8 Matrixexponentialfunktion

Erinnerung: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 \cdot x^0 + 1 x^1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \dots$

Potenzreihe von e^x. Andere Beispiele:

- $\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} = x^0 + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \dots$
- $\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = x^1 + \frac{1}{6} x^3 + \frac{1}{120} x^5 \dots$
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = x^0 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \dots$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)!} = x^1 - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \dots$

Geht auch für Matrizen!

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = A^0 + A^1 + \frac{1}{2} A^2 + \frac{1}{6} A^3 + \frac{1}{24} A^4 + \dots$$

Bsp: (a) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad e^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \dots$

$A^0 = E$ immer!

also $e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$e^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} 1 + \frac{1}{2} + \frac{1}{6} + \dots & 0 \\ 0 & 1 + \frac{1}{2} + \frac{1}{6} + \dots \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$$

(c) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ $\quad e^A = \dots = \begin{pmatrix} 1 + 2 + \frac{1}{2} \cdot 2^2 + \frac{1}{6} \cdot 2^3 + \frac{1}{24} 2^4 + \dots & 0 \\ 0 & 1 + (-3) + \frac{1}{2} (-3)^2 + \dots \end{pmatrix}$

also $e^{\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}} = \begin{pmatrix} e^2 & 0 \\ 0 & e^{-3} \end{pmatrix}$

(ganz analog zu (b))

(d) $$e^{\begin{pmatrix}0&2\\0&0\end{pmatrix}} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0\\0&0\end{pmatrix} + \frac{1}{6}\begin{pmatrix}0&0\\0&0\end{pmatrix} + \dots$$

$$= \begin{pmatrix}1&2\\0&1\end{pmatrix}$$

Fakten: E1 Falls $A \cdot B = B \cdot A$, dann $e^{A+B} = e^A \cdot e^B$

E2 Falls $A = M \cdot D \cdot M^{-1}$; $D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix}$

dann $e^A = M \cdot e^D \cdot M^{-1}$.

E3 D wie oben; $e^D = \begin{pmatrix} e^{d_1} & 0 & \cdots & 0 \\ 0 & e^{d_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{d_n} \end{pmatrix}$

(vgl. Bsp. (c))

E4 $e^{(M^{-1})} = e^{-M}$

Aufgaben dazu laufen auf wenige Matrizen(-typen) hinaus. (Sonst sind sie zu schwierig)

(α) $A = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ $\quad e^A = E = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ (siehe Bsp (α))

(b) $A = E$ $\quad e^A = e \cdot E$ (siehe Bsp (b))

(c) $A = D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix}$ (Diagonalmatrix) $\quad e^A = \begin{pmatrix} e^{d_1} & 0 & \cdots & 0 \\ 0 & e^{d_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{d_n} \end{pmatrix}$

(siehe Bsp (c))

(d) $A = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & \cdots & & 0 \end{pmatrix}$; dann ist $A^n = 0$

$$e^A = E + A + \frac{1}{2}A^2 + \dots + \frac{1}{(n-1)!}A^{n-1}$$

(e) $$e^{\begin{pmatrix} 0 & 1 \\ +1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ +1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & +1 \\ 1 & 0 \end{pmatrix} + \frac{1}{24}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} 1+\frac{1}{2}+\frac{1}{24}+\dots & 1+\frac{1}{6}+\dots \\ 1+\frac{1}{6}+\frac{1}{120}+\dots & 1+\frac{1}{2}+\frac{1}{24}+\dots \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{(2n)!} & \sum \frac{1}{(2n+1)!} \\ \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} & \sum \frac{1}{(2n)!} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix}$$ (vgl. Potenzreihen Seite 1)

(f) $$e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{24}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{120}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \dots$$

(vgl. Seite 1)

$$= \begin{pmatrix} 1-\frac{1}{2}+\frac{1}{24}-\dots & 1-\frac{1}{6}+\frac{1}{120}-\dots \\ -1+\frac{1}{6}-\frac{1}{120} & 1-\frac{1}{2}+\frac{1}{24}-\dots \end{pmatrix} = \begin{pmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{pmatrix}$$

Analog

- $$e^{\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}} = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \qquad (\alpha \in \mathbb{R})$$
- $$e^{\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \qquad (\alpha \in \mathbb{R}^+)$$

(g) $$e^{\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} \alpha^2 & 2\alpha \\ 0 & \alpha^2 \end{pmatrix} + \begin{pmatrix} \alpha^3 & 3\alpha^2 \\ 0 & \alpha^3 \end{pmatrix} + \dots$$

geht wie oben $\left(0+1+2\alpha+3\alpha^2+\dots = \sum_{n=1}^{\infty} \frac{1}{n!} n \cdot \alpha^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\alpha^{n-1} = \sum_{m=0}^{\infty} \frac{1}{m!}\alpha^m = e^{\alpha}\right)$

$$= \begin{pmatrix} e^{\alpha} & e^{\alpha} \\ 0 & e^{\alpha} \end{pmatrix}$$, aber..

leichter geht's mit E1:

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \text{ und } \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

also
$$e^{\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}} = e^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = e^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}} \cdot e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \underset{(c,d)}{=} \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^a & e^a \\ 0 & e^a \end{pmatrix}$$

(h) $e^{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}$ geht analog: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

also
$$e^{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \cdot e^{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & a \cdot e \\ 0 & e \end{pmatrix}$$

(i) z.B. $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ gegeben.

dann ist sicher die Form MDM^{-1}, also E2 & E3 benutzen.