Self–Duality and ⋆–dual tilings

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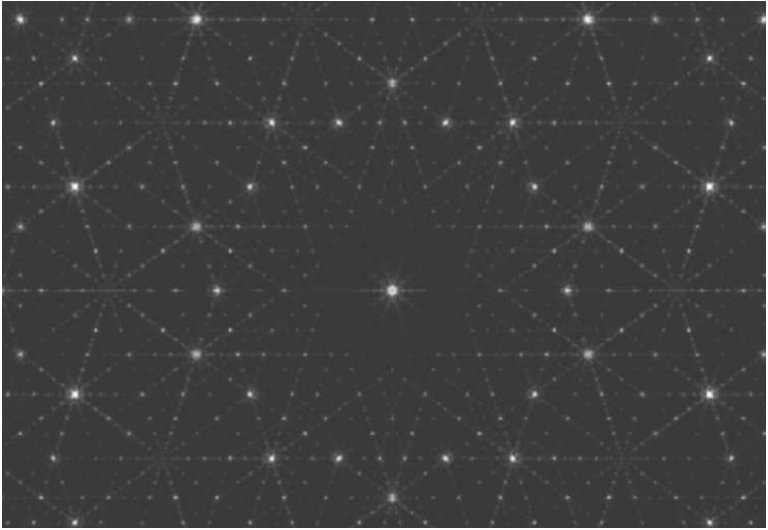
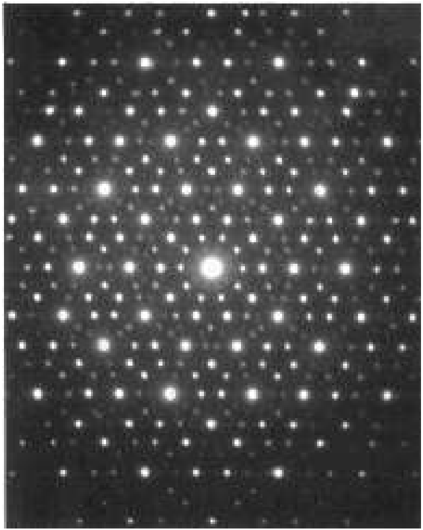
Liège





Substitutions

Symbolic substitution: $\mathcal{A}$ alphabet, $\mathcal{A}^*$ all finite words.

$$\sigma : \mathcal{A} \to \mathcal{A}^*$$

With $\sigma(ab) := \sigma(a)\sigma(b)$, σ extends to $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{Z}}$.

Ex.: $\mathcal{A} = \{a, b\}, \quad \sigma(a) = aba, \quad \sigma(b) = ababa.$

$$a \xrightarrow{\sigma} aba \xrightarrow{\sigma} abaababaaba \xrightarrow{\sigma} abaababaabaabaababaabaababaabaabaababa$$

Tile–substitution:

- $T_1, T_2, \dots T_m$ *prototiles* in $\mathbb{R}^d$,
- $\lambda > 1$ an algebraic integer (the *inflation factor*),
- $\mathcal{D}_{ji}$ $(1 \le i, j \le m)$ *digit sets* (set of translation vectors)

such that

$$\lambda T_i = \bigcup_{j=1}^{m} T_j + \mathcal{D}_{ji}$$

(non-overlapping). This yields a (selfsimilar) *tile–substitution*

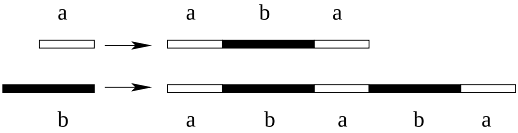
$$\sigma(T_i) := \{ T_j + \mathcal{D}_{ji} \mid j = 1 \dots m \}.$$

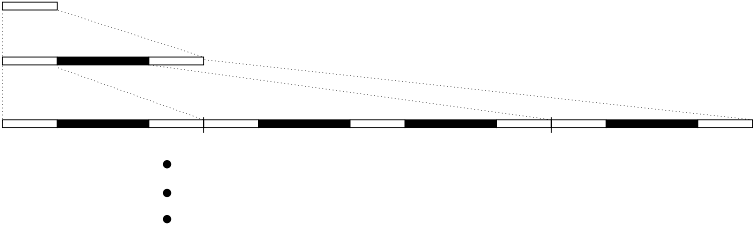
$S = (|D_{ji}|)_{1 \le i,j \le m}$ is the *substitution matrix* (='incidence matrix').

