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ALGEBRAIC K-THEORY OF SPACES,  
CONCORDANCE, AND STABLE HOMOTOPY THEORY

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It is known [7] that there is a splitting, up to homotopy,

$$A(X) \simeq A^S(X) \times \text{Wh}^{\text{DIFF}}(X)$$

as well as another

$$A^S(X) \simeq \Omega^\infty S^\infty(X_+) \times \mu(X) .$$

It will be shown here that the factor  $\mu(X)$  is trivial.

Hence we have

THEOREM.  $A(X) \simeq \Omega^\infty S^\infty(X_+) \times \text{Wh}^{\text{DIFF}}(X) .$

The method of proof is to establish a version of the Kahn-Priddy theorem for  $\mu(X)$ . As  $\mu(X)$  is a homology theory there results a kind of growth condition for the homotopy groups. But  $\mu(X)$  is connected, so the growth condition boils down to zero growth and thus we can conclude that  $\mu(X)$  is trivial.

To explain what is meant by a Kahn-Priddy theorem we have to know about transfer maps. First, there is a

transfer for the algebraic K-theory of spaces:  $A(X)$  is made from spaces over  $X$ , so if  $\tilde{X} \rightarrow X$  is a fibration with fibre of finite type, pullback induces a map  $A(X) \rightarrow A(\tilde{X})$ , cf. [8]. Next, if the fibration is a finite covering projection then the transfer can be considered in the framework of the 'manifold approach' of [7], in particular everything in theorem 1 of that paper is compatible with the transfer. It follows that  $\Omega^\infty S^\infty(X_+)$  and  $A^S(X)$  have transfers for finite covering projections which are compatible to the transfer on  $A(X)$ , and compatible to each other, in the sense that it is possible to fill in the broken arrows so that the following diagram commutes, up to homotopy,

$$\begin{array}{ccccc} \Omega^\infty S^\infty(X_+) & \longrightarrow & A^S(X) & \longrightarrow & A(X) \\ \vdots \downarrow & & \vdots \downarrow & & \downarrow \\ \Omega^\infty S^\infty(\tilde{X}_+) & \longrightarrow & A^S(\tilde{X}) & \longrightarrow & A(\tilde{X}) . \end{array}$$

It could be checked directly that these transfers agree with the usual ones (which are defined for all homology theories) but we will not need this fact.

Let  $\Sigma_n$  denote the symmetric group,  $B\Sigma_n$  its classifying space, and  $E\Sigma_n$  the universal bundle. Let  $\tilde{A}(X)$  denote the reduced part, the factor in the splitting  $A(X) \simeq A(*) \times \tilde{A}(X)$ , i.e.,  $\tilde{A}(X) = \text{fibre}(A(X) \rightarrow A(*))$ .

The transfer gives a map

$$\tilde{A}(B\Sigma_n) \longrightarrow A(B\Sigma_n) \longrightarrow A(E\Sigma_n) \simeq A(*) .$$

Let  $p$  be a prime and let the subscript  $(p)$  denote the localization at  $p$ . Following the method of Segal [2] it was shown in [6] that the Kahn-Priddy theorem is valid for the algebraic K-theory of spaces in the sense that for every  $p$  the map

$$\pi_j \tilde{A}(B\Sigma_p)_{(p)} \longrightarrow \pi_j A(*)_{(p)}$$

is surjective for every  $j > 0$ .

Our main task here will be to show that the analogous map

$$\pi_j \tilde{A}^S(B\Sigma_p)_{(p)} \longrightarrow \pi_j A^S(*)_{(p)}$$

is also surjective. This would follow at once if we knew that the map  $A(X) \rightarrow A^S(X)$  were transfer commuting.

However we do not know this, so we must proceed differently.

In [6] there were constructed maps ('operations')

$$\theta^n : A(*) \longrightarrow A(B\Sigma_n) .$$

They have the property, among others, that the composite of  $\theta^n$  with the transfer map

$$A(B\Sigma_n) \longrightarrow A(E\Sigma_n) \simeq A(*)$$

is homotopic to the polynomial map of  $A(*)$  to itself associated with the polynomial  $x(x-1)\dots(x-n+1)$ .

LEMMA. The map  $A^S(X) \rightarrow A(X)$  is compatible with the operation  $\theta^n$  in the sense that it is possible to fill in the broken arrow so that the diagram

$$\begin{array}{ccc} A^S(*) & \longrightarrow & A(*) \\ \vdots & & \downarrow \\ A^S(B\Sigma_n) & \longrightarrow & A(B\Sigma_n) \end{array}$$

commutes up to (weak) homotopy. Also,  $A^S(*) \rightarrow A(*)$  is a map of ring spaces.

