Algebraic K-theory of topological spaces. II

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The purpose of this paper is to explore the relation between stable homotopy theory and the functor $A(X)$ of the title. The relation turns out to be very simple: The former splits off the latter.

This splitting of $A(X)$ is an unexpected phenomenon. Consider the case where

$$X = * ,$$

a point. In this case we may (and will) take as the definition

$$A(*) = Z \times (\varinjlim_{n,k} B\, Aut(V^k S^n))^+$$

where

$V^k S^n$ = wedge of k spheres of dimension n

$Aut(..)$ = simplicial monoid of pointed homotopy equivalences

$B\, Aut$ = its classifying space

$(...)^+$ = the $+$ construction of Quillen

$\varinjlim_{n,k}$: by suspension, and by wedge with identity maps, respectively.

The artificial factor Z is required to avoid disagreement with other definitions of $A(*)$. Thanks to a theorem of Barratt-Priddy, Quillen, and Segal on the other hand stable homotopy is definable in terms of the symmetric groups,

$$\Omega^\infty S^\infty \simeq Z \times (\varinjlim_{k} B\, \Sigma_k)^+ .$$

Since $\Sigma_k \approx Aut(V^k S^0)$, the map

$$Aut(V^k S^0) \longrightarrow \varinjlim_{n} Aut(V^k S^n)$$

therefore induces a map $\Omega^\infty S^\infty \to A(*)$. It is this map for which the splitting theorem provides a left inverse, up to homotopy.

Let us compare with known facts from algebraic K-theory. There is a map from

$A(*)$ to the algebraic K-theory of the ring of integers,

$$K(Z) \;=\; Z \times (\varinjlim_{k} B\, GL_k(Z))^+ ,$$

it is induced from

$$Aut(V^k S^n) \longrightarrow Aut(H_n(V^k S^n)) \;\approx\; GL_k(Z)$$

This map is a rational homotopy equivalence [14] (an easy consequence of the finiteness of the stable homotopy groups of spheres π_i^S, $i > 0$).

The composite map

$$\Omega^\infty S^\infty \longrightarrow A(*) \longrightarrow K(Z)$$

is the usual map resulting from identification of a symmetric group with a group of permutation matrices. This map has been studied by Quillen [10]. The main result is that

$$\pi^S_{4k+3} \longrightarrow K_{4k+3}(Z)$$

is injective on the image of the J-homomorphism, the subgroup $\mathrm{Im}J_{4k+3}$; in fact, the map is split injective on the odd torsion, and also on the 2-torsion in half the cases (k odd). In the other half it is not. For, Lee and Szczarba [5] have computed $K_3(Z)$ and as a result the map $\mathrm{Im}J_{4k+3} \to K_{4k+3}(Z)$ is, for $k = 0$, the inclusion

$$Z/24 \approx \mathrm{Im}J_3 \approx \pi^S_3 \longrightarrow K_3(Z) \approx Z/48 .$$

Browder [3] has deduced from this that the map is not split for all even k. It also follows from the Lee-Szczarba computation that $\pi_i^S \to K_i(Z)$ is not in general injective, and specifically [3] that

$$Z/2 \approx \pi^S_6 \longrightarrow K_6(Z)$$

is the zero map. To sum up, the relation between π_i^S and $K_i(Z)$ is very interesting, but apparently also very complicated. Certainly the map $\Omega^\infty S^\infty \to K(Z)$ does not split.

One may wonder here how possibly a result can be provable in the 'non-linear' case (the splitting theorem for $A(X)$) but fail to hold in the 'linear' case (algebraic K-theory). The answer is of course that the proof does not really break down in the linear case, it just proves a different result. This result will be discussed at the end of the paper.

Returning to the splitting theorem, to prove it we must in fact prove a stronger result involving the *stabilization of* $A(X)$,

$$A^S(X) \;=\; \varinjlim_{m} \Omega^m \,\mathrm{fibre}(\, A(S^m \wedge X_+) \to A(*) \,)$$

where

X_+ = X with a disjoint basepoint added

fibre(..) = the homotopy theoretic fibre

Ω^m = the m-th loop space,

and where the direct system involves certain naturally defined maps.

There is a natural transformation

$$A(X) \longrightarrow A^S(X)$$

of which one should think of being induced from the identification of $A(X)$ with the 0-th term in the direct system defining $A^S(X)$. (There is a technical point here. The definition of $A(X)$ we use requires that X be connected. So the 0-th term in the direct system is not defined. So the map

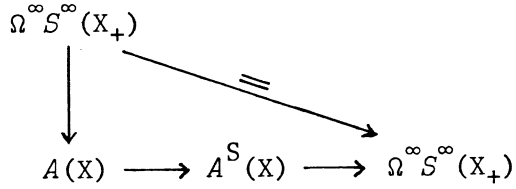
$$A(X) \longrightarrow \Omega \text{ fibre}(A(S^1 \wedge X_+) \to A(*))$$

must be artificially produced. We have to introduce the external pairing for that purpose).

Theorem. There is a natural map, well defined up to weak homotopy,

$$A^S(X) \longrightarrow \Omega^\infty S^\infty(X_+)$$

so that the diagram



commutes up to (weak) homotopy.

Recall that two maps are called weakly homotopic if their restrictions to every compactum are homotopic. 'Weak homotopy' is the price we have to pay for working with stable range arguments.

To produce the required map on $A^S(X)$ is equivalent more or less, in view of the definition of A^S, to producing for highly connected Y a map, defined in a stable range,

$$A(Y) \longrightarrow \Omega^\infty S^\infty(Y) \ .$$

It is not obvious that such a map should exist, and considerable work goes into its construction.

Our method to produce the map is to first manipulate $A(Y)$ in a stable range (section 3). A curious construction of simplicial objects is needed here which will be referred to as the *cyclic bar construction*. The idea for this construction comes from unpublished work of K. Dennis (talk at Evanston conference, January 1976), in fact, the *Hochschild homology* that Dennis uses may be regarded as a linear version of the cyclic bar construction. General facts relating to the cyclic bar construction are assembled in section 2.

Given the manipulation of $A(Y)$ in the stable range, a map $A(Y) \to \Omega^\infty S^\infty(Y)$,

defined in a stable range, may simply be written down (section 4, there are however some technicalities involved here) and it is entirely obvious that this map admits *some* section.

We are then left to show (section 5) that the section is what we want it to be. This requires some preparatory material which is scattered through earlier sections, particularly section 1 which gives a review of some general properties of $A(X)$ and of material involved in the Barratt-Priddy-Quillen-Segal theorem.

§1. Review of $A(X)$ and stable homotopy.

Let X be a simplicial set. We assume X is connected and pointed, so the *loop group* $G(X)$ in the sense of Kan [4] is defined. The geometric realization $|G(X)|$ is a topological group which will be called G for short.

Letting G_+ denote G with a disjoint basepoint added, and $V^k S^n$ the wedge of k spheres of dimension n, we form the G-space

$$V^k S^n \wedge G_+ \qquad (\approx V^k S^n \times G \;/\; * \times G)$$

which should be thought of as a *free pointed G-cell complex with k G-cells of dimension n.*

We consider the simplicial set (= singular complex of the topological space) of G-equivariant pointed maps

$$M_k^n(G) \;=\; Map_G(V^k S^n \wedge G_+,\; V^k S^n \wedge G_+)$$

which may be given the structure of a simplicial monoid, by composition of maps. Further we consider the simplicial monoid of G-equivariant pointed weak homotopy equivalences

$$H_k^n(G) \;=\; Aut_G(V^k S^n \wedge G_+)\ .$$

There is a stabilization map from n to $n+1$, by suspension, hence we can form the direct limit with respect to n. We can also consider a stabilization map from k to $k+1$; in the case of $H_k^n(G)$ it is given by adding the identity map on a new summand in the wedge.

Using the identity element of G we have a canonical map $S^0 \to G_+$. By restriction along this map we obtain an isomorphism

$$Map_G(V^k S^n \wedge G_+,\; V^k S^n \wedge G_+) \xrightarrow{\;\approx\;} Map(V^k S^n,\; V^k S^n \wedge G_+)\ .$$

This isomorphism in turn restricts to an isomorphism from the underlying simplicial set of $H_k^n(G)$ to a union of connected components of $Map(V^k S^n, V^k S^n \wedge G_+)$.

It is suggestive to think of $M_k^n(G)$ as a space of $k \times k$ matrices of some kind. The suggestion is particularly attractive in the limiting case $n = \infty$, for in this

case $M_k^\infty(G)$ is actually homotopy equivalent, in the obvious way, to the product of $k\times k$ copies of

$$M_1^\infty(G) \;=\; \varinjlim_n Map(S^n,\, S^n\wedge G_+) \;=\; \Omega^\infty S^\infty(G_+)\ ,$$

and the composition law on $M_k^\infty(G)$ corresponds, under the homotopy equivalence, to matrix multiplication.