In this talk:

- λ unimodular, real PV (Pisot Vijayaraghavan number).
- Tilings in dimensions $d = 1$ or $d = 2$ only.
- All vertices, maps... can be expressed in $\mathbb{Z}[\lambda]$ ($d = 1$) respectively $\mathbb{Z}[i, \lambda]$ ($d = 2$).

Ex.: $a \longrightarrow aba, \qquad b \longrightarrow ababa$





The equation system

$$\lambda T_i = \bigcup_{i=1}^{m} T_j + \mathcal{D}_{ji}$$

gives rise to the corresponding — graph–directed — iterated function system (IFS)

$$T_i = \bigcup_{i=1}^{m} \lambda^{-1}(T_j + \mathcal{D}_{ji})$$

and vice versa.

The prototiles are the unique compact nonempty solution of the corresponding IFS. (In other words: With this choice: self–similar)

Ex.: $a \rightarrow aba, \qquad b \rightarrow ababa$

$$\mathcal{D} = \begin{pmatrix} \{0, 1+\frac{\sqrt{3}}{3}\} & \{0, 1+\frac{\sqrt{3}}{3}, 2+2\frac{\sqrt{3}}{3}\} \\ \{\frac{\sqrt{3}}{3}\} & \{\frac{\sqrt{3}}{3}, 1+2\frac{\sqrt{3}}{3}\} \end{pmatrix}$$

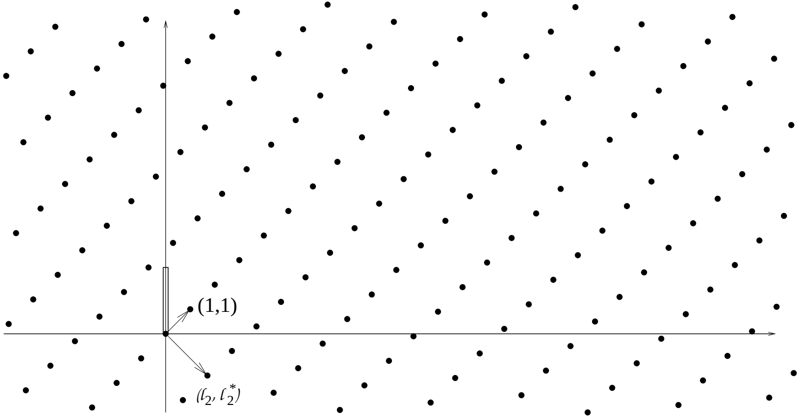
IFS:

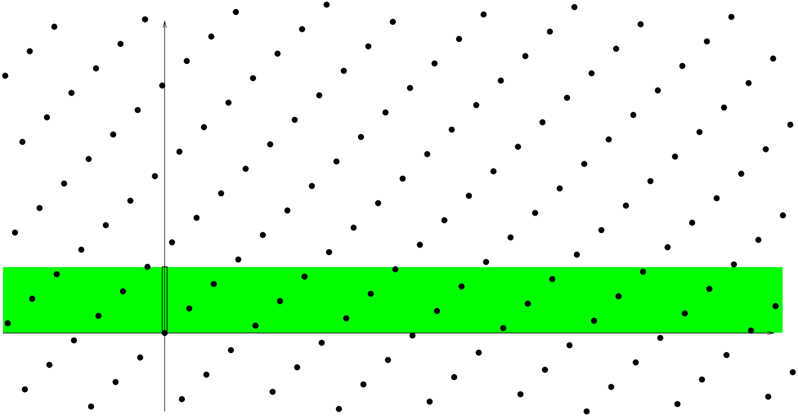
$$a = \beta a + \{0, 1+\frac{\sqrt{3}}{3}\} \quad \cup \quad \beta b + \{\frac{\sqrt{3}}{3}\}$$
$$b = \beta a + \{0, 1+\frac{\sqrt{3}}{3}, 2+2\frac{\sqrt{3}}{3}\} \quad \cup \quad \beta b + \{\frac{\sqrt{3}}{3}, 1+2\frac{\sqrt{3}}{3}\}$$

