

Formal Logic — Last Exercise Sheet 12**Exercise 45: (Tautologies)**

Three out of the following five formulas are tautologies. Which one is, which one is not a tautology? For the tautologies: prove that they are tautologies (either using Theorem 4.2, or the tableau calculus). For the ones that are not tautologies give a structure $\mathcal{A} = (W, R, \alpha)$ and $s \in W$ such that $\mathcal{A}(H_i, s) = 0$.

- (a) $H_1 = \Box F \Rightarrow \Diamond F$
- (b) $H_2 = F \Rightarrow \Diamond F$
- (c) $H_3 = \Box F \Leftrightarrow \neg \Diamond \neg F$
- (d) $H_4 = (\Box F \wedge \Box(F \Rightarrow G)) \Rightarrow \Box G$
- (e) $H_5 = \neg(\Box(F \Rightarrow G) \wedge \Diamond F \wedge \Box \neg G)$

Exercise 46: (According to rank)

Determine the modal rank of the following formulas.

- (a) $\Diamond A \Rightarrow \Diamond \Box(\Diamond A \Rightarrow \Box B)$
- (b) $\neg \Diamond(A \vee \neg \Box \Diamond(A \vee \Box \neg B))$
- (c) $\Box \neg \Diamond(\neg \Diamond B \wedge \Box \Diamond(\Box \neg \Diamond B \vee \neg \Box A)) \wedge \Diamond \Diamond \Diamond A$

Exercise 47: (More tautologies)

Use the tableau calculus for the following tasks. Recall that F is a tautology if and only if $\neg F$ is unsatisfiable.

- (a) Show that $F \Rightarrow \Diamond F$ is satisfiable, but it is not a tautology.
- (b) Prove Rule 2 of Theorem 4.2 by showing that $\neg \Diamond F \Leftrightarrow \Box \neg F$ is a tautology.

Exercise 48: (Neverending story)

A computer program uses integer variables x, y, z . It is known that

F_1 : At each time holds $z \geq 0 \Rightarrow y < 0$.

F_2 : At each time holds $x \geq 0 \Rightarrow z \geq 0$.

F_3 : At each time holds $y \geq 0$.

The program will stop if at some time $x \geq 0$. Express F_1, F_2, F_3 and $G : x \geq 0$ as formulas in modal logic. Show that the program will never stop by applying the tableau calculus to $H = F_1 \wedge F_2 \wedge F_3 \wedge G$ in order to show that H is unsatisfiable.

Additional exercise: (for extra points.)

Prove Rule 5 of Theorem 4.2 again by using the tableau calculus to show that $\Diamond(F \Rightarrow G) \Leftrightarrow (\Box F \Rightarrow \Diamond G)$ is a tautology.