Let $NH_k^n(G)$ denote the nerve (or bar construction) of the simplicial monoid $H_k^n(G)$; it is the simplicial object

$$[m] \longmapsto H_k^n(G) \times \ldots \times H_k^n(G) \quad (\text{m factors})$$

with the usual face structure. Let $B\,H_k^n(G) = |NH_k^n(G)|$ be its geometric realization. Then, by definition,

$$A(X) \;=\; Z \times \Big(\varinjlim_{n,k} B\,H_k^n(G)\Big)^+$$

where $(..)^+$ denotes the + construction of Quillen [9] (recall that G denotes the geometric realization of the loop group of X).

This definition is essentially the same as the first definition of $A(X)$ in [14]. To make the translation one verifies that the space $BH_k^n(G)$ used here is homotopy equivalent to the classifying space of the category used there (this is the content of [14, lemma 2.1], essentially). The requisite arguments are probably well known, a detailed account will be in [15].

The above construction can also be made for any finite n, giving a kind of unstable approximation to $A(X)$. In particular, the case $n = 0$ gives stable homotopy. Indeed, $H_1^0(G) \approx S(G)$ (the singular complex of G) and in general

$$H_k^0(G) \;\approx\; \Sigma_k \int S(G)$$

(wreath product with the symmetric group on k letters). Hence the theorem of Barratt-Priddy, Quillen, and Segal [11] gives a homotopy equivalence

$$(\ \Omega^\infty S^\infty|X_+| \;\simeq\;)\ \ \Omega^\infty S^\infty(BS(G)_+) \;\simeq\; Z \times \Big(\varinjlim_k B\,H_k^0(G)\Big)^+ .$$

The map

$$H_k^0(G) \longrightarrow \varinjlim_n H_k^n(G)$$

therefore induces

$$\Omega^\infty S^\infty|X_+| \longrightarrow A(X)\ .$$

We will need a different description of this map, in a stable range.

Lemma 1.1. The following diagram commutes up to weak homotopy (homotopy on compacta) in which the homotopy equivalence on the right is that of the Barratt-Priddy-Quillen-Segal theorem and the map on the bottom is the natural stabilization map:

$$\begin{array}{ccccc} BH_1^0(G) & \longrightarrow & BH_\infty^0(G) & \longrightarrow & Z \times BH_\infty^0(G)^+ \\ \downarrow \wr & & & & \uparrow \wr \\ |X| & & \longrightarrow & & \Omega^\infty S^\infty |X_+| \end{array}$$

The lemma is, essentially, a quotation from Segal [11]. Before making this explicit we review some material on Γ-spaces. We do this in some detail as the material will also be needed for other purposes, particularly the treatment of pairings below.

(1.2). Γ-spaces. Our reference is Segal [11]; cf. also Anderson [1] for some reformulation. Let $\underline{s}$ denote the basepointed set with s non-basepoint elements $1, \ldots, s$. We recall that a (special) Γ-*space* is a covariant functor F from the category of finite pointed sets to the category of spaces (respectively, the category of (multi-)simplicial sets in our case) which satisfies that $F(\underline{0}) = *$, and which takes sums to products, up to homotopy; this means, if $p_1 : X_1 \vee X_2 \to X_1$ is the retraction which takes X_2 to the basepoint, and p_2 similarly, then

$$(p_{1*}, p_{2*}) : \quad F(X_1 \vee X_2) \longrightarrow F(X_1) \times F(X_2)$$

is a weak homotopy equivalence. The space $F(\underline{1})$ is called the *underlying space* of the Γ-space F.

In our present situation we have for every $n = 0, 1, \ldots,$ or $= \infty$, a Γ-space F_G^n whose underlying space is

$$F_G^n(\underline{1}) = \coprod_k NH_k^n(G) .$$

The higher terms can be obtained by a general procedure of Segal [11, section 2]; the next term is

$$F_G^n(\underline{2}) = \coprod_{k,l} \left(EH_k^n(G) \times EH_l^n(G) \times EH_{k+l}^n(G) \right) / H_k^n(G) \times H_l^n(G)$$

where E denotes a universal bundle (one-sided bar construction) and '/' means quotienting out of the action, and the general term is

$$F_G^n(\underline{s}) = \coprod_{k_1,\ldots,k_s} \left(\prod_{\sigma \subset \underline{s}} EH_{k_\sigma}^n(G) \right) / H_{k_1}^n(G) \times \ldots \times H_{k_s}^n(G)$$

where $k_\sigma = \Sigma_{r \in \sigma} k_r$.

Returning to the general notion of Γ-space, we can extend the functor F, by direct limit and degreewise extension, to a functor defined on the category of pointed simplicial sets. For example if the original functor took values in the category of simplicial sets, the extended functor will take values in the category of bisimplicial sets.

In the special case of a Γ-space which is 'group-valued' (for example this holds if the underlying space is connected) the extended functor is a (reduced) homology

theory; that is, it preserves weak homotopy equivalences, and it takes cofibration sequences to fibration sequences up to homotopy, cf. [1] and e.g. [13] for a more detailed account. In view of a natural transformation $X \wedge F(Y) \to F(X \wedge Y)$ it therefore gives rise to a (connective) loop spectrum

$$|F(\underline{1})| \xrightarrow{\simeq} \Omega|F(S^1)| \ , \quad |F(S^1)| \xrightarrow{\simeq} \Omega|F(S^2)| \ , \ \ldots$$

Our Γ-spaces F_G^n are not group valued in the above sense. In this general case the list of properties must be weakened a bit, namely the extended functor F will not in general produce a fibration sequence from a cofibration sequence unless the latter involves connected spaces only. Thus the spectrum $m \mapsto F(S^m)$ is a loop spectrum only after the first map. The space $F(S^1)$ is equivalent to the underlying space of the Γ-space which in Segal's notation would be called BF, and one of the main general results about Γ-spaces says that it is computable by means of the + construction. Specifically in our situation we have

$$\Omega|F_G^n(S^1)| \ \simeq \ Z \times (\varinjlim_k B\, H_k^n(G))^+ \ .$$

Thus in the cases $n = 0$ and $n = \infty$ we recover $\Omega^\infty S^\infty|X_+|$ and $A(X)$, respectively.

Remark. The latter homotopy equivalence is well defined up to weak homotopy only (for it is obtained by means of an isomorphism of homotopy functors on the category of finite CW complexes [11]). This kind of ambiguity (weak homotopy instead of homotopy) arises frequently in connection with the + construction. It would be tempting to avoid the ambiguity by avoiding the + construction, and specifically by not using the universal property. We could indeed avoid the + construction altogether. But the effort would be in vain. For the stable range arguments that we have to use later on, would re-introduce the ambiguity.

Proof of lemma 1.1. This is a corollary of Segal's proof of the homotopy equivalence of infinite loop spaces

$$\Omega^\infty S^\infty|X_+| \ \simeq \ \Omega|F^0_{G(X)}(S^1)| \ .$$

In [11, proofs of propositions 3.5 and 3.6] Segal does in fact exhibit a specific map of spectra from the suspension spectrum of $|X_+|$ to the spectrum $m \mapsto \Omega|F^0_{G(X)}(S^{m+1})|$ which he then shows is a weak homotopy equivalence of spectra. Since the receiving spectrum is a loop spectrum this map is characterized by the map of first terms which is the composite map

$$B S|G(X)|_+ \longrightarrow \coprod_k BH_k^0(G) \ = \ |F_G^0(S^0)| \longrightarrow \Omega|F_G^0(S^1)|$$

$$B S|G(X)| \xrightarrow{\approx} BH_1^0(G)$$

$$* \xrightarrow{\approx} BH_0^0(G) \ .$$

It is immediate from this that there is a version of lemma 1.1 in which $Z \times BH_\infty^0(G)^+$

has been replaced by $\Omega|F^0_G(S^1)|$. To translate into the form stated, one has to take into account the way the homotopy equivalence between these two spaces arises [11, section 4] and particularly the way that $Z \times BH^0_\infty(G)$ arises as the telescope of $\coprod_k BH^0_k(G)$ and a shift map. □

(1.3). Pairings. Smash product induces a pairing $H^n_k(G) \times H^{n'}_{k'}(G') \longrightarrow H^{n+n'}_{k\cdot k'}(G\times G')$ and therefore also a pairing of Γ-spaces (resp. of their extensions described above)

$$F^n_G(Y) \wedge F^{n'}_{G'}(Y') \longrightarrow F^{n+n'}_{G\times G'}(Y\wedge Y') \ .$$

The pairing is compatible with the natural transformation $Y'' \wedge F^n_G(Y) \longrightarrow F^n_G(Y''\wedge Y)$.