Since  $A^S(*) \rightarrow A(*)$  is a coretraction, up to homotopy, we obtain from the lemma

COROLLARY 1. There is a map  $A^S(*) \rightarrow A^S(B\Sigma_n)$  whose composite with the transfer  $A^S(B\Sigma_n) \rightarrow A^S(*)$  is the polynomial map on  $A^S(*)$  associated with the polynomial  $x(x-1)\dots(x-n+1)$ .

The desired Kahn-Priddy theorem for  $A^S(X)$  now follows from corollary 1 by a formal argument. The argument may be found in the introduction to [6]. (The argument involves an application of Nakayama's lemma, so one has to know the homotopy groups of  $A^S(*)$  are finitely generated. As  $A^S(*)$  is a factor of  $A(*)$  this follows from Dwyer's theorem that the homotopy groups of  $A(*)$  are finitely generated [1].) Thus,

COROLLARY 2. For every prime  $p$ , the (transfer) map

$$\pi_j \tilde{A}^S(B\Sigma_p)_{(p)} \longrightarrow \pi_j A^S(*)_{(p)}$$

is surjective for every  $j > 0$ .

It was explained earlier that the map  $\Omega^\infty S^\infty(X_+) \rightarrow A^S(X)$  is transfer commuting. Hence we know that in the partially defined map of short exact sequences

$$\begin{array}{ccccc} \pi_j \tilde{\Omega}^\infty S^\infty(B\Sigma_{p^+})(p) & \longrightarrow & \pi_j \tilde{A}^S(B\Sigma_p)(p) & \longrightarrow & \pi_j \tilde{\mu}^{(B\Sigma_p)}(p) \\ & & \downarrow & & \\ \pi_j \tilde{\Omega}^\infty S^\infty(p) & \longrightarrow & \pi_j A^S(*) (p) & \longrightarrow & \pi_j \mu^{(*)}(p) \end{array}$$

the left arrow can be filled in. It follows that the right arrow can also be filled in. From corollary 2 we therefore conclude

COROLLARY 3. For every prime  $p$  and for every  $j > 0$  there is a surjective map

$$\pi_j \tilde{\mu}^{(B\Sigma_p)}(p) \longrightarrow \pi_j \mu^{(*)}(p) .$$

*Proof of theorem.*  $\mu(X)$  is a homology theory, so it suffices to show that  $\mu^{(*)}$  is contractible; or that for every prime  $p$  the localization  $\mu^{(*)}_{(p)}$  is. We show by induction on  $j$  that the homotopy groups  $\pi_j \mu^{(*)}_{(p)}$  are trivial.

The induction beginning is provided by the fact that  $\mu^{(*)}$  is connected. In fact,  $\mu^{(*)}$  is known to be 2-connected: this follows from the double splitting theorem together with the fact [5] that the map  $\pi_j \tilde{\Omega}^\infty S^\infty \rightarrow \pi_j A^S(*)$  is an isomorphism for  $j \leq 2$ .

Suppose now that  $j > 0$  and that  $\pi_i \mu^{(*)}_{(p)} = 0$  if  $i \leq j-1$ . By the spectral sequence of a generalized

homology theory we obtain that the reduced group  $\pi_j \tilde{\mu}^{(X)}(p)$  is trivial for every  $X$ . Taking  $X = B\Sigma_p$  we therefore conclude from corollary 3 that there is a surjective map

$$0 = \pi_j \tilde{\mu}^{(B\Sigma_p)}(p) \longrightarrow \pi_j \mu^{(*)}(p) .$$

Hence  $\pi_j \mu^{(*)}_{(p)} = 0$ . This completes the inductive step and hence the proof.

It remains to prove the lemma.

To prove the lemma we need a framework where an explicit description of  $A^S(X)$  and of the map  $A^S(X) \rightarrow A(X)$  are available. The 'manifold approach' of [7] provides such a description in terms of smooth manifolds. Namely, supposing that  $X$  is a manifold, one considers partitions of  $X \times [0,1]$ , that is, triples  $(M,F,N)$  where  $X \times 0 \subset M$ ,  $X \times 1 \subset N$ , and where  $F$  is the common frontier of  $M$  and  $N$ . These form a simplicial category  $h^{\mathcal{P}}(X)$  as described [loc.cit.]. There is a simplicial subcategory  $h^{\mathcal{P}}_k^m(X)$ ; briefly, those partitions where  $M$  is obtained from  $X \times [0,\epsilon]$  by attaching of  $k$   $m$ -handles. It is shown in [7] that  $A(X)$ , or rather a connected component of it, is obtained by the Quillen + construction from the (homotopy) direct limit, with respect to  $n, m, k$ , of the  $h^{\mathcal{P}}_k^m(X \times J^n)$  where  $J$  denotes an interval. It is also shown that  $A^S(X)$  is similarly obtained from the  $\mathcal{P}_k^m(X \times J^n)$ ,

where  $\mathcal{P}_k^m(X)$  denotes the simplicial set of objects of the simplicial category  $h\mathcal{P}_k^m(X)$ . The map  $A^S(X) \rightarrow A(X)$  is thus represented by the inclusion map  $\mathcal{P}_k^m(X \times J^n) \rightarrow h\mathcal{P}_k^m(X \times J^n)$ .

