Formal Logic — Blatt 13

Exercise 49: (Undecidable problem I: PCP)
For each of the following instances of the Post correspondence problem (PCP), find a solution for it or show that it has no solution.

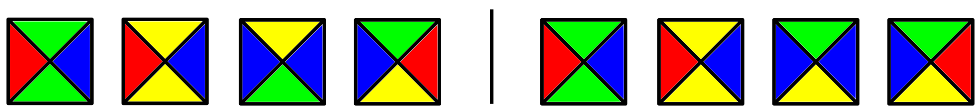
(a) $a_1 = 010, a_2 = 0, a_3 = 11$ and $b_1 = 0, b_2 = 10, b_3 = 01$.

(b) $a_1 = 101, a_2 = 0, a_3 = 01$ and $b_1 = 10, b_2 = 01, b_3 = 010$.

(c) $a_1 = 1, a_2 = 0, a_3 = 010, a_4 = 11$ and $b_1 = 10, b_2 = 10, b_3 = 01, b_4 = 1$.

(d) $a_1 = 10, a_2 = 011, a_3 = 101$ and $b_1 = 101, b_2 = 11, b_3 = 011$.

Exercise 50: (Undecidable problem II: Wang tiles)
For each of the two collections of four Wang tiles below, prove that they can tile the plane (according to the rules: squares are placed vertex-to-vertex, adjacent edges carry the same colour, tiles are not rotated or reflected), or show that there is no such tiling.



Exercise 51: (Undecidable problem III: mortal matrices)
The mortal matrix problem asks for a given set of matrices $A_1, \ldots, A_m \in \mathbb{R}^{n \times n}$ whether there is a way to multiply the matrices such that the result is the matrix containing zeros only. It is allowed to use the same matrix more than once. That is, we are looking for a sequence $i_1, i_2, \ldots, i_k$ with $1 \leqslant i_j \leqslant m$ such that

$$A_{i_1} \cdot A_{i_2} \cdots A_{i_k} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

If this is possible the set $\{A_1, \ldots, A_m\}$ is called *set of mortal matrices*. Decide which of the following sets of matrices are sets of mortal matrices. Give either an example for a product yielding the zero matrix, or give a convincing reason why this is not possible.

(a) $A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(b) $B_1 = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B_3 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) $C_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $C_3 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Aufgabe 52: (Computable First Order Logic)
Monadic first order logic is first order logic without function symbols, and such that each predicate has one input only (that is, only $P(x)$ and so on, not $P(x, y)$ of $P(x, y, z)$ and so on). Let F be a formula in first order logic with the n predicates $P_1, P_2 \ldots, P_n$.

(a) Show: if F is satisfiable, then there is a model $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ for F, such that $U_\mathcal{A}$ has exactly 2^n elements. *(Hint: Merge the elements in $U_\mathcal{A}$ which have the same values $(P_1^\mathcal{A}(x), \ldots, P_n^\mathcal{A}(x))$ into one single element.)*

(b) Justify that the problem "is the formula F in monadic first order logic satisfiable?" is computable.

Send your solutions until Tuesday 27.1.2026 at 14:00 to your tutor.

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