WEIGHTED 1×1 CUT-AND-PROJECT SETS IN BOUNDED DISTANCE TO A LATTICE

DIRK FRETTLÖH AND ALEXEY GARBER

ABSTRACT. Recent results of Grepstad and Larcher are used to show that weighted cut-and-project sets with one-dimensional physical space and one-dimensional internal space are bounded distance equivalent to some lattice if the weight function h is continuous on the internal space, and if h is either piecewise linear, or twice differentiable with bounded curvature.

1. INTRODUCTION

A *Delone set* is a set Λ of points in some metric space $\mathbb{X}$ such that (1) there is $r > 0$ such that each ball of radius r contains at most one point of Λ, and (2) there is $R > 0$ such that each ball of radius R contains at least one point of Λ. Depending on the context, Delone sets are also called separated nets, or (r, R)-sets. Two Delone sets Λ, Λ' in the same metric space are called *bounded distance equivalent* ($\Lambda \overset{\text{bd}}{\sim} \Lambda'$) if there is a bijection $\varphi \colon \Lambda \to \Lambda'$ such that $|x - \varphi(x)|$ is uniformly bounded. In 1993 M. Gromov asked whether any Delone set Λ in $\mathbb{R}^2$ is *bilipschitzequivalent* with $\mathbb{Z}^2$ [8]; i.e., whether there is a bijection from Λ to $\mathbb{Z}^2$ such that the bijection is Lipschitz continuous in both directions. In 1998 D. Burago and B. Kleiner, and independently C. McMullen, gave a negative answer [3, 14]. The analogous question for the hyperbolic plane $\mathbb{H}^2$ was answered positively by Bogopolskii [2] by showing that all Delone sets in $\mathbb{H}^2$ are bounded distance equivalent to each other. Bounded distance equivalence implies bilipschitzequivalence.

Even before that physicists asked whether some given crystallographic or quasicrystallographic Delone set Λ in $\mathbb{R}^2$ or $\mathbb{R}^3$ has an "average lattice" of the form $a\mathbb{Z}^2$; i.e. whether there is $a > 0$ such that $\Lambda \overset{\text{bd}}{\sim} a\mathbb{Z}^2$. A *lattice* in $\mathbb{R}^d$ is the $\mathbb{Z}$-span $\langle v_1, \ldots, v_d \rangle_{\mathbb{Z}}$ of d linearly independent vectors $v_i \in \mathbb{R}^d$. In [4] it is shown that any two lattices in $\mathbb{R}^d$ with equal density are bounded distance equivalent. In [5] a sufficient condition for a cut-and-project set (CPS) being bounded distance equivalent to some lattice with the same density is given. For a definition of a CPS see below. There is no precise mathematical definition of a quasicrystal; but often when speaking of a (mathematical) quasicrystal a CPS set is meant.

Recently bounded distance equivalence of Delone sets did get some attention, see e.g. [13, 7, 9, 10, 11] and references therein. A frequently exploited connection is the correspondence between (certain) CPS and (certain) bounded remainder sets for (discrete) toral rotations. Given a set $S \subset [0, 1)$ and some (irrational) $\alpha > 0$ the *deficiency* (or discrepancy) of S with respect to some $x \in \mathbb{R}$ is

$$D_n(S, x) := \sum_{k=0}^{n-1} 1_S(x + k\alpha \bmod 1) - n\lambda(S),$$

where λ denotes Lebesgue measure on $\mathbb{R}$. A set $S \subset [0, 1)$ is called a *bounded remainder set* (BRS) with respect to α if there is $C > 0$ such that for almost all x we have $\sup |D_n(S, x)| < C$. As we will see, for our purposes the x plays no role; it is included

Date: July 14, 2017.

in the definition only because in some contexts there is an exceptional null-set of x to consider.