Taking Y and Y' to be spheres, we have in particular

$$F^n_G(S^m) \wedge F^{n'}_{G'}(S^{m'}) \longrightarrow F^{n+n'}_{G\times G'}(S^{m+m'})$$

which defines a pairing of spectra because of the compatibility with the structure map $S^1 \wedge F^n_G(S^m) \longrightarrow F^n_G(S^{m+1})$.

Using that, for $m > 0$, we have $A(X) \simeq \Omega^m|F^\infty_{G(X)}(S^m)|$, and using that the weak homotopy equivalence $G(X\times X') \to G(X)\times G(X')$ induces one

$$F^n_{G(X\times X')}(Y) \xrightarrow{\ \sim\ } F^n_{G(X)\times G(X')}(Y) \ ,$$

we thus obtain a pairing, well defined up to (weak) homotopy,

$$A(X) \wedge A(X') \longrightarrow A(X\times X') \ .$$

Note that the pairing could also have been defined more directly in terms of the definition of $A(X)$ by the + construction (similarly to the pairing in K-theory in [6]); with the present definition any desired naturality properties of the pairing are essentially obvious.

The pairing formally implies others. Let $\widetilde{A}(X)$ be the *reduced part* of $A(X)$,

$$\widetilde{A}(X) \ = \ \text{fibre}(\ A(X) \to A(*)\ ) \ .$$

Taking the difference (with respect to the H-space structure) of the identity map on $A(X)$ and the composite map $A(X) \to A(*) \to A(X)$, one obtains the required map in a splitting

$$A(X) \ \simeq \ A(*) \times \widetilde{A}(X) \ .$$

There is a pairing

$$\widetilde{A}(X) \wedge A(Y) \longrightarrow \widetilde{A}(X\wedge Y_+)$$

which is definable as the composite map

$$\widetilde{A}(X) \wedge A(Y) \longrightarrow A(X) \wedge A(Y) \longrightarrow A(X\times Y) \longrightarrow \widetilde{A}(X\times Y/*\times Y) \ ;$$

it satisfies that the following diagram is (weakly) homotopy commutative

$$\begin{array}{ccc} A(X) \wedge A(Y) & \longrightarrow & A(X\times Y) \\ \downarrow & & \downarrow \\ \widetilde{A}(X) \wedge A(Y) & \longrightarrow & \widetilde{A}(X\wedge Y_+) \end{array} .$$

Similarly there is a pairing

$$\widetilde{A}(X) \wedge \widetilde{A}(Y) \longrightarrow \widetilde{A}(X\wedge Y) .$$

There are analogous pairings involving (reduced and/or unreduced) stable homotopy, resp. stable homotopy and $A(X)$. For uniformity of notation we let

$$(\Omega^\infty S^\infty |X_+| \simeq) \quad Q(X) = Z \times (\varinjlim_k B H_k^0(G))^+$$

Lemma 1.4. There is a map $A(X) \to \Omega\widetilde{A}(S^1\wedge X_+)$ so that the diagram

$$\begin{array}{ccc} Q(X) & \xrightarrow{\simeq} & \Omega\widetilde{Q}(S^1\wedge X_+) \\ \downarrow & & \downarrow \\ A(X) & \longrightarrow & \Omega\widetilde{A}(S^1\wedge X_+) \end{array}$$

commutes up to homotopy.

Proof. Let $S^1 \to Q(S^1) \to \widetilde{Q}(S^1)$ be the Hurewicz map from homotopy to stable homotopy (the first map is that of lemma 1.1). Using the above pairings we have a diagram

$$\begin{array}{ccccc} S^1 \wedge Q(X) & \longrightarrow & \widetilde{Q}(S^1) \wedge Q(X) & \longrightarrow & \widetilde{Q}(S^1\wedge X_+) \\ \downarrow & & \downarrow & & \downarrow \\ S^1 \wedge A(X) & \longrightarrow & \widetilde{Q}(S^1) \wedge A(X) & \longrightarrow & \widetilde{A}(S^1\wedge X_+) \end{array}$$

and the adjoint of the composite map on the bottom will have the required property if we can show that the adjoint map

$$Q(X) \longrightarrow \Omega\widetilde{Q}(S^1\wedge X_+)$$

is a homotopy equivalence.

We note here that in treating this $Q(X)$ the necessity of having X connected and pointed is of course an illusion. For

$$N(\Sigma_k \int G(X)) \approx E\Sigma_k \times^{\Sigma_k} NG(X)^k \simeq E\Sigma_k \times^{\Sigma_k} X^k$$

so that we are in the situation of [11] and the term on the right is quite generally defined. Furthermore the pairing extends to this more general situation. Therefore

$$Q(X) \longrightarrow \Omega\widetilde{Q}(S^1\wedge X_+)$$

is in fact a natural transformation from stable homotopy theory to itself, and it

suffices to show it is a homotopy equivalence in the case $X = *$.

Since $Q(*) \to \Omega\widetilde{Q}(S^1)$ extends to a map of spectra it suffices in fact to show that it induces an isomorphism on π_0; equivalently, that its adjoint is surjective on π_1. But from the explicit description of the Hurewicz map (lemma 1.1) we see that the composite map

$$\begin{array}{ccccccc} S^1 \wedge S^0 & \longrightarrow & S^1 \wedge Q(*) & \longrightarrow & Q(S^1) \wedge Q(*) & \longrightarrow & Q(S^1) \\ & & & & \downarrow & & \downarrow \\ & & & & \widetilde{Q}(S^1) \wedge Q(*) & \longrightarrow & \widetilde{Q}(S^1) \end{array}$$

is itself the Hurewicz map, and we are done. □

§2. Simplicial tools.

(2.1). The realization lemma. This asserts that a map of simplicial objects which is a weak homotopy equivalence locally (i.e., the partial map in every degree is a weak homotopy equivalence) is also one globally. We need a version of this for finite connectivity.

We say a map is k-*connected* (or is a k-*equivalence*, by abuse of language) if it induces an isomorphism on π_j for $j < k$, and an epimorphism on π_k.

Lemma 2.1.1. Let $X.. \to Y..$ be a map of bisimplicial sets. Suppose that for every n the map of simplicial sets $X._n \to Y._n$ is k-connected. Then the map $X.. \to Y..$ is also k-connected.

Indeed, recall the argument in the case $k = \infty$, cf. e.g. [16]. One considers the 'skeleton filtration' $X_{(n)}$ of $|X..|$ induced from the second simplicial direction, that is, $X_{(n)}$ is the geometric realization of the bisimplicial subset of $X..$ generated by $X._n$. Then one proves inductively that $X_{(n)} \to Y_{(n)}$ is a k-equivalence using the *gluing lemma*. The same argument works in the case of finite k in view of the following version of the gluing lemma.

Lemma 2.1.2. In the commutative diagram

$$\begin{array}{ccccc} X_1 & \longleftarrow & X_0 & \longrightarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \longleftarrow & Y_0 & \longrightarrow & Y_2 \end{array}$$

let the two left horizontal maps be cofibrations, and suppose that all the vertical maps are k-connected. Then the map of pushouts $X_1 \cup_{X_0} X_2 \to Y_1 \cup_{Y_0} Y_2$ is also k-connected. □

(2.2). Partial monoids. This notion, due to Segal [12], allows a concise description of certain simplicial objects. By definition, a *partial monoid* is a set E together with a partially defined composition law

$$E \times E \supset E_2 \longrightarrow E$$

which is associative in the sense that if one of $(e_1e_2)e_3$ and $e_1(e_2e_3)$ is defined then so is the other and the two are equal. Further there must be a two-sided identity element $*$ and multiplication by $*$ must be everywhere defined, that is,

$$E \vee E \subset E_2 .$$

The simplicial set associated to the partial monoid (we refer to it as the *nerve* of E, notation NE) is given by

$$[n] \longmapsto E_n = \text{set of composable n-tuples}$$

with face and degeneracy maps given in the usual way by composition, resp. by insertion of the identity.

Similarly one has the notion of a *simplicial partial monoid*; its nerve is a bisimplicial set.

For example [12] a pointed simplicial set X can be considered as a simplicial partial monoid in a trivial way, with $X_2 = X \vee X$. The nerve in this case is the simplicial object

$$[n] \longmapsto \underset{\longleftarrow n \longrightarrow}{X \vee \ldots \vee X}$$

whose diagonal simplicial set is a suspension of X.