It has been discussed in [7] that  $h\mathcal{P}_k^m(X \times J^n)$ , the union of the  $h\mathcal{P}_k^m(X \times J^n)$ , has a composition law given by *gluing* (at least in the limit with respect to  $n$ ). The composition law restricts to one on  $\mathcal{P}_k^m(X \times J^n)$  (in the limit again). It results that both  $A^S(X)$  and  $A(X)$  are H-spaces (infinite loop spaces, in fact) and that the map  $A^S(X) \rightarrow A(X)$  is a map of H-spaces. This takes care of the *addition*.

We next come to the *multiplication* or, what is the appropriate general notion, the *exterior pairing*  $A(X) \wedge A(X') \rightarrow A(X \times X')$ . We claim that it restricts to a pairing  $A^S(X) \wedge A^S(X') \rightarrow A^S(X \times X')$ . This is seen by the same argument as before. Namely we check that the pairing is definable in terms of an explicit construction on the simplicial category of partitions. It will therefore restrict to the corresponding construction on the subspace given by the simplicial set of partitions.

The exterior pairing is induced by the fibrewise smash product which to a pair of spaces, over  $X$  and  $X'$ , respectively, associates a space over  $X \times X'$ . We want to represent that, up to homotopy, by a construction with

manifolds. Let  $(M, \dots)$  and  $(M', \dots)$  be partitions in  $\mathcal{P}(X)$  and  $\mathcal{P}(X')$ , respectively. We form the space (a subspace of  $X \times [0, 1] \times X' \times [0, 1]$ )

$$M \times M' \cup X \times [0, \epsilon] \times X' \times [0, 1] \cup X \times [0, 1] \times X' \times [0, \epsilon'] .$$

Then for sufficiently small  $\epsilon$  and  $\epsilon'$  this space has the homotopy type of the fibrewise smash product

$$M \wedge_X \wedge_{X'} M' ,$$

it is a manifold (with corners), and, up to some bending of corners, it defines a partition in  $\mathcal{P}(X \times X' \times [0, 1])$ . We have thus obtained a map, well defined up to some choices ('contractible choices')

$$h\mathcal{P}_k^m(X) \times h\mathcal{P}_k^{m'}(X') \longrightarrow h\mathcal{P}_{k \cdot k}^{m+m'}(X \times X' \times [0, 1])$$

and restricting in the desired way. This completes the account of the multiplication.

The case of the *operations* is a little more delicate, and the verification takes much longer. We need a modification of the 'manifold approach' where the simplicial category of the partitions is replaced by another simplicial category. The modified construction is needed only in the case  $X = *$  which is somewhat easier than the general case. We restrict to that case.

We consider compact smooth submanifolds  $M$  of codimension 0 in euclidean space  $R^d$  containing a neighborhood of the origin. We manufacture a simplicial category from such manifolds. First, we define a

simplicial set  $Q(d)$  where a  $k$ -simplex is a smooth family, parametrized by the simplex  $\Delta^k$ , of manifolds of the type considered. Next we regard  $Q(d)$  as the simplicial set of objects in a simplicial category  $hQ(d)$  (in fact, a simplicial partially ordered set): there is a morphism from  $M$  to  $M'$  if and only if  $M \subset M'$  and if furthermore the two inclusion maps, of boundaries,

$$\partial M \longrightarrow Cl(M'-M) \longleftarrow \partial M'$$

are homotopy equivalences.

We define  $hQ_k^m(d)$  to be the connected component of  $hQ(d)$  which contains the particular  $M$ ,

$M =$  unit disk, with  $k$  unknotted  $m$ -handles attached, and we let  $hQ^m(d)$  denote the union of the  $hQ_k^m(d)$ . We define  $Q_k^m(d)$  (resp.  $Q^m(d)$ ) to be the simplicial set of objects of  $hQ_k^m(d)$  (resp.  $hQ^m(d)$ ).

As in [7] we have to discuss *stabilization* with respect to  $d$ ,  $m$ , and perhaps  $k$ . This is a little technical.

Part of the technicalities is that we should admit now *smooth manifolds with general corners* as described in the appendix to [7]; that is, topological submanifolds of  $R^d$  of codimension 0 which are equipped with suitable extra structure to specify potential smoothings. As explained [loc.cit.] the modification does not alter the homotopy types of the simplicial sets, resp. simplicial categories, which we here consider.

