Formal Logic — Solution to Exercise 14b

Exercise 14: (Prime prefixes)
(b) A number m is called a *prefix* of another number n if m consists of the first digits of n. (E.g., 13 is a prefix of 137, of 1378, and of 1 378 000.) Show that there is an infinite sequence $a_1, a_2, \ldots$ of prime numbers such that for each $i \in \mathbb{N}$ the number a_i is a prefix of a_{i+1}.
(A candidate for such a sequence might, or might not, be $31, 317, 3176269, \ldots$)

This problem connot be solved using the tools of this lecture. In particular, König's Lemma does not yield a solution. The following one was obtained with help of Frederic (thanks!). Another solution can be obtained from this forum topic.

However, the result is true, and it can be proven using the prime number theorem. It tells us that near a number N, aprroximately each $\ln(N)$th number is a prime number. More precisely: let $\pi(x)$ denote the number of prime numbers between 1 and x, then

$$\lim_{x\to\infty} \frac{\pi(x)}{\left[\frac{x}{\log(x)}\right]} = 1.$$

Hence for all $\varepsilon > 0$ there is z such that

$$\left| \frac{\pi(x)}{\left[\frac{x}{\log(x)}\right]} - 1 \right| < \varepsilon, \quad \text{hence}$$

$$\left| \pi(x) - \frac{x}{\ln(x)} \right| < \varepsilon \frac{x}{\ln(x)}, \quad \text{hence}$$

$$-\varepsilon \frac{x}{\ln(x)} < \pi(x) - \frac{x}{\ln(x)}, \quad \text{hence} \frac{x}{ln(x)} - \varepsilon \frac{x}{\ln(x)} < \pi(x).$$

Our problem asks us to find for each prefix p, which is prime, to find some prime number between $x_1 = p\underbrace{00\cdots01}_{k}$ and $x_2 = p\underbrace{99\cdots9}_{k}$. This prime number will be the successor a_{n+1} of $a_n = p$, having p as a prefix. Note that we are done if we can always find k such that $\pi(x_2) - \pi(x_1) \geq 1$.

The inequality above yields

$$\begin{aligned} \pi(x_2) - \pi(x_1) &\geq \frac{x_2}{\ln x_2} - \frac{x_1}{\ln(x_1)} - \varepsilon \frac{x_2}{\ln(x_2)} + \varepsilon \frac{x_1}{\ln(x_1)} \\ &\geq \frac{x_2}{\ln(x_2)} - \frac{x_1}{\ln(x_1)} - \varepsilon \frac{x_2}{\ln(x_2)} = (1-\varepsilon)\frac{x_2}{\ln(x_2)} - \frac{x_1}{\ln(x_1)}. \end{aligned}$$

Because we are counting, $\pi(x_2) - \pi(x_1)$ is an integer. Hence, if $\pi(x_2) - \pi(x_1) > 0$ then it is at least 1. So we need to show that the right hand side is larger than 0. Note that $x_1 = p10^k + 1$ and $x_2 = (p+1)10^k - 1$. Because there are no even prime numbers except 2, we can as well use $p10^k$ and $(p+1)10^k$. Plugging this into the last equation we obtain

$$\pi(x_2) - \pi(x_1) \geq (1-\varepsilon)\frac{(p+1)10^k}{\ln((p+1)10^k)} - \frac{p10^k}{\ln(p10^k)} = (1-\varepsilon)\frac{(p+1)10^k}{\ln(p+1) + k\ln(10)} - \frac{p10^k}{\ln(p) + k\ln(10)}$$

So it suffices to show that the right hand side ist positive. That is, we need to show

$$
\begin{aligned}
&(1-\varepsilon)\frac{(p+1)10^k}{\ln(p+1)+k\ln(10)} - \frac{p10^k}{\ln(p)+k\ln(10)} > 0 \\
\Leftrightarrow &(1-\varepsilon)\frac{(p+1)10^k}{\ln(p+1)+k\ln(10)} > \frac{p10^k}{\ln(p)+k\ln(10)} \\
\Leftrightarrow &(1-\varepsilon)\frac{(p+1)(\ln(p)+k\ln(10))}{p\ln(p+1)+k\ln(10)} > 1
\end{aligned}
$$

It is easy to see that $\lim\limits_{k\to\infty} \frac{(p+1)(\ln(p)+k\ln(10))}{p(\ln(p+1)+k\ln(10))} = \frac{p+1}{p} > 1$. Hence for each δ small enough there is $\frac{(p+1)(\ln(p)+k\ln(10))}{p(\ln(p+1)+k\ln(10))} > \frac{p+1}{p} - \delta$. Choose $0 < \delta < \frac{p+1}{p} - \frac{1}{1-\varepsilon}$ (which is positive if ε is sufficiently small). Then

$$\frac{(p+1)(\ln(p)+k\ln(10))}{p(\ln(p+1)+k\ln(10))} > \frac{p+1}{p} - \delta > 1-\varepsilon,$$

hence $(1-\varepsilon)\frac{(p+1)(\ln(p)+k\ln(10))}{p(\ln(p+1)+k\ln(10))} > 1$. This proves the claim.