A profound theorem of Kesten [12] shows that an interval $[a, b] \subset [0, 1)$ is a BRS for the discrete toral rotation $n\alpha \bmod 1$ on the one-dimensional torus if and only if $b-a \in \mathbb{Z}+\alpha\mathbb{Z}$. Applied to CPS this proves for instance that the Fibonacci sequence, defined by a CPS with lattice $\langle (1,1)^T, (\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})^T \rangle_{\mathbb{Z}}$ and window $[0, \frac{1+\sqrt{5}}{2})$ is bounded distance equivalent to some lattice, whereas the Half-Fibonacci sequence using the same lattice but window $[0, \frac{1+\sqrt{5}}{4})$, is not bounded distance equivalent to any lattice.

In this paper we exploit the connection between continuous toral rotations and weighted cut-and-project sets. Our main result Theorem 4.1 uses two theorems of [6] on continuous toral rotations. It shows that many weighted 1×1 CPS where the window is an interval and the weight function h is continuous and supported on W (hence h equals 0 at the endpoints of the interval) are bounded distance equivalence to some lattice, with no restrictions on the length of the window. This is in strong contrast with the discrete case, see Kesten's theorem mentioned above, respectively the Half-Fibonacci example.

Notation: Throughout the paper, λ denotes d-dimensional Lebesgue measure (where $d = 1$ or $d = 2$, depending on the context). The Dirac measure in x is denoted δ_x.

2. Cut-and-project sets

A *cut-and-project set* (CPS, aka *model set*) Λ is given by a collection of maps and spaces:

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ \Lambda & & \Gamma & & W \end{array}$$

where in general G and H are locally compact abelian groups. Furthermore, Γ is a lattice (i.e., a discrete cocompact subgroup) in $G \times H$, W is a relatively compact set in H, and π_1 and π_2 are projections to G and to H respectively, such that $\pi_1|_\Gamma$ is one-to-one, and $\pi_2(\Gamma)$ is dense in W. Then

$$\Lambda = \{\pi_1(x) \mid x \in \Gamma, \, \pi_2(x) \in W\}$$

is called a CPS.

Throughout this paper we will always have $G = \mathbb{R}$ and $H = \mathbb{R}$, hence we call the resulting CPS sometimes 1×1-CPS in order to distinguish them from CPS where G or H have higher dimension. Anyway, we will refer to these spaces as G and H (rather than $\mathbb{R}$ and $\mathbb{R}$) in order to distinguish the space G supporting the CPS Λ (often called *direct space*) from the space H supporting W (often called *internal space*).

It does not really matter whether Γ is a proper lattice, or a translate of some lattice, since translating the lattice by z yields the same CPS (shifted by $\pi_1(z)$) as translating the window W by $\pi_2(z)$. In general, translating the window corresponds just to changing the CPS Λ to another CPS Λ' that is locally indistinguishable from Λ, in the sense that a copy of each local piece of Λ appears in Λ', and vice versa.

The *density* of a CPS is the average number of points per unit area. It is known that the density of a CPS exists and equals

$$\operatorname{dens} \Lambda = \frac{\lambda(W)}{|\det(M_\Gamma)|}, \tag{1}$$

where M_Γ is the matrix whose columns are the spanning vectors of the lattice Γ. See [1, Thm. 7.2] and references there for details.

Example 2.1. The (symbolic) Fibonacci sequence can be generated by applying the map $\sigma\colon\ a \mapsto ab,\ b \mapsto a$ repeatedly to the letter pair $a|a$: $\sigma(a|a) = ab|ab$, $\sigma^2(a) = aba|aba$,

$\sigma^4(a) = abaababa|abaababa$, $\sigma^6(a) = abaababaabaababaababa|abaababaabaababaababa, \ldots$. This symbolic sequence can be transformed into a Delone set in $\mathbb{R}$ by assigning an interval of length $\tau = \frac{\sqrt{5}+1}{2}$ to a and an interval of length 1 to b. Our Delone set Λ then consists of the endpoints of the intervals. This Delone set can be defined via a CPS, too.

