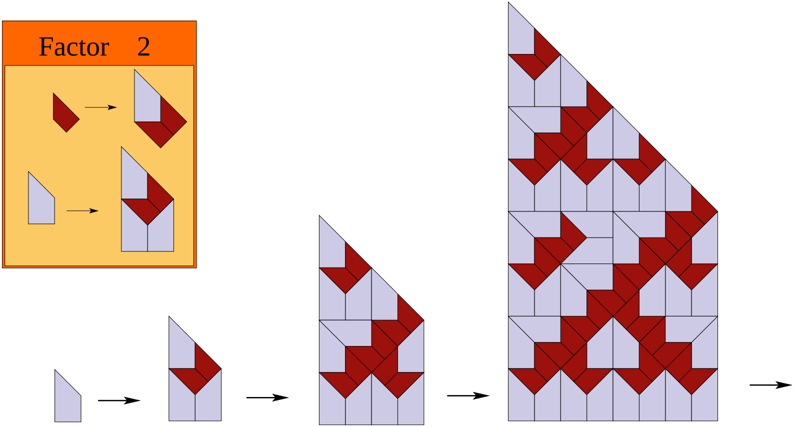
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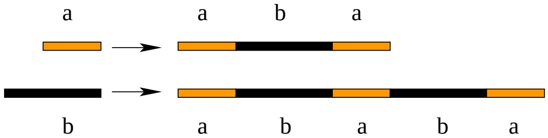
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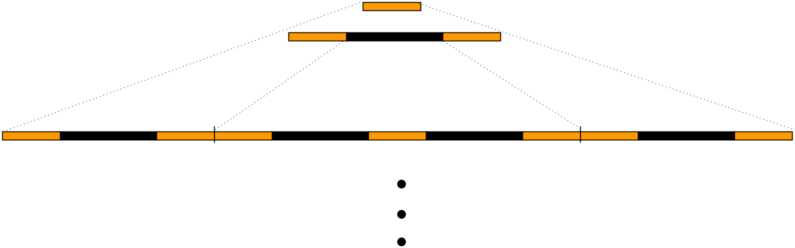