Other examples arise in the following way. Let M be a monoid and A a submonoid of M. Then we can manufacture a partial monoid by declaring that two elements of M shall be composable if and only if at least one of them belongs to the submonoid. Thus $M_2 = M \times A \cup A \times M$, and M_n is what we will refer to as a *generalized wedge*,

$$V^n(M,A) = \text{set of n-tuples of elements in } M, \text{ with at least } (n-1) \text{ elements in } A .$$

Similarly this construction can be made with a simplicial monoid M and a simplicial submonoid A of M.

Lemma 2.2.1. In this situation, if $A \to M$ is $(k-1)$-connected then the inclusion of simplicial objects

$$[n] \longmapsto (V^n(M,A) \longrightarrow M^n)$$

is $(2k-1)$-connected.

Proof. In view of the realization lemma (2.1.1.) it suffices to show that for every n the inclusion $V^n(M,A) \to M^n$ is $(2k-1)$-connected. This is certainly true if n

is either 0 or 1 as the inclusion is an isomorphism in those cases. The case $n = 2$ follows from the following remark.

A map of simplicial sets is $(k-1)$-connected if and only if its geometric realization is homotopy equivalent to an inclusion of CW complexes $K \to X$ so that $X \smallsetminus K$ has no cells of dimension $< k$. Let similarly $Y \smallsetminus L$ have no cells of dimension < 1. Then $X\times Y \smallsetminus X\times LUK\times Y$ has no cells of dimension $< k+1$, and therefore the map $X\times LUK\times Y \to X\times Y$ is $(k+1-1)$-connected.

The general case follows inductively by factoring the inclusion suitably and using the same remark and the gluing lemma. □

Finally we will need to consider, in this framework of partial monoids, the notion of *semi-direct product*.

Suppose first that F is a monoid (which we think of as multiplicative) and that E is another (which we think of as additive). Let F act from both sides, and compatibly, on E (in other words, if F^{op} denotes the opposite monoid of F then $F \times F^{op}$ acts on E from the left, say). In this situation, the semi-direct product

$$F \ltimes E$$

is the monoid of pairs (f,e) with multiplication given by the formula

$$(f,e)(f',e') = (ff',\ ef' + fe')$$

Remark. In case this looks unfamiliar, consider the case where F is a group. Here one can rewrite in the usual form, as follows. Write

$$(f,e) = (f,\ f\tilde{e})$$

where $\tilde{e} = f^{-1}e$. Then

$$\begin{aligned}(f,f\tilde{e})(f',f'\tilde{e}') &= (ff',\ f\tilde{e}f' + ff'\tilde{e}') \\ &= (ff',\ (ff')f'^{-1}\tilde{e}\ f' + (ff')\tilde{e}')\end{aligned}$$

and hence with $[f,\tilde{e}] = (f,f\tilde{e})$ the multiplication is given by the formula

$$[f,\tilde{e}][f',\tilde{e}'] = [ff',\ f'^{-1}\tilde{e}f' + \tilde{e}']$$

This ends the remark. □

Suppose now that E is a partial monoid on which the monoid F acts compatibly from both sides. We need to assume that E is *saturated* with respect to the action in the sense that the following condition is satisfied: for every pair (e,e') whose sum is defined, and for every f, the sums of the four pairs

$$(fe,e')\ ,\quad (ef,e')\ ,\quad (e,fe')\ ,\quad (e,e'f)$$

must also be defined (they need not however be related in any particular way). Under this assumption the formula $(f,e)(f',e') = (ff',ef'+fe')$ carries over to define a

partial monoid $F \ltimes E$ with underlying set $F \times E$ and with $(F \ltimes E)_2 \approx F \times F \times E_2$.

We will be especially concerned with the particular case where E is a pointed set X considered as a partial monoid in a trivial way. In this case $(F \ltimes E)_n$ is the generalized wedge

$$(F \ltimes X)_n = V^n(F\times X, F\times *) \approx F^n \times (X \vee \ldots \vee X) .$$

In particular $(F \ltimes X)_2 \approx F\times F\times(X\vee X)$, and the partial composition law is given by the case distinction

$$(f,x)(f',*) = (ff', xf')$$

$$(f,*)(f',x) = (ff', fx) .$$

All of the above extends to (and will be used in) a simplicial framework.

(2.3). The cyclic bar construction. Let F be a monoid which acts on a set X both from the left and the right, and compatibly. The *cyclic bar construction* is defined to be the simplicial set

$$N^{cy}(F,X) , \qquad [k] \longmapsto F \times \ldots \times F \times X \quad (\longleftarrow k \longrightarrow)$$

with face maps

$$d_0(f_1,\ldots,f_k, x) = (f_2,\ldots,f_k, xf_1)$$

$$d_i(f_1,\ldots,f_k, x) = (f_1,\ldots,f_if_{i+1},\ldots,f_k, x) \qquad \text{if } 0 < i < k$$

$$d_k(f_1,\ldots,f_k, x) = (f_1,\ldots,f_{k-1}, f_kx)$$

Similarly if F is a simplicial monoid and X a simplicial set, the cyclic bar construction is defined in the same way, giving a bisimplicial set.

The cyclic bar construction may be regarded as a generalization of the two-sided bar construction. Indeed, the latter may be identified to the special case of the former where X is the product of two factors of which the first has a left F-struture and the second a right F-structure, respectively.

As another example consider the case of a (simplicial) group acting on its underlying (simplicial) set from either side by multiplication. Then the map which in degree k is

$$(g_1,\ldots,g_k, g) \longmapsto (g_1,\ldots,g_k, g(g_1\cdots g_k))$$

defines an isomorphism from $N^{cy}(G,G)$ to the one-sided bar construction of G acting on itself by conjugation. The latter represents the free loop space of NG.

The case of main concern to us arises in the situation where a (simplicial) monoid F acts on a (simplicial) partial monoid E in such a way that the semi-direct product $F \ltimes E$ is defined. In this situation F will also act on the nerve NE in

such a way that the cyclic bar construction $N^{cy}(F,NE)$ is defined. We denote by $\mathrm{diag}N^{cy}(F,NE)$ the simplicial (resp. bisimplicial) set resulting from diagonalizing the two N-directions of the latter.

Lemma 2.3.1. There is a natural map

$$u: \ \mathrm{diag}\ N^{cy}(F,NE) \longrightarrow N(F \ltimes E) \ .$$

The map u is an isomorphism if F acts invertibly. If $\pi_0 F$ is a group then u is a weak homotopy equivalence.

Proof. In the formulas to follow we will suppose for simplicity of notation that F and E are a monoid and partial monoid, respectively, rather than a simplicial monoid and simplicial partial monoid. In the general case the formulas are exactly the same except that a dummy index has to be added everywhere.

By definition, $\mathrm{diag}\ N^{cy}(F,NE)$ is the simplicial set (resp. simplicial object in the general case)

$$[n] \longmapsto \underset{\longleftarrow n \longrightarrow}{F\times\ldots\times F \times E_n} \subset \underset{\longleftarrow n \longrightarrow}{F\times\ldots\times F} \times \underset{\longleftarrow n \longrightarrow}{E\times\ldots\times E}$$

with face maps taking $(f_1,\ldots,f_n;\ e_1,\ldots,e_n)$ to

$$\begin{aligned} d_0(..) &= (f_2,..,f_n;\ e_2f_1,..,e_nf_1) \\ d_i(..) &= (f_1,..,f_if_{i+1},..,f_n;\ e_1,..,e_i{+}e_{i+1},..,e_n) \ , \qquad 0 < i < n \\ d_n(..) &= (f_1,..,f_{n-1};\ f_ne_1,..,f_ne_{n-1}) \ , \end{aligned}$$

while $N(F \ltimes E)$ is given by

$$[n] \longmapsto (F \ltimes E)_n \subset \underset{\longleftarrow\ n \longrightarrow}{F\times E \times \ldots \times F\times E}$$

with face maps taking $(f_1,e_1;\ldots;\ f_n,e_n)$ to

$$\begin{aligned} d_0(..) &= (f_2,e_2;..;\ f_n,e_n) \\ d_i(..) &= (f_1,e_1;..;\ f_if_{i+1},\ e_if_{i+1}{+}f_ie_{i+1};\ldots;\ f_n,e_n) \ , \qquad 0 < i < n \\ d_n(..) &= (f_1,e_1;..;\ f_{n-1},e_{n-1}) \ . \end{aligned}$$

We define $u_n(f_1,..,f_n;\ e_1,..,e_n)$ to be

$$(\ f_1,\ (f_1..f_n)e_1(f_1)\ ;\ f_2,\ (f_2..f_n)e_2(f_1f_2)\ ;\ \ldots\ ;\ f_n,\ (f_n)e_n(f_1..f_n)\ )\ ,$$

then the collection of maps u_n forms a simplicial map u as one checks. Here is the situation for face maps: evaluating on $(f_1,..,f_n;\ e_1,..,e_n)$ we obtain

$$\begin{aligned} (d_0u_n:) &\quad (\ f_2,\ (f_2..f_n)e_2(f_1f_2)\ ;\ \ldots\ ;\ f_n,\ (f_n)e_n(f_1..f_n)\ ) \\ (u_{n-1}d_0:) &\quad (\ f_2,\ (f_2..f_n)(e_2f_1)(f_2)\ ;\ldots;\ f_n,\ (f_n)(e_nf_1)(f_2..f_n)\ ) \end{aligned}$$

and similarly with $d_n u_n$ and $u_{n-1} d_n$, further if $0 < i < n$ then

$$(d_i u_n :) \quad (..; f_i f_{i+1}, (f_i..f_n)e_i(f_1..f_i)f_{i+1} + f_i(f_{i+1}..f_n)e_{i+1}(f_1..f_{i+1}) ; ..)$$
$$(u_{n-1} d_i :) \quad (..; f_i f_{i+1}, ((f_i f_{i+1})f_{i+2}..f_n)(e_i + e_{i+1})(f_1..f_{i-1}(f_i f_{i+1})) ; ..) ,$$

thus the identities for iterated face maps are satisfied.