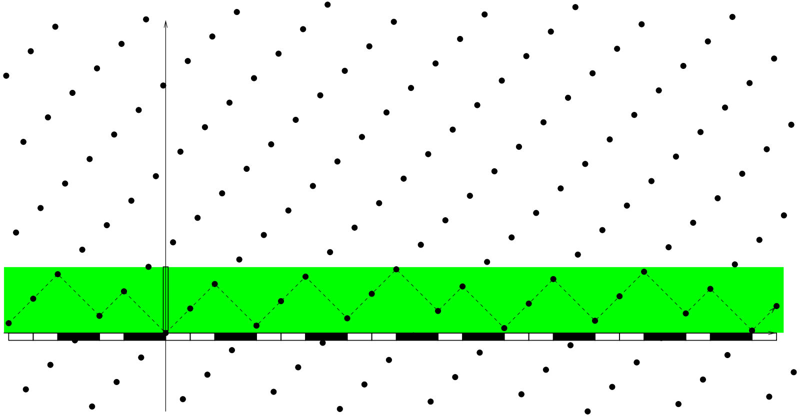
Solution: $a = [0, \frac{\sqrt{3}}{3}], \qquad b = [0, 1]$

Model Sets and Rauzy fractals

Another way to generate tilings.


(1,1)
(ℓ_2, ℓ_2^*)





$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^{d+e} & \xrightarrow{\pi_2} & \mathbb{R}^e \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

- Λ a *lattice* in $\mathbb{R}^{d+e}$
- π_1, π_2 *projections*
  - $\pi_1|_\Lambda$ injective
  - $\pi_2(\Lambda)$ dense
- W *compact*
  - cl(int(W))= W
  - $\mu(\partial(W)) = 0$

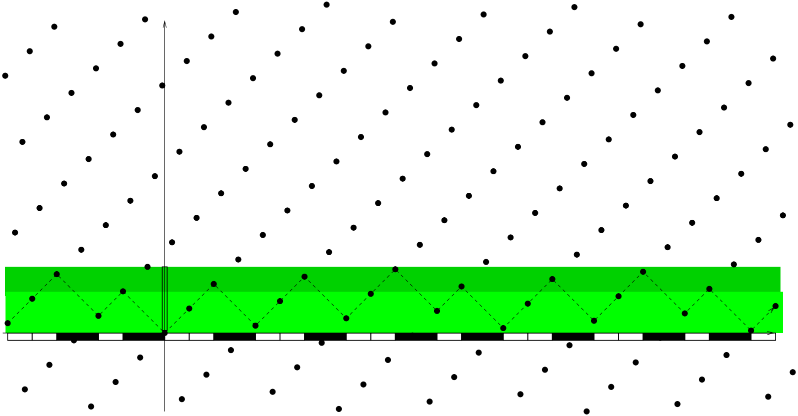
Then $V = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *model set*.

The *star map* : $\star : \pi_1(\Lambda) \to \mathbb{R}^e, \; x^\star = \pi_2 \circ {\pi_1}^{-1}(x)$

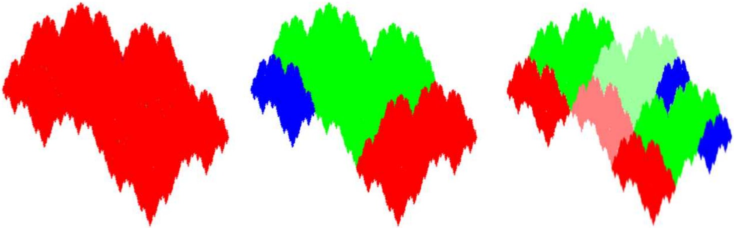
Given a substitution tiling which *is* a model set:

Tiling $\rightsquigarrow$ point set V; $\overline{V^\star} = W$

(the *window* or *Rauzy fractal*).