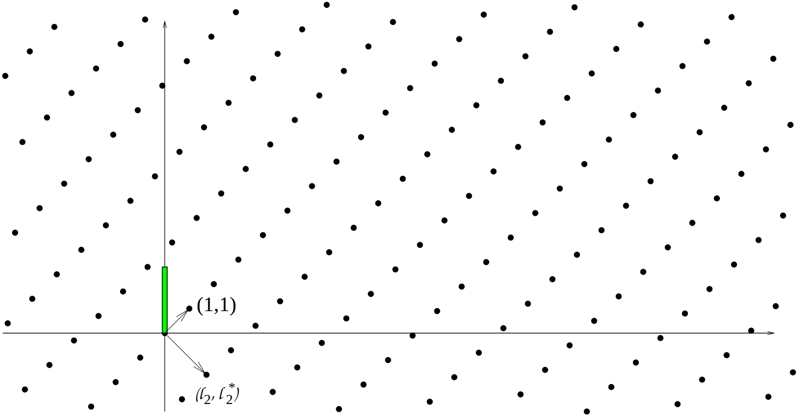
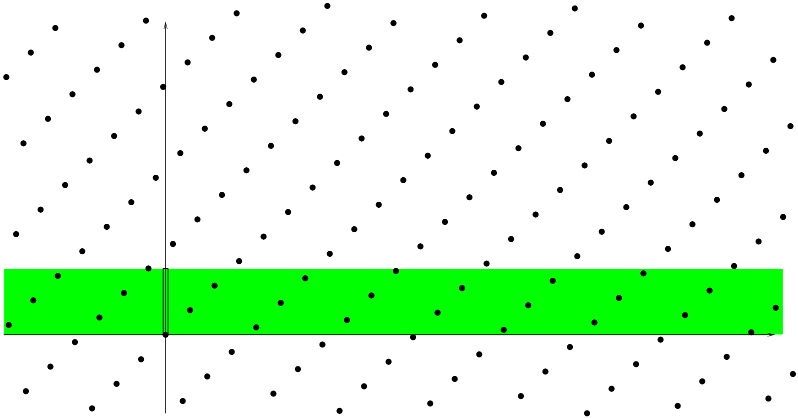
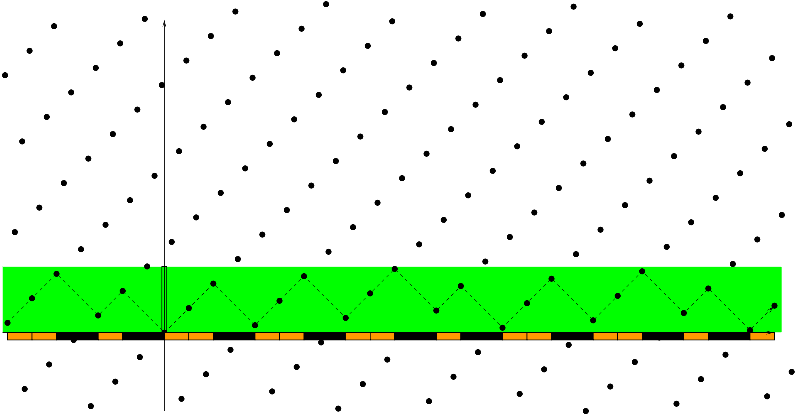
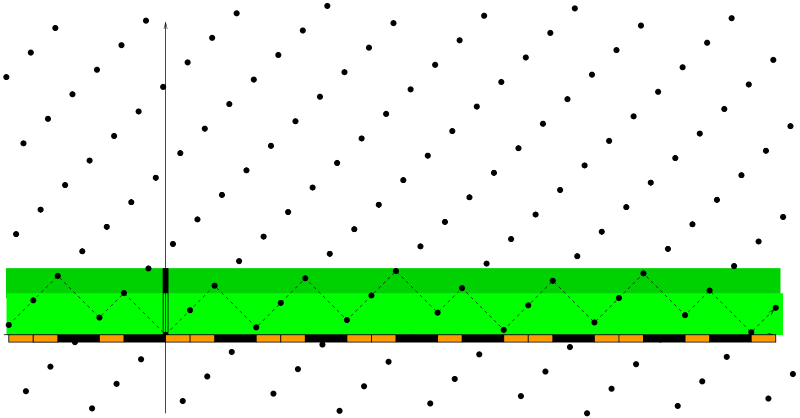
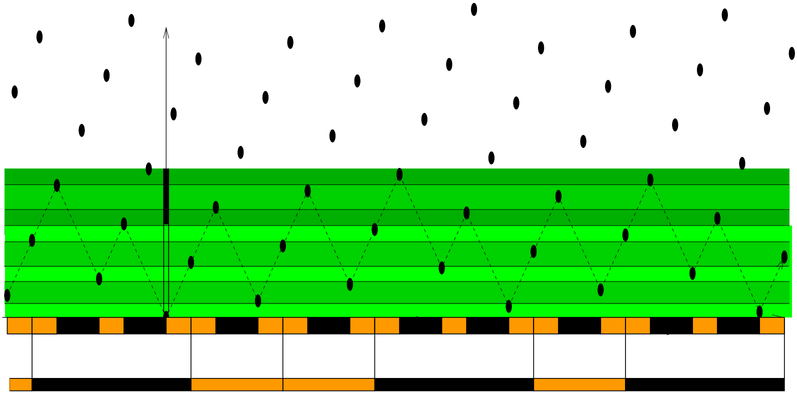