If the two actions of F on E are invertible then each of the maps u_n is an isomorphism, therefore u is an isomorphism in this case.

Suppose now that $\pi_0 F$ is a group. Then any action of F is homotopy invertible, that is, if F acts, from the left say, on X then the shearing map

$$\begin{aligned} F \times X &\longrightarrow F \times X \\ (f,x) &\longmapsto (f, fx) \end{aligned}$$

is a weak homotopy equivalence. Therefore in order to show the map u_n is a weak homotopy equivalence it suffices to write it as a composite of maps each of which is isomorphic to a shearing map. But u_n is isomorphic to the composite map

$$F \times ... \times F \times E_n \xrightarrow{u_n} (F \times E)_n \xrightarrow{\approx} F \times ... \times F \times E_n$$

and the latter may be factored

$$l_1 \cdots l_n \, r_n \cdots r_2 \, r_1 \qquad \text{(composition from right to left)}$$

where r_i is the restriction of the map

$$F \times ... \times F \times E \times ... \times E \longrightarrow F \times ... \times F \times E \times ... \times E$$
$$(f_1, ..., f_n; e_1, ..., e_n) \longmapsto (f_1, ..., f_n; e_1, .., e_{i-1}, e_i f_i, .., e_n f_i)$$

and where l_i is similarly defined using the left action.

Thus each of the maps u_n is a weak homotopy equivalence. In view of the realization lemma therefore the entire map u is a weak homotopy equivalence, too. The proof is complete. □

§3. Manipulation in a stable range.

In the theorem below we will suppose that X is highly connected and, for technical reasons, that it actually be given as a suspension. While there is no canonical way to suspend a simplicial set, a choice can of course be made universally. Our present choice is to be made so that $G(SX)$ is the free simplicial group generated by the non-basepoint simplices of X [4]. The geometric realization of the canonical map $X \to G(SX)$ then represents $|X| \to \Omega S|X|$ and is $(2m-1)$-connected if X is $(m-1)$-connected.

If V, W are pointed topological spaces we denote $Map(V,W)$ the pointed simplicial set (= the singular complex of the topological space) of pointed maps from V to W, and $H(V)$ the simplicial monoid of pointed (weak) self-homotopy equivalences of V. In a context of G-equivariant maps the analogous notions are indicated by a subscript G.

The simplicial monoid $H(V^k S^n)$ acts from the left on the pointed simplicial set $Map(V^k S^n, V^k S^n \wedge |X|)$, by composition of maps. But it also acts from the right in view of the canonical map

$$\begin{aligned} H(V^k S^n) &\longrightarrow H(V^k S^n \wedge |X|) \ , \\ h &\longmapsto h \wedge id_{|X|} \end{aligned}$$

and the two actions are compatible. Hence the cyclic bar construction, cf. (2.3),

$$N^{cy}(H(V^k S^n), Map(V^k S^n, V^k S^n \wedge |X|))$$

is defined.

Theorem 3.1. Let X be a pointed simplicial set which is m-connected, $m \geqslant 0$. Let SX be its suspension. Then the two spaces

$$N H_{|G(SX)|}(V^k S^n \wedge |G(SX)|_+)$$

and

$$N^{cy}(H(V^k S^n), Map(V^k S^n, V^k S^n \wedge |SX|))$$

are naturally q-equivalent, where

$$q = \min(n-2, 2m+1) \ ;$$

that is, there is a chain of natural maps connecting these two spaces, and all the maps in the chain are q-connected.

Naturality here refers to n and k, and the X variable. We will also need a further piece of naturality which we record in the following addendum.

Addendum 3.2. There is a chain of $(2m+1)$-equivalences between $NG(SX)$ and SX, and a transformation from this chain to the one of the theorem with the property that the first map in the transformation is the composite of $NG(SX) \xrightarrow{\sim} NH_1^0(|G(SX)|)$ with the inclusion $NH_1^0(|G(SX)|) \to NH_k^n(|G(SX)|)$ (cf. lemma 1.1); and the last map in the transformation is given by the composite map

$$SX \xrightarrow{\simeq} S|SX| \xrightarrow{\approx} Map(V^1 S^0, V^1 S^0 \wedge |SX|) \longrightarrow Map(V^k S^n, V^k S^n \wedge |SX|)$$

together with the identification of the latter space with the term in degree 0 of $N^{cy}(..)$.

The proof of the theorem will occupy this section. The addendum will be noted as we go along. The chain of maps will consist of five maps; it could be reduced to four as the first two maps are composable. Each of the maps will be described in its

own subsection.

(3.3). The first map. The simplicial monoid of the theorem,

$$H_{|G(SX)|}(V^k S^n \wedge |G(SX)|_+) ,$$

can be considered as a simplicial partial monoid by declaring that multiplication of elements in a fixed degree is possible if and only if at most one of them is outside the simplicial submonoid

$$H(V^k S^n) .$$

Thus the nerve of the simplicial monoid contains as a simplicial subobject the nerve of that simplicial partial monoid (the situation of lemma 2.2.1). The inclusion map will be our first map.

To verify the asserted connectivity, and also for its own sake, we do some rewriting now. As pointed out in section 1, the canonical map $S^0 \to |G(SX)|_+$ induces an isomorphism from the underlying simplicial set of the simplicial monoid to a union of connected components of the simplicial set of maps

$$Map(V^k S^n, V^k S^n \wedge |G(SX)|_+) ;$$

we denote this union of components by

$$\overline{Map}(..) .$$

Clearly the isomorphism is compatible with the left and right actions of $H(V^k S^n)$. Further the inclusion of the underlying simplicial set of the simplicial submonoid $H(V^k S^n)$ corresponds, under the isomorphism, to the natural inclusion

$$H(V^k S^n) \longrightarrow \overline{Map}(V^k S^n, V^k S^n \wedge |G(SX)|_+)$$

induced from $S^0 \to |G(SX)|_+$.

But it is only those two bits of structure, the latter inclusion and the left and right actions of $H(V^k S^n)$, which matter in the structure of the simplicial partial monoid. Therefore its nerve may be described as the simplicial object given by *generalized wedges* (cf. (2.2) for this notation),

$$[p] \longmapsto V^p(\overline{Map}(V^k S^n, V^k S^n \wedge |G(SX)|_+), H(V^k S^n)) .$$

The inclusion into the nerve of the original simplicial monoid is (2m+1)-connected by lemma 2.2.1, for the inclusion

$$H(V^k S^n) \longrightarrow \overline{Map}(V^k S^n, V^k S^n \wedge |G(SX)|_+)$$

is m-connected since $S^0 \to |G(SX)|_+$ is.

This finishes the account of the first map. Concerning the addendum, the first map in that chain is given by the analogous inclusion

$$[p] \longmapsto (V^p(G(SX), G(*)) \longrightarrow G(SX)^p) .$$

(3.4). The second map. The inclusion $X \to G(SX)$ induces one

$$\overline{Map}(V^k S^n, V^k S^n \wedge |X|_+) \longrightarrow \overline{Map}(V^k S^n, V^k S^n \wedge |G(SX)|_+)$$

where we are continuing to denote by $\overline{Map}$ a suitable union of connected components of Map, and the latter inclusion is $(2m+1)$-connected since the former is.

The inclusion is compatible with the left and right actions of $H(V^k S^n)$. It is also compatible with the inclusion of the underlying simplicial set of $H(V^k S^n)$, for the natural map $S^0 \to |X|_+$ given by the basepoint of X satisfies that

$$\begin{array}{ccc} S^0 & \longrightarrow & |X|_+ \\ & \searrow \quad \swarrow & \\ & |G(SX)|_+ & \end{array}$$

commutes.