The corresponding CPS has $G = \langle (1,0)^T \rangle_{\mathbb{R}}$, $H \langle (0,1)^T \rangle_{\mathbb{R}}$, $W = [-\frac{1}{\tau}, 1[\subset H$, lattice $\Gamma = \langle \binom{1}{1}, \binom{\tau}{-\tau^{-1}} \rangle_{\mathbb{Z}}$, and π_1 and π_2 are orthogonal projections to G, respectively to H.

Weighted CPS are a generalisation of the notion of a CPS. A *weighted* CPS is a Dirac comb $\sum_{x \in \Lambda} h(x^\star)\delta_x$, where $h \colon W \to \mathbb{R}$ is continuos, and $x^\star := \pi_2(\pi_1^{-1}(x))$. Here, $\pi_1^{-1}(x)$ makes sense since $\pi_1|_\Gamma$ is one-to-one. A weighted CPS with constant weight function $h(x) = 1$ for all $x \in W$(and $h(x) = 0$ for $x \notin W$) is just an ordinary CPS, viewed as a measure. Weighted Dirac combs and weighted CPS are relevant in the study of diffraction properties of CPS, see [1] and references therein. It is easy to see that the density formula (1) for CPS generalises to weighted CPS as follows:

$$\operatorname{dens} \Lambda = \frac{\int_W h(t)dt}{|\det(M_\Gamma)|}. \tag{2}$$

3. BRS for continuous rotations and weighted CPS

In order to utilize the results of [6] we generalise the notion of bounded distance equivalence from point sets to measures.

Definition 3.1. Two measures μ, ν on $\mathbb{R}$ are *bounded distance equivalent*, if there is $C > 0$ such that for all $a, b \in \mathbb{R}$ with $a < b$

$$|\mu([a,b]) - \nu([a,b])| < C.$$

Since a point set Λ in $\mathbb{R}$ can be identified with a measure $\sum_{x \in \Lambda} \delta_x$ it is not hard to see that this definition reduces for Delone sets to the definition of bounded distance equivalence above. Nevertheless, we spell out the details in the proof of the next lemma.

Lemma 3.2. *Two Delone sets Λ, Λ' in $\mathbb{R}$ are bounded distance equivalent as point sets if and only if the corresponding Dirac combs $\omega = \sum_{x \in \Lambda} \delta_x$ and $\omega = \sum_{x' \in \Lambda'} \delta_{x'}$ are bounded distance equivalent as measures.*

Proof. Without loss of generality let $\Lambda = \{\ldots, x_{-1}, x_0 = 0, x_1, \ldots\}$ (with $x_i < x_j$ if $i < j$) and $\Lambda' = \{\ldots, x'_{-1}, x'_0 = 0, x'_1, \ldots\}$ (with $x'_i < x'_j$ if $i < j$). Let ω respectively ω' be the corresponding Dirac combs.

If there is a bounded distance bijection between Λ and Λ' then $x_i \mapsto x'_i$ is a bounded distance bijection, too. Hence there is $C' > 0$ such that $|x_i - x'_i| < C'$ for all i.

Let $x'_{i+\ell}$ be the largest $x' \in \Lambda'$ with $x' < x_i$. By the Delone property the interval $[x'_i, x_i]$ contains at most $\frac{|x_i - x'_i|}{r}$ points of Λ', hence

$$|\ell| \le \frac{|x_i - x'_i|}{r} < \frac{C'}{r}.$$

Thus the difference

$$|\omega([a,b]) - \omega'([a,b])| = |\sum_{x \in \Lambda \cap [a,b]} \delta_x - \sum_{x' \in \Lambda' \cap [a,b]} \delta_{x'}|$$

is bounded by the number of points $x_i \in [a,b]$ such that $x'_i \notin [a,b]$ (or vice versa). Thus

$$|\omega([a,b]) - \omega'([a,b])| < 2C'r,$$

where C' and r depend only on Λ and Λ'.