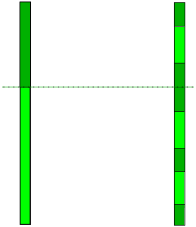

(1,1)
(ℓ_2, ℓ_2^*)

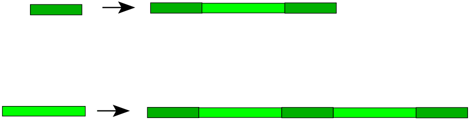












Dual substitution

Notions of duality in dim 1

- Natural decomposition method
- Inverse substitution
- Galois-dual (star-dual)
- Dual maps of substitutions

Inverse substitution

For $d = 1$, two letters:

View the substitution σ as an endomorphism of the free group F_2 on 2 letters.

Ex.: $\sigma(a) = aba, \quad \sigma(b) = ababa.$

If $\sigma \in \mathrm{Aut}(F_2)$ then σ^{-1} defines another substitution.

Ex. cont.: $\sigma^{-1}(a) = ab^{-1}a, \quad \sigma^{-1}(b^{-1}) = ab^{-1}ab^{-1}a.$

In this case, essentially the same substitution.

(up to an (outer) automorphism $\tau : a \to a, b \to b^{-1}$)

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Galois-dual

Consider a tile-substitution ($d = 1$, 2 tiles):

- prototiles T_1, T_2 intervals,
- $\lambda > 1$ the inflation factor,
- $\mathcal{D}_{ji}$ $(1 \leq i, j \leq 2)$ digit sets (set of translation vectors)

such that

$$\lambda T_1 = T_1 + \mathcal{D}_{11} \cup T_2 + \mathcal{D}_{21}$$

$$\lambda T_2 = T_1 + \mathcal{D}_{12} \cup T_2 + \mathcal{D}_{22}$$

(non-overlapping). This yields a selfsimilar tile–substitution

$$\sigma(T_i) = \{T_j + \mathcal{D}_{ji} \mid j = 1, 2\}.$$

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$S = (|\mathcal{D}_{ji}|)_{1 \le i,j \le m}$ is the substitution matrix (='incidence matrix').

The Perron-Frobenius eigenvector of S is the inflation factor λ.

For simplicity, let $\det(S) = 1$. Then, the inflation factor λ is a quadratic algebraic integer.

Let $\nu = \lambda^{-1}$, its algebraic conjugate.

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$$\sigma^*(T_i^*) = \{ T_j^* + \mathcal{D}_{ij}^* \mid j = 1, 2 \}.$$

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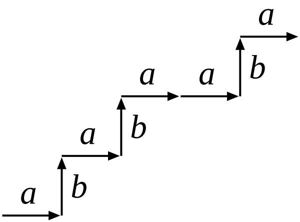
Dual maps of substitutions

Realize a sequence *ababaaba* as a path in $\mathbb{Z}^2$.



Define the space $\mathcal{G}_1$ of all formal (finite) sums

$$\sum_k n_k(x_k, i_k), \qquad (n_k \in \mathbb{Z}, x_k \in \mathbb{Z}^2, i_k = 1, 2)$$

where (x, i) represents a path from x to $x + e_i$.

Here, a substitution reads

$$\begin{aligned} E_1(\sigma) : (0,1) &\mapsto (0,1) + (e_1, 2) + (e_1 + e_2, 1) \\ (0,2) &\mapsto (0,1) + (e_1, 2) + (e_1 + e_2, 1) \\ &\quad + (2e_1 + e_2, 2) + (2e_1 + 2e_2, 1) \end{aligned}$$

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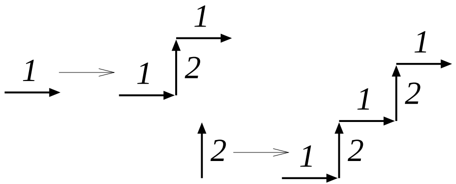

The dual $E_1^*(\sigma)$ is defined on the dual space $\mathcal{G}_1^*$ of $\mathcal{G}_1$.

Here: $\mathcal{G}_1^* \cong \mathcal{G}_1$.

As usual, $\langle v, \phi \rangle = \phi(v)$ for $v \in \mathcal{G}_1, \ \phi \in \mathcal{G}_1^*$.

Then $E_1^*(\sigma)$ is defined by $\langle v, E_1^*(\sigma)\phi \rangle = \langle E_1(\sigma)v, \phi \rangle$.

Explicit formula, here:

$$E_1^*(\sigma)(x, i^*) = \sum_{n,j: W_n^{(j)} = i} \left(S^{-1}(x - fP_n^{(j)}), j^* \right)$$

(S the substitution matrix, $P_n^{(i)}$ ($W_n^{(i)}$) prefix (type) of the n-th letter in $\sigma(i)$, f abelianization map of $\{a, b\}$.)

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Ex. (cont.)

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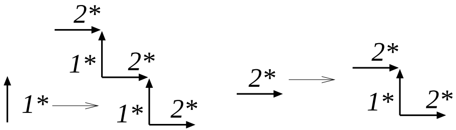


(now, (x, i^*) represents a path from $x + e_i$ to $x + e_i + e_i^*$).

Notions of duality in dim 1

- Natural decomposition method
- Inverse substitution
- Galois-dual (star-dual)
- Dual maps of substitutions

All notions are equivalent in dim 1

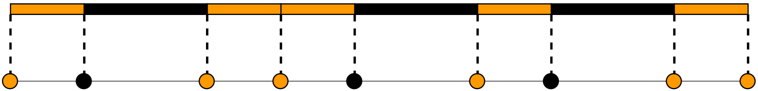
w.r.t. the tilings they define.