Therefore the nerve of the simplicial partial monoid considered before, contains another,

$$[p] \longmapsto V^p(\overline{Map}(V^k S^n, V^k S^n \wedge |X|_+), H(V^k S^n)) .$$

The inclusion is our second map.

To show the map is $(2m+1)$-connected it suffices, by the realization lemma, to show this in each degree p. The case $p = 1$ was noted before. It implies the general case in view of the gluing lemma (2.1.2) and induction.

This finishes the account of the second map. Concerning the addendum, the second map in that chain is given by a similar inclusion, namely

$$[p] \longmapsto (V^p(X,*) \longrightarrow V^p(G(SX),G(*))) .$$

(3.5). The third map. The pointed simplicial set $Map(V^k S^n, V^k S^n \wedge |X|)$ can be considered as a simplicial partial monoid in a trivial way, and the simplicial monoid $H(V^k S^n)$ acts on it from both sides, and compatibly. Hence the product

$$H(V^k S^n) \times Map(V^k S^n, V^k S^n \wedge |X|)$$

can be given the structure of a simplicial partial monoid, namely the semi-direct product in the sense of (2.2).

The pair of maps $|X|_+ \to S^0$, $|X|_+ \to |X|$ induces a map of simplicial partial monoids whose underlying map of simplicial sets is

$$\overline{Map}(V^k S^n, V^k S^n \wedge |X|_+) \longrightarrow H(V^k S^n) \times Map(V^k S^n, V^k S^n \wedge |X|) .$$

We show this map is $(n-2)$-connected. Indeed, since X is connected (we assumed this in the theorem) this map is the restriction to a union of connected components of the map

$$Map(V^k S^n, V^k S^n \wedge |X|_+) \longrightarrow Map(V^k S^n) \times Map(V^k S^n, V^k S^n \wedge |X|) ,$$

so it suffices to show the latter map is (n-2)-connected. We treat the case $k = 1$ first.

<u>Lemma</u>. The map $\Omega^n S^n(|X| \cup *) \to \Omega^n S^n \times \Omega^n S^n |X|$ is (n-2)-connected.

<u>Proof</u>. The long exact sequence of stable homotopy groups of the cofibration sequence

$$S^n(S^0) \longrightarrow S^n(|X| \cup *) \longrightarrow S^n|X|$$

decomposes into split short exact sequences. As $\pi_i S^n Y \to \pi_i^S S^n Y$ is an isomorphism for $i \leqslant 2n-2$ it follows that $S^n(|X| \cup *) \to S^n \times S^n|X|$ induces an isomorphism on homotopy groups for $i \leqslant 2n-2$. The assertion results by taking loop spaces. □

The case $k = 1$ being established, the case of general k now follows from the isomorphism

$$Map(V^k S^n, Y) \xrightarrow{\approx} (\ Map(S^n, Y)\ )^k$$

and the (n-1)-equivalence

$$Map(S^n, V^k S^n \wedge Y') \longrightarrow (\ Map(S^n, S^n \wedge Y')\ )^k$$

induced from the (2n-1)-equivalence

$$(S^n \wedge Y') \vee \ldots \vee (S^n \wedge Y') \longrightarrow (S^n \wedge Y') \times \ldots \times (S^n \wedge Y') \ .$$

The map of simplicial partial monoids induces a map of their nerves. In the notation of generalized wedges, this is a map from

$$[p] \longmapsto V^p(\ \overline{Map}(V^k S^n, V^k S^n \wedge |X|_+),\ H(V^k S^n)\ )$$

to

$$[p] \longmapsto V^p(\ H(V^k S^n) \times Map(V^k S^n, V^k S^n \wedge |X|),\ H(V^k S^n) \times *\ ) \ .$$

This map is (n-2)-connected for every p (the gluing lemma reduces the assertion to the case $p = 1$ which was verified above) and therefore the entire map is also (n-2)-connected by the realization lemma. This is our third map.

Concerning the addendum, the third map in that chain is the identity map on

$$[p] \longmapsto V^p(|X|, *) \ .$$

(3.6). <u>The fourth map</u>. Considering the pointed simplicial set $Map(V^k S^n, V^k S^n \wedge |X|)$ as a simplicial partial monoid in a trivial way, and forming the nerve of the latter, we obtain the simplicial object

$$[p] \longmapsto \underbrace{Map(V^k S^n, V^k S^n \wedge |X|) \vee \ldots \vee Map(V^k S^n, V^k S^n \wedge |X|)}_{\longleftarrow\ p\ \longrightarrow}$$

which we denote by

$$\Sigma\ Map(V^k S^n, V^k S^n \wedge |X|) \ .$$

It inherits compatible left and right actions of the simplicial monoid $H(V^kS^n)$, so we can form the cyclic bar construction

$$N^{cy}(H(V^kS^n), \Sigma Map(V^kS^n, V^kS^n \wedge |X|)) ,$$

a trisimplicial set. Our fourth map is provided by lemma 2.3.1. It is the weak homotopy equivalence whose source is

$$\text{diag } N^{cy}(H(V^kS^n), \Sigma Map(V^kS^n, V^kS^n \wedge |X|))$$

(diagonal along the N- and Σ-directions) and whose target is identical to the target of the third map, namely the nerve of the simplicial partial monoid given by the semi-direct product of $H(V^kS^n)$ acting on $Map(V^kS^n, V^kS^n \wedge |X|)$.

Concerning the addendum, the fourth map in that chain is again the identity map on

$$([p] \longmapsto V^p(|X|,*)) \qquad (= \Sigma |X|) .$$

(3.7). The fifth map. Partial geometric realization takes the bisimplicial set

$$\Sigma Map(V^kS^n, V^kS^n \wedge |X|)$$

to the simplicial topological space

$$S^1 \wedge Map(V^kS^n, V^kS^n \wedge |X|)$$

and the canonical map from the latter to

$$Map(V^kS^n, V^kS^n \wedge S^1 \wedge |X|) \quad \approx \quad Map(V^kS^n, V^kS^n \wedge |SX|)$$

is (2m+1)-connected. The induced map from (the partial geometric realization of)

$$N^{cy}(H(V^kS^n), \Sigma Map(V^kS^n, V^kS^n \wedge |X|))$$

to

$$N^{cy}(H(V^kS^n), Map(V^kS^n, V^kS^n \wedge |SX|))$$

is therefore also (2m+1)-connected, by the realization lemma. This is our fifth map.

Concerning the addendum, the fifth map in that chain is the isomorphism from (the geometric realization of)

$$\Sigma |X|$$

to

$$|SX| .$$

The proof of the theorem and its addendum are now complete. □

§4. The stabilization of $A(X)$.

We will need the following elementary properties of the functor $A(X)$. Namely, it

(i) takes n-equivalences to n-equivalences if n is at least 2 ,

(ii) satisfies a version of homotopy excision, namely for $m, n \geq 2$, $k \leq m+n-2$, it preserves (m,n)-*connected* k-*homotopy cartesian squares*, that is, commutative squares

$$\begin{array}{ccc} V & \longrightarrow & W \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in which the horizontal (resp. vertical) arrows are m-connected (resp. n-connected) and the map $\mathrm{fibre}(V \to X) \to \mathrm{fibre}(W \to Y)$, or equivalently the map $\mathrm{fibre}(V \to W) \to \mathrm{fibre}(X \to Y)$, is (k+1)-connected.

These properties are propositions 2.3 and 2.4 of [14]. Their proofs are actually easiest with the definition of $A(X)$ used here.

We note here that $A(X)$ can be a functor on the nose, not just up to homotopy. In our present context we may simply point to the possibility of performing the + construction uniformly (for example by attaching a single 2-cell and 3-cell to $BH(V^5S^0)$). In particular the above maps of homotopy fibres are well defined.