After the modification (which, by abuse, we suppress from the notation) we have a map

$$\begin{aligned} hQ(d) &\longrightarrow hQ(d+1) \\ M &\longmapsto M \times [-1, +1] . \end{aligned}$$

So, by using that map, we can form the stabilization with respect to dimension,

$$\varinjlim_d hQ(d) .$$

We can obtain a homotopy equivalent simplicial category  $hQ'(d)$  by restricting to those submanifolds  $M$  of  $R^d$  which satisfy  $D^d(1) \subset \text{Int}(M)$ ,  $M \subset \text{Int}(D^d(2))$  where  $D^d(r)$  denotes the disk of radius  $r$ . In view of the isomorphism of  $Cl(D^d(2)-D^d(1))$  with  $S^{d-1} \times [0,1]$  we obtain an isomorphism of  $hQ'(d)$  with one of the simplicial categories of [7],

$$\begin{aligned} hQ'(d) &\longrightarrow h\mathcal{P}(S^{d-1}) \\ M &\longmapsto Cl(M - D^d(1)) . \end{aligned}$$

and hence a homotopy equivalence

$$h\mathcal{P}(S^{d-1}) \longrightarrow hQ(d) .$$

It restricts to other homotopy equivalences  $\mathcal{P}(S^{d-1}) \rightarrow Q(d)$ ,  $h\mathcal{P}^m(S^{d-1}) \rightarrow hQ^m(d)$ , and so on.

The stabilization map  $hQ(d) \rightarrow hQ(d+1)$  corresponds, under the homotopy equivalence, to a map

$$h\mathcal{P}(S^{d-1}) \longrightarrow h\mathcal{P}(S^d) .$$

Up to homotopy, that map factors through  $h\mathcal{P}(D^d)$ , so we have a homotopy equivalence

$$\varinjlim_d hQ(d) \simeq \varinjlim_d h\mathcal{P}(D^d) .$$

Similarly we have homotopy equivalences

$$\lim Q(d) \simeq \lim \mathcal{P}(D^d) , \text{ and so on.}$$

As a result, therefore, theorem 1 of [7] may be restated to say, among other things, that the inclusion map

$$\varinjlim_d Q_k^m(d) \longrightarrow \varinjlim_d hQ_k^m(d)$$

is an approximation to the map  $A^S(\ast) \rightarrow A(\ast)$  .

As regards the limits with respect to  $m$  and  $k$  , there is the happy technical point that the details don't really matter. The reason is that, as we already know, the map  $A^S(X) \rightarrow A(X)$  is a coretraction, up to homotopy; this will allow us to restrict the necessary checking, below, to a checking on representatives only. All we need to know about those limits, therefore, is that they exist in some weak sense; say, as homotopy direct limits with respect to stabilization maps which exist only after geometric realization and are well defined up to (weak) homotopy, and compatible to each other. Thus we may simply take the stabilization maps of [7] and transport them to the present situation by means of the homotopy equivalences above.

The simplicial set  $Q^m(d)$  (resp. the simplicial category  $hQ^m(d)$  ) has an additional structure, namely it is a *partial monoid* in the sense of [3] with respect to gluing. (The monoid is only partial because the result of

the gluing should be a manifold again, and should be of the correct type.) As a result,

$$\varinjlim_d Q^m(d) \text{ (resp. } \varinjlim_d hQ^m(d) \text{ )}$$

is the underlying space of a  $\Gamma$ -space in the sense of [4].

Let  $B_\Gamma(\lim(h)Q^m(d))$  denote the realization of that  $\Gamma$ -space. Then the loop space  $\Omega_{B_\Gamma}(\lim(h)Q^m(d))$  serves as a 'group completion' for the H-space  $\lim(h)Q^m(d)$  , and there is a map

$$\lim(h)Q^m(d) \longrightarrow \Omega_{B_\Gamma}(\lim(h)Q^m(d))$$

which, up to (weak) homotopy, is universal for H-maps of  $\lim(h)Q^m(d)$  into group-like H-spaces [4]. The map

$$\text{holim}_m \Omega_{B_\Gamma}(\varinjlim_d Q^m(d)) \longrightarrow \text{holim}_m \Omega_{B_\Gamma}(\varinjlim_d hQ^m(d))$$

may be identified, by [7] and the homotopy equivalences  $\lim Q(d) \simeq \lim \mathcal{P}(D^d)$  , etc., above, to the map  $A^S(\ast) \rightarrow A(\ast)$  .

We claim now that to show the existence of the broken arrow in the diagram

$$\begin{array}{ccc} A^S(\ast) & \longrightarrow & A(\ast) \\ \vdots & & \downarrow \\ A^S(B\Sigma_n) & \longrightarrow & A(B\Sigma_n) \end{array}$$

it will suffice to show that the arrow exists if the source

$$A^S(\ast) \simeq \text{holim}_m \Omega_{B_\Gamma}(\varinjlim_d Q^m(d))$$

is replaced by just

$$Q^m(d) .$$

This is seen by the following series of reductions. First, suppose the broken arrow can always be found if restricted to

$$\Omega_{B\Gamma}(\varinjlim_d Q^m(d)) .$$

Then, for varying  $m$ , these arrows are automatically compatible to each other, up to homotopy: this follows from the fact that  $A^S(B\Sigma_n) \rightarrow A(B\Sigma_n)$  is a coretraction, up to homotopy. As a result, the arrows can therefore be assembled to a map of the homotopy direct limit, and the resulting diagram is weakly homotopy commutative (the two composite arrows are homotopic when restricted to any compactum).

