

Formal Logic — Exercise Sheet 1**Exercise 1: (Satisfiable, tautology, equivalent)**

Consider the formulas

$$F_1 = (A \wedge B) \vee \neg A, \quad F_2 = A \Rightarrow (A \Rightarrow B), \quad F_3 = ((A \Rightarrow B) \wedge A) \Rightarrow B.$$

List all partial formulas of F_1 . Then decide for each of the three formulas F_1, F_2, F_3 : is the formula a tautology? Is the formula satisfiable? If it is, give a satisfying valuation $\mathcal{A}$ for the respective formula. Are F_1 and F_2 equivalent? Are F_1 and F_3 equivalent? Are F_2 and F_3 equivalent?

Exercise 2: (Elves and Vampires)

In a distant world there are only two types of inhabitants: elves, who always tell the truth, and vampires, who always lie. All problems (a)-(e) of this exercise take place in this world. You, as a visitor in this distant world, cannot tell elves and vampires apart by their looks, but the elves and vampires know exactly who is what.

(a) In this problem there are two people, A and B, each of whom is either an elf or a vampire. A makes the statement: “At least one of us is a vampire.” What are A and B?

(b) Suppose A says, “I am a vampire, and B isn’t.” What are A and B?

(c) Now we have three people, A, B, C, each of whom is either an elf or a vampire. A and B make the following statements:

A: All of us are vampires.

B: Exactly one of us is an elf.

What are A, B, C?

(d) Again we have three people A, B and C, each of whom is an elf or a vampire. Two people are said to be of the same type if they are both elves or both vampires. A and B make the following statements:

A: B is a vampire.

B: A and C are of the same type. What is C?

(e) Again three people A, B, and C. A says “B and C are of the same type.” Someone then asks C, “Are A and B of the same type?” What does C answer?

Exercise 3: (Playing with formulas)

(a) The “exclusive or” $A \oplus B$ is defined via $\mathcal{A}(A \oplus B) = 1$ if and only if $\mathcal{A}(A) \neq \mathcal{A}(B)$. Find a formula including $\oplus$ that is a tautology; and find a formula including $\oplus$ that is unsatisfiable.

(b) Let F be a formula F where no atomic formula occurs twice in F . Prove that, if F has exactly n symbols (not counting parentheses: $(,)$), then F has exactly n partial formulas.

Exercise 4: (Sufficient sets of operators)

(a) Prove that for each formula F there is an equivalent formula G using only $\neg$ and $\wedge$.

(b) Prove that there is some formula F such that there is no formula G equivalent to F , where G uses only $\wedge$ and $\vee$.

(c) NOR, aka “not or”, aka $\downarrow$, is defined by $A \downarrow B = \neg(A \vee B)$. Prove that for each formula F there is an equivalent formula G using only NOR.

(Hint to (a) and (c): if you can express $\neg$ and $\wedge$ and $\vee$ by combinations of the respective operator(s) then you are done.)