Conversely, if $|\omega([-n,n]) - \omega'([-n,n])| < C$ for all n, then the number of points in $\Lambda \cap [-n,n]$ deviates at most by C from the number of points in $\Lambda' \cap [-n,n]$. If $x_i \in [-n,n]$ but $x_i' \notin [-n,n]$, then $[x_i, x_i'[$ can contain at most $\frac{|x_i'-x_i|}{r}$ points of Λ'; again by the Delone property of Λ'. Hence

$$\frac{|x_i' - x_i|}{r} < C \text{ respectively } |x_i' - x_i| < Cr.$$

where C and r depend only on Λ and Λ'. The same holds for x_i, x_i' with $i < 0$. □

The paper [6] studies BRSs of the continuous analogue of the discrete toral rotations above. We state two definitions from [6], slightly simplified for our purposes.

Definition 3.3. Let $x = (x_1, x_2) \in [0,1]^2$, and let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We say that the function $X\colon [0,\infty) \mapsto [0,1]^2$ defined by

$$X(t) = (x_1 + t \bmod 1, x_2 + \alpha t \bmod 1)$$

is the two-dimensional continuous irrational rotation with slope α and starting point x.

The notion of deficiency translates as follows.

Definition 3.4. Let $P \subset [0,1]^2$ be an arbitrary measurable set with Lebesgue measure $\lambda(P)$. We say that P is a *bounded remainder set* (BRS) for the continuous irrational rotation with slope $\alpha > 0$ and starting point $x = (x_1, x_2) \in [0,1]^2$ if the distributional error

$$\Delta_t(P, \alpha, x) = \int_0^t 1_P\left(x_1 + s \bmod 1, x_2 + \alpha s \bmod 1\right)\, ds - t\lambda(P) \tag{3}$$

is uniformly bounded for all $t > 0$. Here, 1_P denotes the characteristic function for the set P.

The following simple observation will be useful in the sequel. It can be shown easily by spelling out the definition (resp., definitions, since it holds in both cases, discrete toral rotations and continuous toral rotations).

Lemma 3.5. *Let P, P' be BRSs. If $P \cap P' = \varnothing$ then the union $P \cup P'$ is a BRS, too. If $P' \subset P$ then the difference $P \setminus P'$ is a BRS, too.*

Two of the main results in [6] are the following.

Theorem 3.6. *For almost all $\alpha > 0$ and every $x \in [0,1]^2$, every polygon $P \subset [0,1]^2$ with no edge of slope α is a BRS for the continuous irrational rotation with slope α and starting point x.*

Theorem 3.7. *For almost all $\alpha > 0$ and every $x \in [0,1]^2$, every convex set $P \subset [0,1]^2$ whose boundary ∂P is a twice continuously differentiable curve with positive curvature at every point is a BRS for the continuous irrational rotation with slope α and starting point x.*

To a BRS P and an irrational slope α as above one can associate a weighted CPS as follows; see also Figure 1. The direct space is $G = \binom{1}{\alpha}\mathbb{R}$, the internal space is the orthogonal complement $H = \binom{1}{\alpha}^{\perp}$ of G in $\mathbb{R}^2$. The projections π_1 and π_2 are the orthogonal projections to G, respectively to H, and $W = \pi_2(P)$. Since P is connected, W is a line segment in H, so we have $W = [h_1, h_2]$ for some $h_i \in H$. Because of the properties of P (either positive curvature, or no slope in direction α) there is exactly one point $z \in [0,1]^2$ such that $\pi_2(z) = h_1$. Let $\Gamma \subset G \times H$ be $z + \mathbb{Z}^2$. Hence Γ is not actually a lattice here, but a translation of the lattice $\mathbb{Z}^2$. This makes no difference, see the remark in the definition