Let S^m denote a suitable simplicial set representing the m-sphere, and let

$$D_1^m \cup_{S^{m-1}} D_2^m \xrightarrow{\approx} S^m$$

be a decomposition into hemispheres. Then for any X the diagram

$$\begin{array}{ccc} S^{m-1}\wedge X_+ & \longrightarrow & D_1^m\wedge X_+ \\ \downarrow & & \downarrow \\ D_2^m\wedge X_+ & \longrightarrow & S^m\wedge X_+ \end{array}$$

being (m-1,m-1)-connected, is (2m-4)-homotopy cartesian by the homotopy excision theorem. In view of the above therefore the map

$$\mathrm{fibre}(\, A(S^{m-1}\wedge X_+) \to A(D_2^m\wedge X_+)\,) \longrightarrow \mathrm{fibre}(\, A(D_1^m\wedge X_+) \to A(S^m\wedge X_+)\,) \simeq \Omega\ \mathrm{fibre}(\, A(S^m\wedge X_+) \to A(*)\,)$$

is (2m-3)-connected. Thus we have a spectrum

$$m \longmapsto \mathrm{fibre}(\, A(S^m\wedge X_+) \to A(*)\,) ,$$

and we define $A^S(X)$ to be its telescope,

$$A^S(X) = \varinjlim \Omega^m\ \mathrm{fibre}(\, A(S^m\wedge X_+) \to A(*)\,) .$$

The map $|X| \to A(X)$ (lemma 1.1) is a natural transformation if we write it in

the form $|NG(X)| \to A(X)$, therefore it is compatible with the stabilization process and induces a map

$$\Omega^\infty S^\infty |X_+| \longrightarrow A^S(X) \ .$$

Theorem 4.1. There is a map

$$A^S(X) \longrightarrow \Omega^\infty S^\infty |X_+| \ ,$$

well defined up to weak homotopy, so that the composite map

$$\Omega^\infty S^\infty |X_+| \longrightarrow A^S(X) \longrightarrow \Omega^\infty S^\infty |X_+|$$

is weakly homotopic to the identity map.

The proof of the theorem will occupy this section. The first step is to rewrite $A^S(X)$ in terms of the cyclic bar construction. We abbreviate

$$C_k^n(X) = N^{cy}(H(V^k S^n), Map(V^k S^n, V^k S^n \wedge |X|))$$

$$C(X) = \varinjlim_{n,k} C_k^n(X) \ .$$

Lemma 4.2. The chain of maps of theorem 3.1 induces a homotopy equivalence between

$$A^S(X)$$

and

$$\varinjlim_m \Omega^m \text{ fibre}(C(S^m \wedge X_+) \to C(*))$$

where the maps in the latter direct system are, up to homotopy, given by Ω^{m-1} applied to the vertical homotopy fibres of the stabilization diagram

$$\begin{array}{ccc} |C(S^{m-1}\wedge X_+)|^+ & \longrightarrow & |C(D_1^m \wedge X_+)|^+ \\ \downarrow & & \downarrow \\ |C(D_2^m \wedge X_+)|^+ & \longrightarrow & |C(S^m \wedge X_+)|^+ \ . \end{array}$$

The homotopy equivalence itself is well defined up to weak homotopy.

Proof. In order to get theorem 3.1 to apply to all the terms in the stabilization diagram, we replace the variables $S^m \wedge X_+$, $D_1^m \wedge X_+$, etc., by their suspensions $S(S^m \wedge X_+)$, $S(D_1^m \wedge X_+)$, etc. This can be accounted for in the end by passing to loop spaces.

In view of the naturality with respect to n, k, and the X variable, theorem 3.1 induces, for every m, a chain of natural transformations of stabilization diagrams before the + construction. By performing the + construction uniformly (for example, by attaching a 2-cell and 3-cell to $|N\, H(V^5 S^0)|$ which is contained in everything in sight) we obtain from this another chain of natural transformations of stabilization diagrams, and all the diagrams involved are still strictly commutative. So the requisite maps of homotopy fibres are well defined, and we obtain a chain of transformations connecting the m-th map of the original direct system to the m-th map of the new direct system.

By splicing these, for varying m, we obtain a chain of transformations between the original direct system and the new one. As the connectivity of the transformations increases with m, we obtain in the limit a chain of weak homotopy equivalences. To show the latter is well defined up to weak homotopy, it suffices to show that the chain of maps is well defined up to homotopy if everything in sight is replaced by a term in its Postnikov tower. But if we replace by the m-th terms in the Postnikov towers then our original direct system becomes essentially constant (the maps are weak homotopy equivalences from number m+3 on). Consequently, in view of the connectivity of the transformations, the other direct systems also become essentially constant. So the chain of maps between those terms in the Postnikov towers comes from a chain of maps at some finite stage, and this is well defined up to homotopy. □

We note that the addendum 3.2 provides a description of the map $\Omega^\infty S^\infty |X_+| \to A^S(X)$ in terms of our new definition of $A^S(X)$.

Before proceeding we state a lemma which will be needed presently.

Lemma 4.3. Suppose that Y is (m-1)-connected. Then the map

$$Map(\vee^k S^n, S^{n+m}) \wedge Map(S^{n+m}, S^{n+m}\wedge Y) \longrightarrow Map(\vee^k S^n, S^{n+m}\wedge Y)$$

given by composition, is (3m-1)-connected. Similarly, in the case $k = 1$, we obtain a (3m-1)-connected map if we compose the other way, that is, consider the map

$$Map(S^{n+m}, S^{n+m}\wedge Y) \wedge Map(S^n, S^{n+m}) \longrightarrow Map(S^{n+m}, S^{n+2m}\wedge Y)$$

obtained by stabilizing the second factor to $Map(S^{n+m}\wedge Y, S^{n+2m}\wedge Y)$, and composing.

Proof. The first map is isomorphic to the upper horizontal map in the commutative diagram

$$\begin{array}{ccc}
Map(S^n, S^{n+m})^k \wedge Map(S^{n+m}, S^{n+m}\wedge Y) & \longrightarrow & Map(S^n, S^{n+m}\wedge Y)^k \\
\uparrow & & \uparrow \\
\vee^k S^0\wedge S^m \wedge Map(S^{n+m}, S^{n+m}\wedge Y) & & \\
\uparrow & & \\
\vee^k S^0\wedge S^m \wedge Map(S^m, S^m\wedge Y) & \longrightarrow & \vee^k S^0\wedge(S^m\wedge Y) \\
 & \swarrow & \uparrow = \\
 & & (\vee^k S^0\wedge S^m)\wedge Y
\end{array}$$

The arrow on the right is (4m-1)-connected. Each of the two arrows on the left and the diagonal arrow on the bottom is the smash product of a (2m-1)-connected map with the identity on an (m-1)-connected space, hence (3m-1)-connected. So we must have the asserted connectivity of the first map.

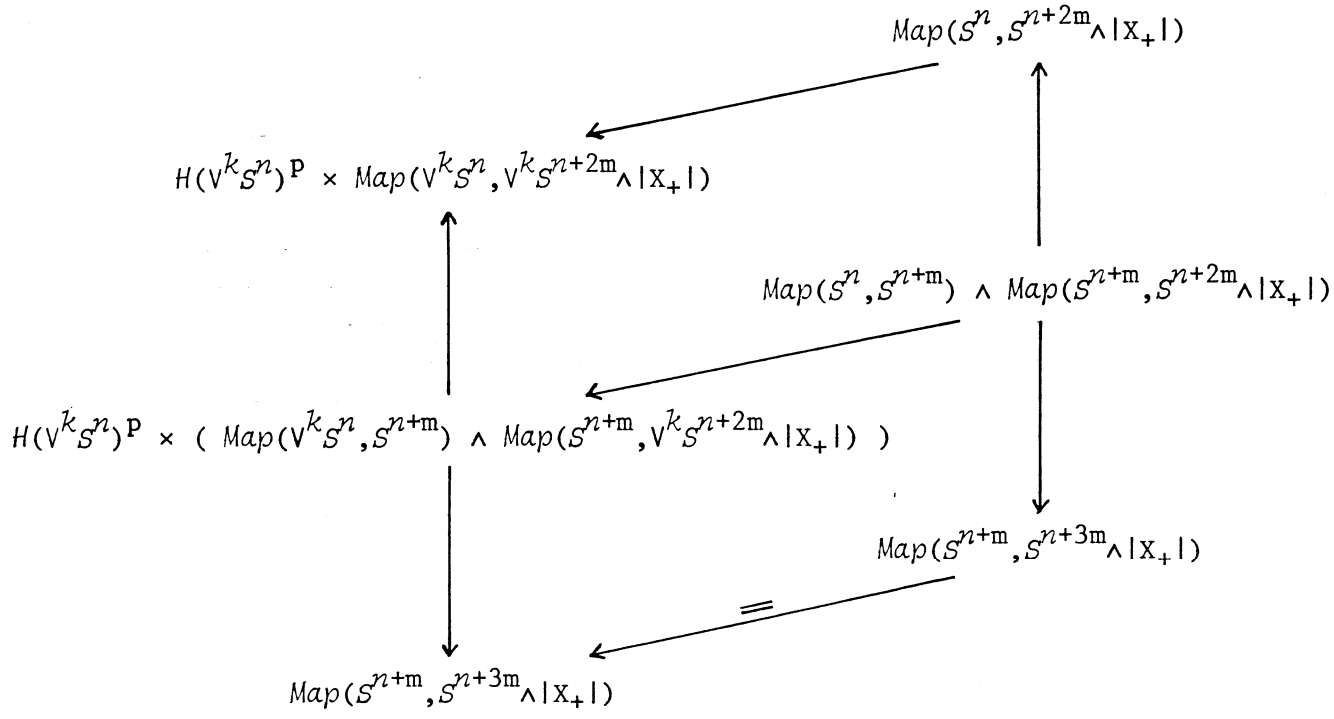
The second map is part of the commutative diagram

$$\begin{array}{ccc}
Map(S^{n+m},S^{n+m}\wedge Y)\wedge Map(S^{n},S^{n+m}) & \longrightarrow & Map(S^{n+m},S^{n+2m}\wedge Y)\\
\uparrow & & \uparrow\\
Map(S^{n+m},S^{n+m}\wedge Y)\wedge Map(S^{0},S^{m}) & & \\
\uparrow & & \\
Map(S^{0},Y)\wedge Map(S^{0},S^{m}) & \xrightarrow{\ \approx\ } & Map(S^{0},S^{m}\wedge Y)
\end{array}$$

and the same kind of connectivity considerations apply as before. □

Returning to the proof of the theorem, we will proceed in two steps. In the first step (4.4 below) we represent, in a stable range, the asserted map by a chain of two maps of which one is highly connected and has to be inverted. By taking into account some more data it will be immediate that the map is a retraction up to homotopy in that range. In the second step (4.5 below) we discuss the stabilization procedure.