(Inverse subtitution: two letters only)

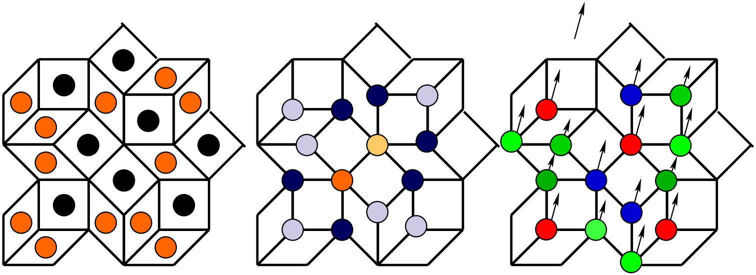
Notions of duality in dim > 1

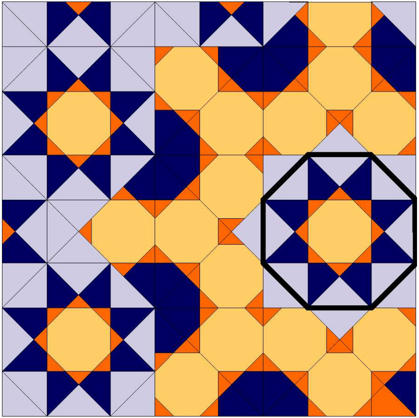
- Natural decomposition method, problem: vertices vs tiles
  - Control points
  - Vertex star type
  - Generic direction
  - ...
- ~~Inverse substitution~~
- Galois-dual (star-dual)
- Dual maps of substitutions

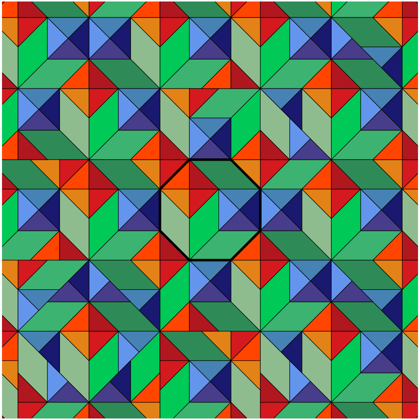
In dim 1 it is clear how to identify tiles and vertices:



In dim > 1 not.



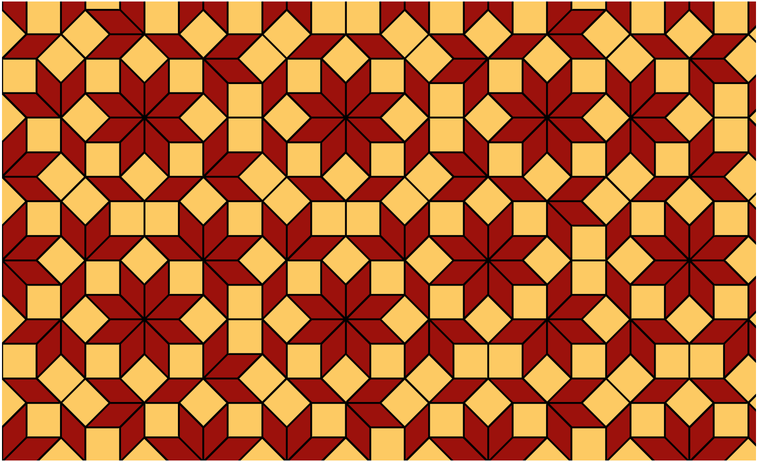


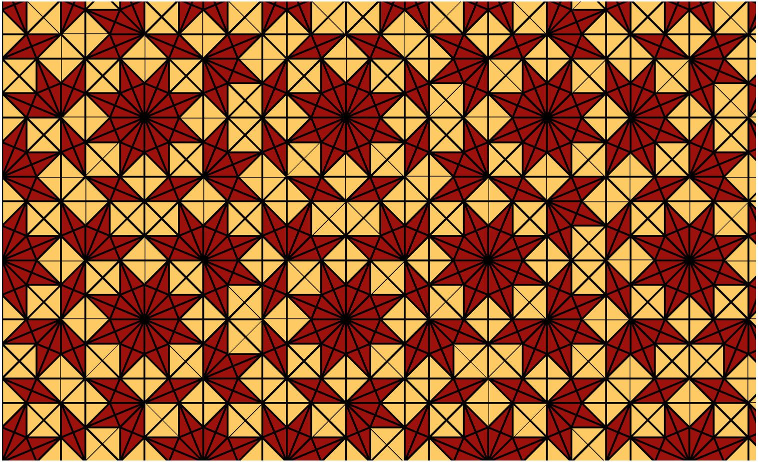


Both dual tilings are not Ammann-Beenker tilings.