For the next reduction we have to invoke the universal property of 'group completion', we must therefore keep track of H-space structures. The maps  $\theta^n$  can all be assembled into a single map

$$\theta : A(*) \longrightarrow \prod_n A(B\Sigma_n)$$

which is an H-map with respect to the additive structure on  $A(*)$  and a suitable H-space structure on the product [6].

That H-space structure is manufactured from the exterior pairings

$$A(B\Sigma_p) \wedge A(B\Sigma_q) \longrightarrow A(B\Sigma_p \times B\Sigma_q) \longrightarrow A(B\Sigma_{p+q})$$

together with the additive structure. Now both of these exist, compatibly, on  $A^S$  (the account above gives this

only in the compact case, but the general case follows by an exhaustion argument from this). Hence we have a map of H-spaces

$$\prod_n A^S(B\Sigma_n) \longrightarrow \prod_n A(B\Sigma_n) ,$$

and both of these are in fact group-like H-spaces by an easy formal argument [6]. It results that in the diagram

$$\begin{array}{ccc} \Omega_{B\Gamma}(\varinjlim_d Q^m(d)) & \longrightarrow & A(*) \\ \downarrow & & \downarrow \\ \prod_n A^S(B\Sigma_n) & \longrightarrow & \prod_n A(B\Sigma_n) \end{array}$$

all the solid arrows are H-maps. Now suppose that the broken arrow can be filled in if restricted to  $\varinjlim_d Q^m(d)$ . Using the fact that the bottom arrow is a coretraction, up to homotopy, we obtain that the filled-in arrow is necessarily an H-map. Hence, by the universal property for H-maps into group-like H-spaces, we conclude that the arrow extends to  $\Omega_{B\Gamma}(\varinjlim_d Q^m(d))$ .

Finally, suppose that in the diagram

$$\begin{array}{ccc} \varinjlim_d Q^m(d) & \longrightarrow & A(*) \\ \downarrow & & \downarrow \\ A^S(B\Sigma_n) & \longrightarrow & A(B\Sigma_n) \end{array}$$

the broken arrow can always be found if restricted to  $Q^m(d)$ . By the coretraction argument again, the arrows are then compatible and assemble to a map of the homotopy direct limit with respect to  $d$ , which is homotopy equivalent to the actual direct limit.

It remains to see that the desired factorization exists if restricted to  $\mathcal{Q}^m(d)$ .

Recall from [6] that the map  $\theta^n: A(*) \rightarrow A(B\Sigma_n)$  is defined in terms of the functor from pointed spaces to free pointed  $\Sigma_n$ -spaces,

$$Y \longmapsto \theta^n(Y) = Y^n / (\text{coordinate axes} \cup \text{fat diagonal}) .$$

In detail,  $\theta^n(Y) = Y^n / \xi^n(Y)$  where  $\xi^n(Y)$  is the subspace of the tuples  $(y_1, \dots, y_n)$  having the property that at least one of the  $y_i$  is equal to the basepoint or that at least two of the  $y_i$  are equal to each other.

In [6] the construction has been done simplicially. We want the topological version here. There are routine ways to pass back and forth between the simplicial and the topological contexts [8]. But there is a little extra technical point. Namely if  $Y$  is allowed to be a topological space of the pointed homotopy type of a CW complex, there is, unfortunately, no reason to suppose that the diagonal map  $Y \rightarrow Y^2$  is a cofibration. So the above definition of  $\theta^n(Y)$  would not give the correct homotopy type. On the other hand, for the purposes of [6] one is interested in  $\theta^n$  only up to homotopy (up to weak homotopy equivalence of functors, to be precise). There is therefore a variety of ways to correct the defect. For example one can combine  $\theta^n$  with some correction functor, such as the geometric realization of the singular complex.

Another technical point is the remark that it is not really necessary to work with pointed spaces throughout. Specifically, we want the following modification here. Fix a (weakly) contractible  $\Sigma_n$ -space  $W$ . Consider the category of the  $\Sigma_n$ -spaces having the  $\Sigma_n$ -homotopy type, in the strong sense, of a finite free  $\Sigma_n$ -CW complex relative to  $W$ . Then there are functors between the pointed situation and the  $W$ -situation given by product with  $W$  and by quotienting out  $W$ , respectively. These functors induce homotopy equivalences of the respective subcategories of weak homotopy equivalences.

