Formal Logic — Exercise Sheet 9

Exercise 32: (Guess the cardinality)
Give the cardinality of each of the following sets. You do not need to justify your answer.

(a) $\mathbb{N}^2$
(b) $\mathbb{R}^3$
(c) $\mathcal{P}(\mathcal{P}(\mathbb{R}))$
(d) $\mathcal{P}_{fin}(\mathbb{N})$
(e) the set of all infinite 0-1-words: $\{u_1 u_2 \ldots \mid u_i \in \{0,1\}\}$
(f) the set of all finite 0-1-words: $\{u_1 u_2 \ldots u_n \mid u_i \in \{0,1\}, n \in \mathbb{N}_0\}$
(g) $H(F)$ (Herbrand universe of some formula in first-order logic)
(h) the set $\{f : \mathbb{R} \to \mathbb{R}\}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$.

The term in (d) denotes all finite subsets of $\mathbb{N}$, i.e. $\mathcal{P}_{fin}(\mathbb{N}) := \{M \subset \mathbb{N} \mid M \text{ finite }\}$. In (e) the set contains all infinite words (strings) made from the letters 0 and 1, for instance 010101010...., 101000111010..., 00000000... and so on. In (f) the set contains all finite words with letters 0 or 1, i.e.: 0, 1, 00, 01, 10, 11, 000, 001, 010, ...; including the empty word ϵ with zero letters.

Exercise 33: (Prove the cardinality)
Give the cardinality of each of the following sets and prove your answer.

(a) $\mathcal{P}_{fin}(\mathbb{N})$.
(b) The set of all infinite 0-1-words: $\{u_1 u_2, \ldots \mid u_i \in \{0,1\}\}$.
(c) the set of all finite 0-1-words: $\{u_1 u_2 \ldots u_n \mid u_i \in \{0,1\}, n \in \mathbb{N}_0\}$.
(d) The set of all different Turing machines.
(e) The set of all sequences $(a_n)_{n\in\mathbb{N}}$ with values $a_n \in \mathbb{R}$.

Exercise 34: (Infinite models)
Prove Theorem 3.2 by:

(a) Show that $F = \forall x\, P(x, f(x)) \wedge \forall y\, \neg P(y,y) \wedge \forall x\, \forall y\, \forall z\, ((P(x,y) \wedge P(y,z) \Rightarrow P(x,z))$ is satisfiable by finding a model for F.

(b) Show that F is not satisfiable for any $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ where $U_{\mathcal{A}}$ is finite.

Exercise 35: (Tricky bijection)
Your aim is to show that $|[0,1]| = |\mathbb{R}|$.

(a) Find a bijection f from the closed interval $[0,1]$ into the open interval $]0,1[$.

(b) Find a bijection from the open interval $]0,1[$ into the open interval $]-1,1[$.

(c) Find a bijection from the open interval $]-1,1[$ into $\mathbb{R}$.

Since the composition of bijective maps is again bijective, the map $h \circ g \circ f$ is the desired bijection between $[0,1]$ and $\mathbb{R}$.
Extra points are granted for careful proofs that the maps you give are really bijections. *(Hint: in each case you can show that your map is (a) injective, and (b) surjective; or you can give the inverse map f^{-1} and show $f(f^{-1}(x) = x)$*

Hand in your solutions until 18.12.2018 at 14:00 in post box 2183 in V3,
or via email to the tutor.

Tutor: Oliver Tautz tschmidt@techfak.uni-bielefeld.de