Not even MLD to them. (Why? Fractal windows)

On the other hand, the Galois dual of the Ammann-Beenker tiling is MLD to Ammann-Beenker:





$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

- Λ a lattice in $\mathbb{R}^d \times H$ (i.e. cocompact discrete subgroup)
- π_1, π_2 projections
  - $\pi_1|_\Lambda$ injective
  - $\pi_2(\Lambda)$ dense
- W compact
  - cl(int(W))$= W$
  - $\mu(\partial(W)) = 0$

Then $V = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) model set.

The star map : $\quad \star : \pi_1(\Lambda) \to \mathbb{R}^e, \; x^\star = \pi_2 \circ {\pi_1}^{-1}(x)$

Given a substitution tiling which is a cut-and-project tiling:

Tiling $\rightsquigarrow$ point set V; $\overline{V^\star} = W$

(the window or Rauzy fractal).

From the substitution:

$$\lambda T_i = \bigcup_{j=1}^{m} T_j + \mathcal{D}_{ji}$$

one obtains an IFS:

$$T_i = \bigcup_{j=1}^{m} \lambda^{-1}(T_j + \mathcal{D}_{ji})$$

The unique compact nonempty solution: the prototiles.

And...

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$$V_j = \bigcup_{i=1}^{m} \lambda V_i - \mathcal{D}_{ij}$$

A - non-unique - solution: A tupel of point sets $(V_1, V_2, ...V_m)$,

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$$T_i = \bigcup_{j=1}^{m} \lambda^{-1}(T_j + \mathcal{D}_{ji}) \quad (1) \qquad T_i^\star = \bigcup_{j=1}^{m} (\lambda^{-1})^\star (T_j^\star + \mathcal{D}_{ji}^\star) \quad (3)$$

$$V_j = \bigcup_{i=1}^{m} \lambda V_i - \mathcal{D}_{ij} \quad (2) \qquad V_j^\star = \bigcup_{i=1}^{m} \lambda^\star V_i^\star - \mathcal{D}_{ij}^\star \quad (4)$$

(1) and (4): IFS with unique solutions.

(1): the prototiles of the original tiling.

(4): the window, and the prototiles of the dual tiling.

(2) and (3): Discrete point sets, MLD to the original (2) and the dual (3) tiling.

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How does the star-map act:

If the inflation factor λ is an algebraic unit, then

$$\lambda^\star = (\lambda_1, \lambda_2, \ldots, \lambda_N)$$

where λ_i are the algebraic conjugates of λ.

If the inflation factor λ is an integer, then

$$\star : \mathbb{Z}^d \to (\mathbb{Z}_p)^d, \quad x^\star = x$$

($\mathbb{Z}_p$ p-adic integers)

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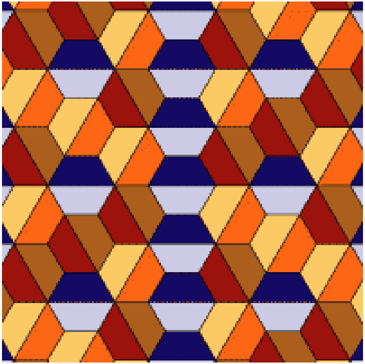
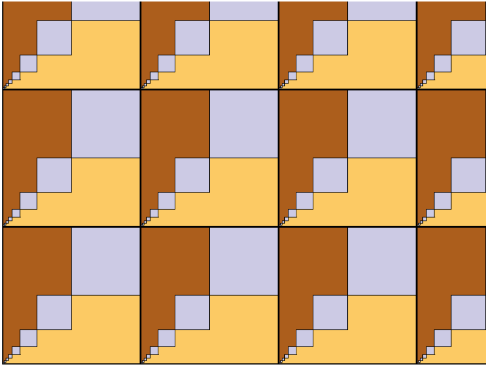
Ex.: Halfhex tiling .... and its dual





in $\mathbb{R}^2$ in $(\mathbb{Q}_2)^2$

Conclusion:

- In dim 1, most concepts of 'dual substitution tiling' are equivalent.
- In dim > 1, concepts diverge. In particular, there is no satisfying concept of the natural decomposition method.
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