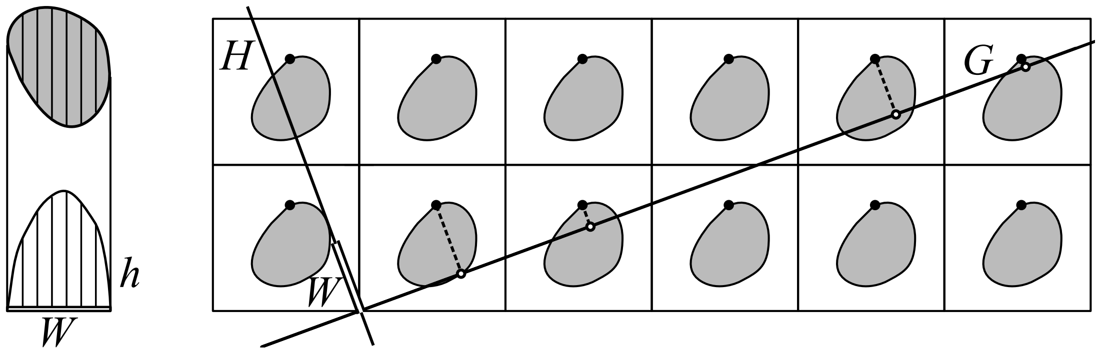

FIGURE 1. A CPS tailored to BRS for continuous toral rotations. The direct space G is the line $(1, \alpha)^T\mathbb{R}$. The internal space H is the orthogonal complement of G in $\mathbb{R}^2$. The CPS consists of the projected points $z + g$ of $z + \mathbb{Z}^2$ (black points) where G intersects the adjacent convex set $y + P$ ($g \in \mathbb{Z}^2$). The weighted CPS is obtained by attaching to each point $x = \pi_1(z + g) \in \Lambda$ the length of the intersection of G with $g + P$. (The weights are not shown in the image.) Hence the weight function h on W is given by the width of P in direction G (indicated on the left).

of a CPS in Section 2. Since α is irrational, $\pi_1|_\Gamma$ is one-to-one, and $\pi_2(\Gamma)$ is dense in W. Let Λ be the CPS defined by these data.

The map $h\colon H \to \mathbb{R}$ is defined by letting $h(\pi_2(y))$ (for $y \in \mathbb{R}^2$) be the length of $\binom{1}{\alpha}\mathbb{R} \cap (y + P)$. Clearly, h vanishes outside W, and each P fulfilling either the conditions of Theorem 3.6 or of Theorem 3.7 yields a map h that is continuous on H: the support of h is W, and $h(h_1) = h(h_2) = 0$. Hence $\sum_{x\in\Lambda} h(x^\star)\delta_x$ is a weighted CPS. (Recall that $x^{star} = \pi_2(\pi_1^{-1}(x))$.)

Conversely, given a weighted CPS Λ with data $G = \binom{1}{\alpha}\mathbb{R}, H = \binom{1}{\alpha}^\perp, \Gamma = \mathbb{Z}^2, W = [a, b], \Gamma = \mathbb{Z}^2, h$, we can apply the opposite construction to obtain a candidate for a BRS with respect to a continuous rotation on the torus. One possible problem is that the window for Λ may be too large to fit into a standard fundamental domain of the lattice $\mathbb{Z}^2$. One way to handle this is to split the "big" CPS into smaller ones.

Lemma 3.8. *A CPS Λ with lattice $\Gamma = \mathbb{Z}^2$, $G = \binom{1}{\alpha}\mathbb{R}$, $H = G^\perp$, and $W = [a, b] \subset H$ is the union of n^2 CPS with lattice translates $\Gamma_{k,\ell} = (k, \ell)^T + n\mathbb{Z}^2$ $(0 \le k, \ell \le n - 1)$, and the same G, H, W.*

Hence we assume without loss of generality in the following that W fits into the interior of the projection of the fundamental domain $[0, 1)^2$ of $\mathbb{Z}^2$ along G. Otherwise we split the CPS into n^2 smaller ones as in the lemma above for appropriate large enough n.