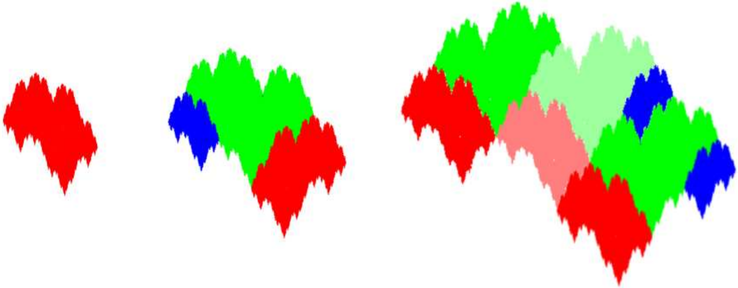


$$\sigma: \quad S \to ML, \quad M \to SML, \quad L \to LML$$

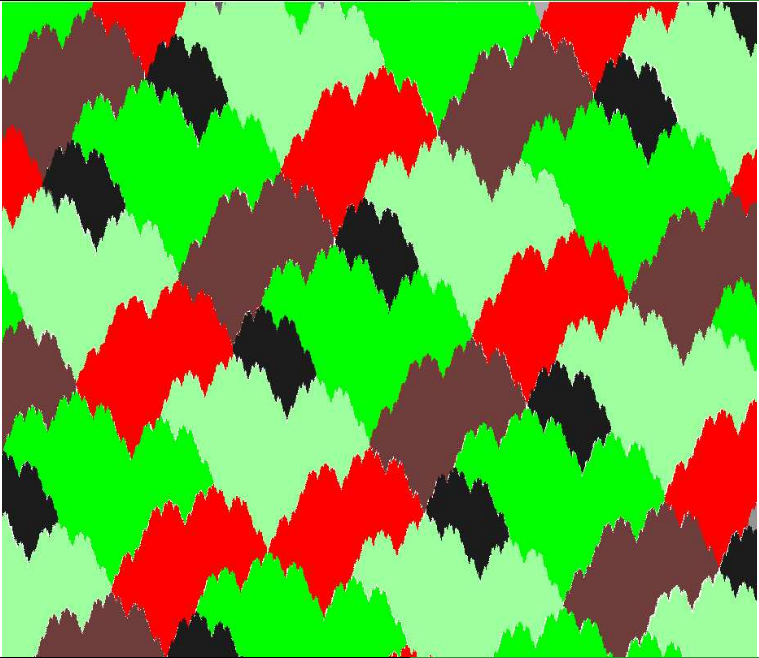


The *natural decomposition* $\rightsquigarrow$ IFS.

$$\sigma: \quad S \to ML, \quad M \to SML, \quad L \to LML$$



The natural decomposition, resp. its IFS $\rightsquigarrow$ the *dual* substitution tiling. (See Sing, Sirvent-Wang,...)



The dual substitution tiling $\mathcal{T}'$ defines a family of tilings, the *tiling space* $\mathbb{X}_{\mathcal{T}'}$.

A(nother) way to compute the dual tiling:

Let $\mathcal{D}$ be the digit set for $\mathcal{T}$

(where $\mathcal{T}$ arises from a model set. E.g., the vertices of $\mathcal{T}$ are a model set.)

Then $(\mathcal{D}^\star)^T$ defines a new substitution: $\sigma^\star$.