As a result we may modify  $\theta^n$  by allowing the basepoint of  $\theta^n(Y)$  to be blown up into some contractible  $\Sigma_n$ -space  $W$ .

Specifically, therefore, we have the following representative, up to homotopy, of  $\theta^n$  on the simplicial category  $h\mathcal{Q}^m(d)$ . Let  $W$  be defined as the subspace of  $(\mathbb{R}^d)^n$  given by the union of the coordinate axes and of the fat diagonal; that is,  $W = \xi^n(\mathbb{R}^d)$  in the notation used before. Then for  $M$  in  $h\mathcal{Q}^m(d)$  the map is given by

$$M \longmapsto M^n \cup \xi^n(\mathbb{R}^d) .$$

We want to modify the construction a little further so that the result can be a manifold (with corners). Let  $N_\epsilon(\xi^n(\mathbb{R}^d))$  denote the  $\epsilon$ -neighborhood of  $\xi^n(\mathbb{R}^d)$ . Suppose, for the moment, we know that  $M^n$  is in general



position with respect to  $\xi^n(\mathbb{R}^d)$ . Under this assumption we obtain that, for sufficiently small  $\epsilon$ ,

$$M^n \cup N_\epsilon(\xi^n(\mathbb{R}^d))$$

is indeed a manifold, and is essentially independent of  $\epsilon$ .

By modifying the construction some more, we can re-interpret its result as a partition in the sense of [7] (cf. above). Namely let  $\delta < \epsilon$  and suppose that  $\tau$  is sufficiently large. Then

$$(M^n \cup N_\epsilon(\xi^n(\mathbb{R}^d))) - \text{Int}(N_\delta(\xi^n(\mathbb{R}^d)))$$

defines a  $\Sigma_n$ -equivariant partition in

$$N_\tau(\xi^n(\mathbb{R}^d)) - \text{Int}(N_\delta(\xi^n(\mathbb{R}^d))) \approx \partial N_\delta(\xi^n(\mathbb{R}^d)) \times [0,1]$$

and hence a partition in  $\mathcal{P}(C)$  where  $C$  is the orbit space

$$C = \partial N_\delta(\xi^n(\mathbb{R}^d)) / \Sigma_n.$$

Thus, if the assumption of general position could be generally justified, we would have obtained a factorization

$$\begin{array}{ccccc} Q^m(d) & \longrightarrow & hQ^m(d) & \longrightarrow & A(*) \\ \vdots & & \vdots & & \uparrow \theta^n \\ \mathcal{P}(C) & \longrightarrow & h\mathcal{P}(C) & & \\ \vdots & & \vdots & & \\ A^S(C) & \longrightarrow & A(C) & & \\ \swarrow & & \searrow & & \\ A^S(B\Sigma_n) & \longrightarrow & A(B\Sigma_n) & & \end{array}$$

where we use the sublemma below to provide the broken arrows in the middle; i.e., to show that  $\mathcal{P}(C)$  and  $h\mathcal{P}(C)$

(rather than just  $\mathcal{P}^m(C)$  and  $h\mathcal{P}^m(C)$ ) relate naturally to  $A^S(C)$  and  $A(C)$ , respectively. To complete the proof of the lemma it thus remains to establish that sublemma and to justify the appeal to general position, above.

As to the latter, there is certainly no problem as far as the part of  $\xi^n(\mathbb{R}^d)$  coming from the coordinate axes is concerned: we just restrict  $(h)Q^m(d)$  to the homotopy equivalent simplicial subset (resp. -category) of the  $M$  containing the disk  $D^d(1)$  in its interior (and being contained in the interior of  $D^d(2)$ ; the latter has the effect of ensuring that one and the same  $\tau$ , above, will do).

Concerning the remaining part of  $\xi^n(\mathbb{R}^d)$ , the fat diagonal, there is a potential problem only at such points  $(y_1, \dots, y_n) \in M^n$  where one or more of the  $y_i$  are boundary points of  $M$ .

We will take for granted that in fact there is no such problem at all in the following special case: the case where near all the  $y_i$  concerned, the boundary is actually flat (i.e., there is a neighborhood of  $y_i$  in  $\mathbb{R}^d$  inside of which  $M$  looks like euclidean half-space, up to a rigid motion).

More precisely, what we take for granted in this case is the following: that near such a point,  $M^n \cup N_\epsilon(\xi^n(\mathbb{R}^d))$  is a manifold (with corners) and essentially independent of

$\epsilon$  (i.e., varies with  $\epsilon$  in a locally trivial way); the  $\epsilon$  here is allowed to be a sufficiently small constant  $> 0$  or, more generally, a function which is  $C^1$ -close to such a constant.

Our theme will now be that in the general case there is no problem either, and that we can convince ourselves of this by means of suitably chosen isomorphisms to compare with the special case.

