Formal Logic — Exercise Sheet 11

Exercise 41: (Football experts)
Formulate the following statements as formulas in modal logic. Moreover, show that the four formulas are not simultaneously satisfiable. That is, show that under each structure at least one of these statements is wrong.

(a) "Werder Bremen will win all remaining matches"

(b) "Borussia Dortmund will not win all remaining matches"

(c) "Whenever Bayern München will not win its match then Werder Bremen will not win its match the same weekend"

(d) "Whenever Bayern München wins its match then Borussia Dortmund will win its match the same weekend"

(We assume that all matches are played on weekends, and that for each team match number i takes place on weekend number i.)

Exercise 42: (Relations and directed graphs)
Visualise the following relations as directed graphs $G = (W, R)$. I.e., the nodes of G are the elements of W, the edges of G are the (ordered!) elements of R.

(a) $W = \{1,2,3,4,5\},\ R = \{(n,m) \mid n,m \in W,\ |n-m| \geq 2\}$

(b) $W = \{0,1,2,3\},\ R = \{(n,m) \mid n,m \in W,\ n+m \bmod 2 = 0\}$

(c) $W = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\},\ R = \{(n,m) \mid n,m \in W,\ n \subseteq m\}$

(d) $W = \{1,2,3,4,5,6,7,8\},\ R = \{(m,n) \mid \frac{n}{m} \in \mathbb{N}\}$

State for each of the four relations (W,R) whether they are reflexive, and/or symmetric, and/or transitive.

Exercise 43: (Frames)
Given the frame (W,R) with $W = \{0,1,2,3,4,5\}$ and $R = \{(m,n) \mid m < n\}$ a relation on W. Let the valuation $\alpha : \{A,B\} \times W \to \{0,1\}$ be given by

$$\alpha(A,n) = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{else} \end{cases} \qquad \alpha(B,n) = \begin{cases} 1 & \text{if } n < 4 \\ 0 & \text{else} \end{cases}$$

Determine the truth values $\mathcal{A}(F_1,0), \mathcal{A}(F_1,1), \mathcal{A}(F_2,1), \mathcal{A}(F_3,1), \mathcal{A}(F_3,3)$ of the following formulas.

(a) $F_1 = \diamond\Box A$

(b) $F_2 = \Box \diamond A$

(c) $F_3 = \Box(B \Rightarrow A)$

Now do the same for the same valuation α on the frame (W, R') with $W = \{0,1,2,3,4,5\}$ and the relation $R' = \{(m,n) \mid n = m+1 \bmod 6\}$ on W.

(More next page.)

Exercise 44: (More rules)
(a) Prove Rule 5 of Theorem 4.1.
(You may use rules 1,2,6 and 7 in order to transform one of the formulas into the other.)

(b) Prove Rule 4 of Theorem 4.1 by showing that $\Box(F \Rightarrow G) \Rightarrow (\diamond F \Rightarrow \diamond G)$ is a tautology.
(You may use rules 1,2,6 and 7 in order to tranform this expression into one like $F \vee \neg F \vee \ldots$*; this is obviously a tautology.)*

(c) Show that the following variant of Rule 6 in Theorem 4.1:

$$(\diamond F \wedge \diamond G) \equiv \diamond(F \wedge G)$$

does not hold; for instance, by providing a counterexample.

Send your solutions until Tuesday 17.1.2023 at 14:00 to your respective tutor.

Please indicate the name of the tutor on your solution sheet.
Your solutions have to be in a single file (pdf or similar). Multiple jpeg files (photos) do not count.

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