(See Thurston, Gelbrich, Vince)

Ex.:

$$\mathcal{D} = \begin{pmatrix} \{0, 1+\frac{\sqrt{3}}{3}\} & \{0, 1+\frac{\sqrt{3}}{3}, 2+2\frac{\sqrt{3}}{3}\} \\ \{\frac{\sqrt{3}}{3}\} & \{\frac{\sqrt{3}}{3}, 1+2\frac{\sqrt{3}}{3}\} \end{pmatrix}$$

$$(\mathcal{D}^\star)^T = \begin{pmatrix} \{0, 1-\frac{\sqrt{3}}{3}\} & \{-\frac{\sqrt{3}}{3}\} \\ \{0, 1-\frac{\sqrt{3}}{3}, 2-2\frac{\sqrt{3}}{3} & \{-\frac{\sqrt{3}}{3}, 1-2\frac{\sqrt{3}}{3}\} \end{pmatrix}$$

Claim: The tiling spaces $\mathbb{X}_{\mathcal{T}'}$ and $\mathbb{X}_{\mathcal{T}^\star}$ are equal, at least if $d = e = 1$, two letters.

($\mathcal{T}^\star$ a tiling generated by $\sigma^\star$)

Some $\star$–dual tilings (better: tiling spaces):

- Fibonacci: Fibonacci itself (self–dual!)
- $a \to aaaab, \quad b \to aaab$: $\quad c \to cd, \quad d \to dcdcdcd$
- Ex. above $a \to aba, \quad b \to ababa$: Again, itself (self–dual!)

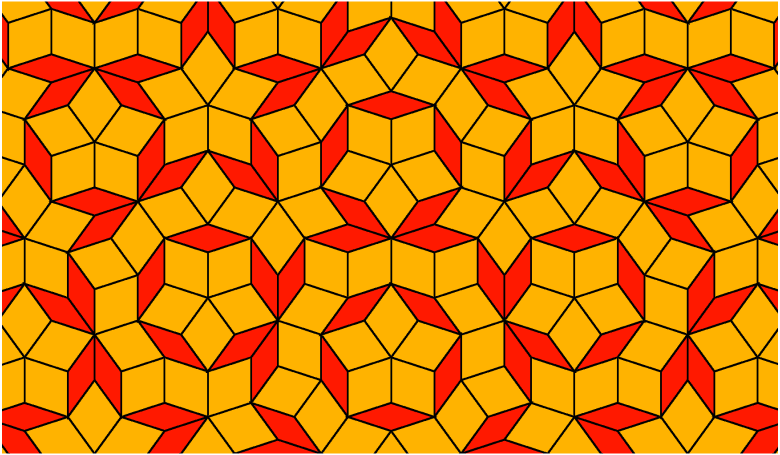
Ex.: Penrose tiling, version with Robinson triangles:

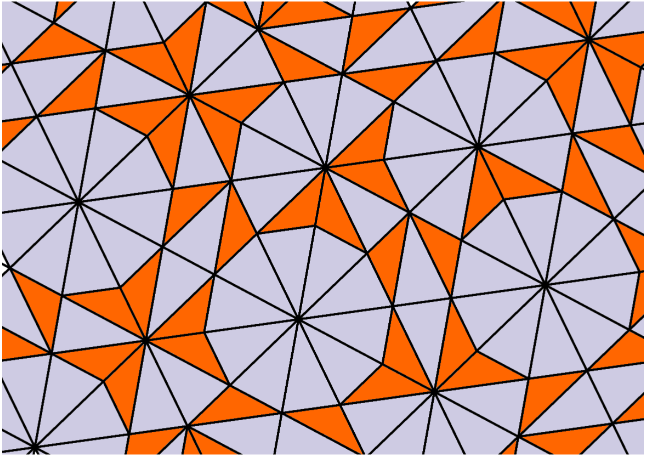
- 40 prototiles up to translations
- 2 prototiles up to isometries

So it is better to work with isometries instead of digit sets.