To this end we note that we can restrict  $(h)Q^m(d)$  to the homotopy equivalent simplicial subset (resp. -category) of the manifolds which are actually smooth rather than smooth with general corners [7] as so far. Thus, given  $M$ , and given any  $n$ -tuple  $(y_1, \dots, y_n) \in M^n$  we can find a 'trivializing' diffeomorphism of  $R^d$  whose effect is to make the boundary  $\partial M$  flat near the points  $y_1, \dots, y_n$ . We can, and will, assume here that in first approximation the diffeomorphism is the identity at each of the points  $y_1, \dots, y_n$ . But this implies that the induced diffeomorphism of  $(R^d)^n$  is  $C^1$ -close to the identity map near the point  $(y_1, \dots, y_n)$ . Thus we obtain, locally, the desired comparison.

To draw the desired conclusion globally, we must impose a condition of uniformity on the construction. For example it would suffice to know that the trivializing diffeomorphisms could be found out to a certain distance.

uniformly, and  $C^1$ -close to the identity, again uniformly. But this is no problem. For it suffices to treat a finite number of (parametrized families of)  $M$ 's at a time. This is enough to give the factorization on one finite piece at a time (i.e., finite simplicial subcategory in the case of  $hQ^m(d)$ , resp. finite simplicial subset in the case of  $Q^m(d)$ ) and is therefore enough to give, eventually, a factorization up to weak homotopy.

We are left to show now

SUBLEMMA. *If  $X$  is a manifold then*

- (1) *there is a natural map  $h\mathcal{P}(X) \rightarrow A(X)$  extending, up to the sign  $(-1)^m$ , the map  $hQ^m(X) \rightarrow A(X)$ ;*
- (2) *there is a factorization*

$$\begin{array}{ccc} \mathcal{P}(X) & \longrightarrow & h\mathcal{P}(X) \\ \vdots & & \downarrow \\ A^S(X) & \longrightarrow & A(X) \end{array}$$

The first part is in effect in [7]: the required map may be given in terms of a composite map

$$h\mathcal{P}(X) \longrightarrow h\mathcal{K}_{hf}(X) \longrightarrow A(X)$$

where  $h\mathcal{K}_{hf}(X)$  denotes the simplicial category of weak homotopy equivalences of retractive spaces over  $X$  of homotopically finite type, and where the second map is from [8].

The content of the sublemma really is that this map can be described without the auxiliary use of a homotopy theoretic device such as  $h_{hf}^{\mathcal{P}}(X)$ . . This amounts to showing that the 'manifold approach' [7] can be extended to cover the  $\mathcal{P}$ . construction of [8] (or rather a technical variant referred to as the  $\mathcal{T}$ . construction in section 1.3 of [8]). Given then that theorem 1 of [7] can be restated in terms of that construction, the compatibility asserted in part (2) of the sublemma will be an automatic consequence.

Let  $X$  be a manifold. We consider submanifolds  $M \subset X \times [0,1]$  as in the definition of the partitions [7], but now we consider sequences of such,

$$M_0 \subset M_1 \subset \dots \subset M_n .$$

These form a simplicial set  $\mathcal{P}_n(X)$ . We make it into a simplicial category in two ways which we will denote by the prefixes  $i$  and  $h$ , respectively. In each case there will be a morphism from  $\{M_p\}$  to  $\{M'_p\}$  only if  $M_p \subset M'_p$  for all  $p$  and if  $M'_p \cap M_q$  is not bigger than  $M_p$  for all  $p < q$ .

In  $i\mathcal{P}_n(X)$  there is, by definition, a morphism from  $\{M_p\}$  to  $\{M'_p\}$  if and only if, in addition to the above conditions, we have

$$M_p \cup M'_0 = M'_p$$

for all  $p$ . Note there is really no condition on the inclusion  $M_0 \rightarrow M'_0$ . Thus, for example, the category

$i\mathcal{P}_0(X)$  is contractible. In general, the role of these morphisms is to provide a systematic way of ignoring the  $M_0$ 's in those filtrations. (By abuse one could think of a morphism in  $i\mathcal{P}_n(X)$  as an isomorphism of the nonexistent quotient-manifolds  $\{M_p/M_0\} \rightarrow \{M'_p/M'_0\}$ .)

In  $h\mathcal{P}_n(X)$  there is a morphism from  $\{M_p\}$  to  $\{M'_p\}$  if and only if, in addition to the above conditions, we have that the inclusion maps

$$M_p \cup M'_0 \rightarrow M'_p$$

are homotopy equivalences. (We omit using a refined condition here, as in the definition of the simplicial category of partitions; stably, in the limit, such a distinction would not be essential anyway.)