Now we choose a compact set $P \subset [0, 1]^2$ such that for $z \in W$ the value $h(z)$ equals the length of $(z + \binom{1}{\alpha}\mathbb{R}) \cap P$. (For instance, if $h(z) \ge 0$, then P can be the region between the graph of $\frac{1}{2}h(z)$ and the graph of $-\frac{1}{2}h(z)$.) Now again, the values of h may be too large to fit P into $[0, 1)^2$. Hence, if needed, we may rescale h by some appropriate factor $c' > 0$ such that P fits into $[0, 1)^2$.

Lemma 3.9. *Let $\omega = \sum_{x\in\Lambda} h(x^\star)\delta_x$ and P be as in the preceding construction. The weighted CPS ω is bounded distant equivalent to $c\lambda$ for some $c > 0$, if and only if P is a BRS with respect to α.*

Proof. We compare $\Delta_t(P,\alpha)$ with $\sum\limits_{\substack{x\in\Lambda\\0\le x\le t}} h(x^\star) - t\int\limits_{h_1}^{h_2} h(s)ds$. By construction we have $\lambda(P) = \int\limits_{h_1}^{h_2} h(s)ds$. Also by construction, $h(x^\star)$ is the width of the intersection of the line segment $\{(s,\alpha s)^T \mid \lfloor x\rfloor \le s \le \lfloor x\rfloor + 1\}$. So for $t\in\mathbb{N}$ we have

$$\sum_{\substack{x\in\Lambda\\0\le x\le t}} h(x^\star) - t\int_{h_1}^{h_2} h(s)ds = \int_0^t 1_P(s \bmod 1, \alpha s \bmod 1)dt - t\lambda(P)$$

Hence the right hand side is uniformly bounded if and only if the left hand side is. □

Remark 3.10. The authors of [6] give a precise meaning to the "almost all" in Theorems 3.6 and 3.7. Namely, the results hold for all α whose continued fraction expansion $\alpha = [a_0; a_1, a_2, \cdots]$ satisfies

$$\sum_{\ell=0}^{m} \frac{a_{\ell+1}}{q_\ell^{1/2}} \sum_{k=1}^{\ell+1} a_k < C, \tag{4}$$

where C is a constant independent of m. Here, $(q_\ell)_{\ell\ge 0}$ is the sequence of best approximation denominators for α. In particular this implies that the results hold for all $\alpha = [a_0; a_1, a_2, \cdots]$ where the a_i are uniformly bounded by some constant c. This follows from the fact that the q_n grow at least as fast as τ^n (where $\tau = \frac{\sqrt{5}+1}{2}$). Then the sum above is less than the convergent sum

$$\sum_{\ell=0}^{\infty} \frac{c}{\tau^{\ell/2}}(\ell+1)c.$$

Since many 1×1 CPS in the literature use quadratic irrationals for the slope α, and quadratic irrationals have periodic continued fraction expansion, these results apply to most cases of 1×1 CPS studied in the literature.

4. Main results

Using the results from the last section we can now prove the following result.

Theorem 4.1. *Let Λ be a 1×1 CPS with lattice $\Gamma = \mathbb{Z}^2$, $G = \binom{1}{\alpha}\mathbb{R}$ and $H = G^\perp$, window $W = [a,b] \subset H$, and $h \in C(H)$ with support W (i.e., h vanishes outside W, and $h(x) \ne 0$ for $x \in W$). Furthermore, let α fulfill the condition* (4) *in Remark 3.10.*

(1) *If h is piecewise linear, or*
(2) *if h is twice differentiable on W, and h'' is uniformly bounded on W,*

then the weighted Dirac comb $\omega = \sum\limits_{x\in\Lambda} h(x^\star)\delta_x$ is bounded distance equivalent to $m\lambda$, where $m = \int\limits_a^b h(t)dt$.

