Dr. Dirk Frettlöh 2.11.2021

Formal Logic — Exercise Sheet 4

Exercise 13: (Variations of König's Lemma)
(a) Show that König's Lemma is wrong if we do not require that each vertex has finitely many neighbours, even if each level of the tree is not empty (I.e., for each $n \in \mathbb{N}$ there is a vertex in the tree with distance n from the root). Hence: construct a counterexample, that is, find a tree T that has infinitely many vertices, each level of T is not empty, and T does not contain any infinite path.

(b) Let T be a tree such that each vertex has only finitely many neighbours, and such that T does not contain any infinite path. Show that T has only finitely many vertices.

Exercise 14: (Grinding and primes)
(a) In the massive online role playing game HYDRA™ you have to fight monsters with a large number of heads. Whenever you kill a monster with $n \geq 2$ heads it spawns some finite number of monsters (possibly millions) with $n-1$ heads. A monster with only one head will just die on being killed. Assume that the game starts with one single monster with N heads, and your task is to kill this monster and all other monsters spawned in the process described above. Show that the task can be done with finitely many kills for any $N \in \mathbb{N}$.

(b) A number m is called a *prefix* of another number n if m consists of the first digits of n. (E.g., 13 is a prefix of 137, of 1378, and of 1 378 000.) Show that there is an infinite sequence $a_1, a_2, \ldots$ of prime numbers such that for each $i \in \mathbb{N}$ the number a_i is a prefix of a_{i+1}.
(A candidate for such a sequence might, or might not, be $31, 317, 3176269, \ldots$)

Exercise 15: (Modus tollens and resolution are consequences)
(a) Prove the modus tollens. That is, show that $\{F \Rightarrow G, \neg G\} \models \neg F$, if F and G are two formulas in propositional logic.

(b) Prove that $\{F \vee L, G \vee \neg L\} \models F \vee G$, if F and G are formulas and L is some literal. (This is Lemma 1.9)

Exercise 16: (Resolvents)
Construct the CNF of the following formula F and determine $Res^1(F)$ and $Res^2(F)$. Is $\square \in Res^3(F)$? Is F satisfiable?

$$F = (A \vee \neg(B \wedge \neg C)) \wedge (B \vee C) \wedge (A \Rightarrow C) \wedge (C \Rightarrow B) \wedge \neg C.$$

Send your solutions until Tue 9.11.2021 at 14:00 to the tutor who sent you the correction of your last solutions.

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