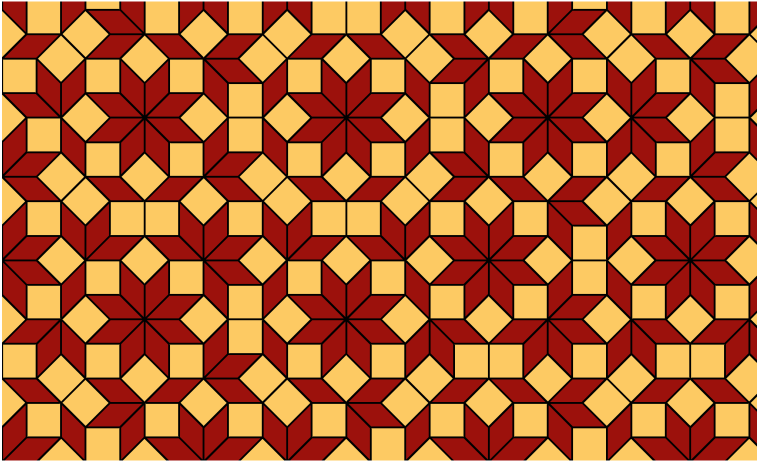
(Allow reflections and rotations.)

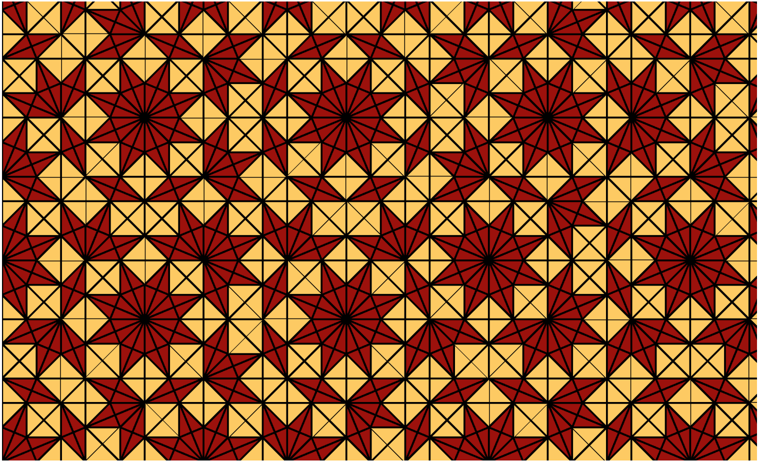
Use cyclotomic number fields $\mathbb{Z}[\xi]$, $\xi = e^{2\pi i/n}$.





Ex.: Ammann–Beenker





Self–duality

- The dual of the Penrose tiling: the Tübingen triangle tiling (different w.r.t. diffraction, dynamics, top. properties of $\mathbb{X}_{\mathcal{T}}$).
- The dual of the Ammann-Beenker tiling: A very similar tiling!

Def.: (preliminary)

A substitution is *self–dual* (with respect to $\star$–duality), if $\mathbb{X}_\sigma = \mathbb{X}_{\sigma^\star}$.

Necessary:

- Factor λ is of algebraic degree 2.
- Substitution matrix $M^T = PMP^{-1}$.

For two tiles (or letters), in any dim:

This gives a characterization of all possible substitution matrices

(λ unimodular!).

$$\begin{pmatrix} k & m \\ (k^2 \pm 1)/m & k \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} m & k \\ k & (k^2 \pm 1)/m \end{pmatrix} \qquad k, m \geq 1, \; m | k$$

Connections to automorphisms

Case $d = 1$, two letters.

The symbolic substitution σ defines an endomorphism of the free group F_2 on 2 letters.

Ex. $\sigma(a) = aba, \quad \sigma(b) = ababa.$

If $\sigma \in \mathrm{Aut}(F_2)$ then σ^{-1} defines another substitution.

Ex. cont.: $\sigma^{-1}(a) = ab^{-1}a, \quad \sigma^{-1}(b^{-1}) = ab^{-1}ab^{-1}a.$

This is the same, up to an (outer) automorphism

$\tau : a \rightarrow a, b \rightarrow b^{-1}.$

In all examples so far: $\sigma^{-1} \sim \tau \circ \sigma^{\star}$.

$(\tau \in \langle s, t \rangle$, essentially a permutation of letters)

This is no surprise. What I learned last week: This follows from a paper of Hiromi Ei (2003).

M. Baake & JAG Roberts (2001) showed a necessary condition for σ being self–dual ('reversing symmetries').

V. Berthé (preprint) has a necessary & sufficient criterion for σ being self–dual ($d = 1$, 2 letters).