We also need a simplicial subset  $\mathcal{P}_n^h(X)$  and corresponding simplicial subcategories  $i\mathcal{P}_n^h(X)$  and  $h\mathcal{P}_n^h(X)$ . Again we omit using a refined condition, and we let  $\mathcal{P}_n^h(X)$  denote the simplicial set of the sequences  $M_0 \subset \dots \subset M_n$  where all the inclusions are homotopy equivalences.

For varying  $n$ , these simplicial objects assemble to bisimplicial objects. We assert that, in the limit with respect to dimension, the square

$$\begin{array}{ccc} i\mathcal{P}_n^h(X) & \longrightarrow & i\mathcal{P}_n(X) \\ \downarrow & & \downarrow \\ h\mathcal{P}_n^h(X) & \longrightarrow & h\mathcal{P}_n(X) \end{array}$$

may be identified to the square on p. 149 of [7].

$$\begin{array}{ccc}
 N_{\Gamma}(\varinjlim_{\mathcal{d}} \mathcal{K}(X \times J^{\mathcal{d}})) & \longrightarrow & N_{\Gamma}(\varinjlim_{\mathcal{d}} \mathcal{P}^m(X \times J^{\mathcal{d}})) \\
 \downarrow & & \downarrow \\
 N_{\Gamma}(\varinjlim_{\mathcal{d}} h\mathcal{K}(X \times J^{\mathcal{d}})) & \longrightarrow & N_{\Gamma}(\varinjlim_{\mathcal{d}} h\mathcal{P}^m(X \times J^{\mathcal{d}}))
 \end{array}$$

or rather the square obtained from that by homotopy direct limit with respect to  $m$ .

This is seen as follows. First, one shows the square is homotopy cartesian; the method is that of proposition 5.1 of [7], essentially. Next, there is a natural transformation from the latter square to the former, and it will suffice to show that the transformation is a homotopy equivalence on three of the four corners. This is trivially true in the case of the lower left corner (both terms are contractible by the initial object argument). In the case of the upper left corner one uses a degreewise argument, namely one shows that for every  $n$  there is a homotopy equivalence, in the limit with respect to dimension, between  $i\mathcal{P}_n^h(X)$  on the one hand and the  $n$ -fold product  $(\varinjlim_{\mathcal{d}} \mathcal{K}(X \times J^{\mathcal{d}}))^n$  on the other. And finally, in the case of the lower right corner, one reduces, by proposition 5.4 of [7] and an analogue of that in the filtered case at hand, to the map

$$\varinjlim_m N_{\Gamma}(h\mathcal{K}^m(X)) \longrightarrow h\mathcal{K}_{hf}(X)$$

which is a homotopy equivalence by [8].

Since  $i\mathcal{P}_0(X)$  is contractible, the '1-skeleton' of the simplicial object  $[n] \mapsto i\mathcal{P}_n(X)$  may be identified, up to homotopy, to the suspension of  $i\mathcal{P}_1(X)$ . By adjointness there is therefore a map into the loop space,

$$i\mathcal{P}_1(X) \longrightarrow \Omega |i\mathcal{P}_*(X)| .$$

In view of the above assertion (the comparison of diagrams) that loop space may be identified, in the limit with respect to dimension, to  $A^S(X)$ . But also

$$i\mathcal{P}_1(X) \simeq \mathcal{P}(X) ,$$

at least in the limit with respect to dimension (by the initial object argument, essentially). Thus we obtain the required factorization

$$\begin{array}{ccc}
 \mathcal{P}(X) & \longrightarrow & h\mathcal{P}(X) \\
 \vdots & & \downarrow \\
 A^S(X) & \longrightarrow & A(X) .
 \end{array}$$

This completes the argument.

*Remarks.* The vanishing of  $\mu(X)$  is equivalent to the statement that a certain map  $Wh^{Comb}(X) \rightarrow Wh^{DIFF}(X)$  is a homotopy equivalence [7]. The statement may be regarded as a stable version (stable with respect to dimension) of Igusa's theorem that *Higher singularities of smooth functions are unnecessary*. It is therefore not surprising that, conversely, the vanishing of  $\mu(X)$  may be deduced from Igusa's theorem.

There is still another proof of the vanishing of  $\mu(X)$ , using quite different methods again. Namely there is a method, due to Goodwillie, to obtain information about  $Wh^{DIFF}$  of a highly connected map. There is another method to obtain information about  $A$  of a highly connected map. The computations obtained by these two methods are, of course, similar looking. But, and this is the point, they are not quite identical: the only way to avoid a contradiction is to conclude that  $\mu(X)$  must be trivial.

These computations also provide a generalization of the vanishing of  $\mu(X)$ . The ultimate statement is that, generally, stable K-theory may be identified to Hochschild homology provided that the latter is understood, throughout, over the universal ground ring,  $\Omega^{\infty} S^{\infty}$ . The case of the ground ring  $\Omega^{\infty} S^{\infty}$  itself here is precisely the statement that  $\mu(*)$  (and hence  $\mu(X)$  in general) is trivial.

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