Proof. Let us first assume that h is twice differentiable on W, and h'' is uniformly bounded on W. Choose a compactly supported twice differentiable f, such that the support of h is contained in the interior of the support of f, and such that there is $c_0 > 0$ such that the second derivative of f is less than $-c_0$. (For instance, f may be the width function of an appropriate big circle.) Choose $c_1 > 0$ such that the second derivative of $c_1 f - h$ is bounded away from 0. I.e., there is $c_2 < 0$ such that for all $t \in W$ holds: $(c_1 f(t) - h(t))'' < c_2$. Then $c_1 f - h$ is twice differentiable, $c_1 f - h$ has negative second derivative less than $c_2 < 0$, and consequently $c_1 f - h$ is convex. At the endpoints of W the function $c_1 f - h$ has vertical tangents, but its graph has positive curvature at these points because the curvature will coincide with the curvature of a circle.

Both $c_1 f$ and $c_1 f - h$ yield convex sets P, P' that fulfill the conditions of Theorem 3.7: The convex set P for $c_1 f$ is just an ellipse. As P' we might again choose the region between the graphs of $\pm\frac{1}{2}(c_1 f - h)$. So both P and P' yield BRS. By Lemma 3.5 the difference $P \setminus P'$ of two BRS P, P' with $P' \subset P$ is again a BRS, hence h corresponds to a BRS, too. By Lemma 3.9 the claim follows.

The case of piecewise linear f is handled analogously. Note that if h is piecewise linear and continuous on H, then the corresponding polygon P has no edge parallel to $\binom{1}{\alpha}$. $\square$

Since Lemma 3.9 and Lemma 3.5 imply that the sum $\mu_1+\mu_2$ of two measures μ_1, μ_2 that are bounded distance equivalent with $c_1\lambda$, respectively $c_2\lambda$, is bounded distance equivalent to $(c_1 + c_2)\lambda$, the following result is immediate.

Corollary 4.2. *Any linear combination of Dirac combs as in Theorem 4.1 is again bounded distance equivalent to $c\lambda$, for some appropriate $c > 0$.*

Theorem 4.1 holds for almost all α, more precisely: for all α fulfilling Equation (4). In particular, Theorem 4.1 holds for all α with bounded values in their continued fraction expansion. However, there is no particular example of an algebraic number of degree larger than two where it is known whether the values in its continued fraction expansion are bounded. Fortunately, many 1×1 CPS in the literature arise from two-letter substitutions. The slope α for a CPS for some two letter substitution is always a quadratic irrational, compare for instance with Example 2.1. Since quadratic irrationals have periodic continued fraction expansions, Theorem 4.1 holds for all quadratic irrationals α.

Unfortunately, the most natural way to describe a CPS for a two-letter substitution is to use a different lattice than $\mathbb{Z}^2$, namely the one spanned by the vectors $(1,1)^T, (\beta, \beta')$, where $1, \beta$ are the natural tile lengths, and β' is the algebraic conjugate of β, see [1] for details. Hence $\beta\beta' = \frac{p}{q} \in \mathbb{Q}$.

Corollary 4.3. *Let β be a quadratic irrational. Let Λ be a weighted 1×1 CPS with $G = \mathbb{R}$, $\Gamma = \langle (1,1)^T, (\beta, \beta')\rangle_{\mathbb{Z}}$, the window $W = [a, b]$ an interval in H and h as in Theorem 4.1. Then the Dirac comb $\omega = \sum_{x\in\Lambda} h(x^\star)\delta_x$ is bounded distance equivalent to $m\lambda$ where $m = \frac{1}{|\beta-\beta'|}\int_a^b h(t)dt$.*

Proof. The lattice Γ can be mapped to the standard integer lattice $\mathbb{Z}^2$ by applying some matrix M, where $M^{-1} = \begin{pmatrix} 1 & \beta \\ 1 & \beta' \end{pmatrix}$. Hence $M = \frac{1}{\beta-\beta'}\begin{pmatrix} -\beta' & \beta \\ 1 & -1 \end{pmatrix}$. The slope α of Theorem 4.1 is then

$$\alpha = M\binom{1}{0} = \frac{1}{\beta - \beta'}\binom{-\beta'}{1}.$$

Hence

$$\alpha\mathbb{R} = \binom{-\beta'}{1}\mathbb{R} = \binom{\frac{-p}{q}}{\beta}\mathbb{R}.$$

