Formal Logic   —   Exercise Sheet 7

Exercise 25: (Models and non-models)
Which of the following structures are models for

$$F = \forall x\, \exists y\, P(x,y) \;\wedge\; \exists x\, \neg P(x,x),$$

which are not? Please justify your answers.

(a) $U_{\mathcal{A}} = \mathbb{N}$, $P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N},\; m \le n\}$

(b) $U_{\mathcal{A}} = \mathbb{N}$, $P^{\mathcal{A}} = \{(n, n+1) \mid n \in \mathbb{N}\}$

(c) $U_{\mathcal{A}} = \mathrm{Pot}(\mathbb{N})$, (that is the set of all subsets of $\mathbb{N}$), $P^{\mathcal{A}} = \{(A,B) \mid A, B \subseteq \mathbb{N},\; A \subseteq B\}$

(d) $U_{\mathcal{A}} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ differentiable.}\}$, $P^{\mathcal{A}} = \{(f,g) \mid f = g'\}$

Exercise 26: (Not a law)
Show that the two formulas in (a) (respectively, in (b)) are not equivalent to each other by
providing a (counter-)example for each.

(a) $(\forall x\, F) \vee (\forall x\, G) \quad \not\equiv \quad \forall x (F \vee G)$,

(b) $\exists x\, (F \wedge G) \quad \not\equiv \quad \exists x\, F \wedge G$, where x is free in G.

Exercise 27: (Structures and models II)
Consider the formulas

$$F = \forall x\, \forall y\, \forall z\, \Big(\neg P(x,x) \wedge \big(P(x,y) \Rightarrow \neg P(y,x) \big) \wedge \big(P(x,y) \wedge P(y,z) \Rightarrow P(x,z) \big) \Big).$$

Find two structures $\mathcal{A}_1, \mathcal{A}_1'$ for F that are not models for F, one with a finite universe $U_{\mathcal{A}_1}$,
one with an infinite universe $U_{\mathcal{A}_1'}$.

Moreover, find two models $\mathcal{A}_2, \mathcal{A}_2'$ for F, one with a finite universe $U_{\mathcal{A}_2}$, one with an infinite
universe $U_{\mathcal{A}_2'}$.

Please justify why $\mathcal{A}_1 \not\models F$, $\mathcal{A}_2 \not\models F$, $\mathcal{A}_1' \models F$, $\mathcal{A}_2' \models F$, respectively.

Exercise 28: (Prenexification)
Consider the formulas

$$F = \forall x\, P(x) \wedge \neg \forall y\, \exists z\, \big(P(y) \Rightarrow P(z) \big), \quad G = \exists x\, \forall y\, \forall z\, \big((P(x,y) \vee \neg P(y,z)) \big) \wedge \neg \forall x\, \exists y\, \big(P(x,y) \vee \forall z\, \neg P(x,z) \big).$$

Establish the Prenex normal form of F and G. Note that it might be necessary to rename
some of the variables first.