Because of the symmetry of $\mathbb{Z}^2$ the slope $(\frac{-p}{q}, \beta)^T$ yields the same CPS as the slope $(\frac{p}{q}, \beta)^T$, respectively the slope $(1, \frac{q}{p}\beta)^T$. Hence the slope α equals $\frac{q}{p}\beta$. In particular, α is a quadratic irrational as well. Furthermore, M preserves the properties of h. By Theorem 4.1 the resulting CPS Λ' is bounded distance equivalent to $c'\lambda$ for some appropriate c'. Since the original CPS is just the image of Λ' under some (regular) linear map, Λ is also bounded distance equivalent to $m\lambda$ for some appropriate m. By the density formula for weighted CPS (2) holds $m = \frac{1}{|\det(M^{-1})|} = \frac{1}{|\beta-\beta'|}\int_a^b h(t)dt$. $\square$

References

[1] M. Baake, U. Grimm: *Aperiodic Order*, Cambridge Univ. Press, Cambridge (2013).

[2] O.V. Bogopolski: Infinite commensurable hyperbolic groups are bi-Lipschitz equivalent, *Algebra i Logika* 36 (1997) 259-272; English transl. *Algebra and Logic* 36 (1997) 155-163.

[3] D. Burago, B. Kleiner: Separated nets in Euclidean space and Jacobians of bi-Lipschitz maps, *Geom. Funct. Anal.* 8 (1998) 273-282

[4] M. Duneau, C. Oguey: Bounded interpolation between lattices, *J. Phys. A: Math. Gen.* 24 (1991) 461-475.

[5] M. Duneau, C. Oguey: Displacive transformations and quasicrystalline symmetries, *J. Phys. France* 51 (1990) 5-19.

[6] S. Grepstad, G. Larcher: Sets of bounded remainder for the continuous irrational rotation on $[0,1)^2$, arXiv:1603.00207.

[7] S. Grepstad, N. Lev: Sets of bounded discrepancy for multi-dimensional irrational rotation, *Geom. Funct. Anal.* 25 (2015) 87-133.

[8] M. Gromov: *Geometric Group Theory, Asymptotic Invariants of Finite Groups*, LMS Lecture Note Series 182, Cambridge University Press (1993).

[9] A. Haynes: Equivalence classes of codimension one cut-and-project nets, *Ergodic Theory Dyn. Syst.* 3 (2016) 816-831.

[10] A. Haynes, H. Koivusalo: Constructing bounded remainder sets and cut-and-project sets which are bounded distance to lattices, *Israel J. Math.* (2015) 189-201.

[11] A. Haynes, M. Kelly, H. Koivusalo: Constructing bounded remainder sets and cut-and-project sets which are bounded distance to lattices II, *Indag. Math.* 28 (2017) 138-144.

[12] H. Kesten: On a conjecture of Erdös and Szüsz related to uniform distribution mod 1, *Acta Arithmetica* XII (1966) 193-212.

[13] M. Laczkovich: Uniformly spread discrete sets in $\mathbb{R}^d$, *J. London Math. Soc.* 46 (1992) 39-57.

[14] C. McMullen: Lipschitz maps and nets in Euclidean space, *Geom. Funct. Anal.* 8 (1998) 304-314.

Technische Fakultät, Bielefeld University
E-mail address: `dirk.frettloeh@math.uni-bielefeld.de`

School of Mathematical & Statistical Sciences, The University of Texas Rio Grande Valley, 1 West University Blvd., Brownsville, TX 78520, USA.
E-mail address: `alexeygarber@gmail.com`