(4.4). The representative in the stable range. The relevant data are displayed on the following diagram. The diagram shows the part in degree p of a commutative diagram of simplicial objects. Two of these simplicial objects are given by the cyclic bar construction (the upper and middle terms in the left column), the four others are trivial simplicial objects.



Two of the maps require comment, these are the lower vertical maps in the diagram. The one on the right is given by composition of maps *after switch of factors*. The one on the left similarly involves a switch of factors. It is the unique map of

quotient spaces induced by the following sequence of maps,

$$H(V^k S^n) \times \ldots \times H(V^k S^n) \times Map(V^k S^n, S^{n+m}) \times Map(S^{n+m}, V^k S^{n+2m} \wedge |X_+|)$$

$$\downarrow \text{(switch of factors)}$$

$$Map(S^{n+m}, V^k S^{n+2m} \wedge |X_+|) \times H(V^k S^n) \times \ldots \times H(V^k S^n) \times Map(V^k S^n, S^{n+m})$$

$$\downarrow \text{(smash product with identity maps)}$$

$$Map(S^{n+m}, V^k S^{n+2m} \wedge |X_+|) \times H(V^k S^{n+2m} \wedge |X_+|) \times \ldots \times Map(V^k S^{n+2m} \wedge |X_+|, S^{n+3m} \wedge |X_+|)$$

$$\downarrow \text{(composition of maps)}$$

$$Map(S^{n+m}, S^{n+3m} \wedge |X_+|)$$

The map is compatible with the structure maps of the cyclic bar construction. This fact, indeed, is the reason why we are using the cyclic bar construction.

Remark. The left column of the diagram really describes nothing else but a homotopy theoretic version of the *trace map*, at least in the case $p = 0$. Indeed, let R be a commutative ring and P a projective of finite type over R. Then the trace map

$$\mathrm{Hom}_R(P,P) \longrightarrow R$$

is given by the diagram

$$\mathrm{Hom}(P,P) \xleftarrow{\approx} \mathrm{Hom}(P,R) \otimes \mathrm{Hom}(R,P) \xrightarrow{\approx} \mathrm{Hom}(R,P) \otimes \mathrm{Hom}(P,R) \longrightarrow \mathrm{Hom}(R,R) \approx R$$

in which the first arrow has to be inverted, and the last arrow is given by composition of maps. In the case of general p, the left column is a version of the map *trace of the product matrix*

$$(\mathrm{Is}(P))^p \times \mathrm{Hom}(P,P) \longrightarrow R$$

which is given by the diagram

$$(\mathrm{Is}(P))^p \times \mathrm{Hom}(P,P) \xleftarrow{\approx} (\mathrm{Is}(P))^p \times \mathrm{Hom}(P,R) \otimes \mathrm{Hom}(R,P) \longrightarrow \mathrm{Hom}(R,R)$$

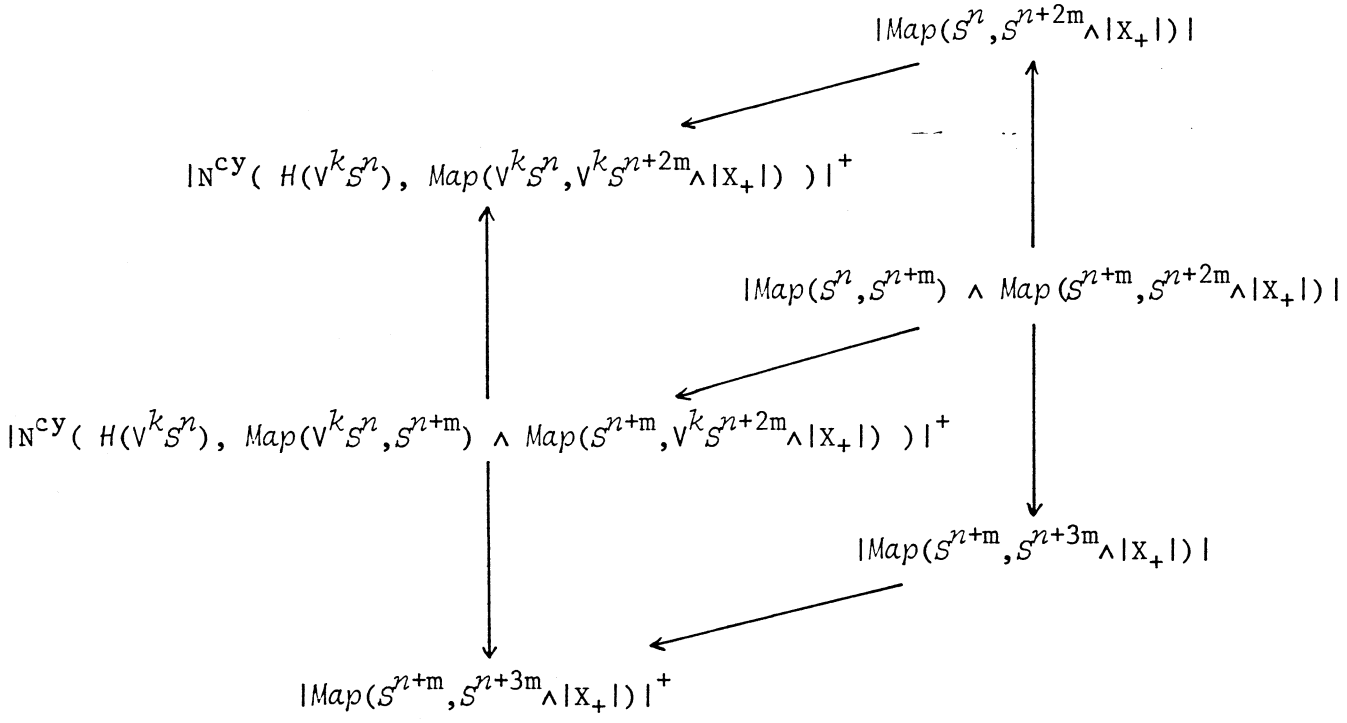
$$(g_1,\ldots,g_p,\ f) \longleftarrow\!\shortmid\ (g_1,\ldots,g_p,\ f_1 \otimes f_2) \longmapsto f_2 g_1 \ldots g_p f_1$$

This ends the remark.

Concerning the relevance of the diagram of simplicial objects described, we will

eventually have to pass to loop spaces, namely the (2m)-th loop spaces. Thus any required connectivities must increase faster than 2m. This is indeed the case. *The map on the upper left is* (3m-1)*-connected*: The realization lemma reduces us to showing this in every degree p in which case it is the content of the first part of lemma 4.3. Thus the left column does represent a map, defined in a stable range, from top to bottom. *This map is a retraction up to homotopy, in that range.* This information is provided by the rest of the diagram since the two vertical maps on the right are (3m-1)-connected by lemma 4.3 again. *The coretraction involved* (the upper horizontal map) *is a representative* (before the + construction, in a stable range) *of the map* $\Omega^\infty S^\infty |S^{2m}\wedge X_+| \to A(S^{2m}\wedge X_+)$. As noted before, this is the content of the addendum 3.2.

Passing to geometric realization and performing the + construction to the terms on the left, we obtain the diagram



The + construction is possible if k is at least 5 and it can be done uniformly with regard to the upper and middle space on the the left by attaching a pair of cells to the common subspace $|N(H(V^5S^0))|$. It preserves the connectivity of the upper map on the left (by the gluing lemma). The + construction on the bottom term on the left refers to the induced attaching of the pair of cells (a pushout). As the original term had abelian fundamental group, the + construction does not change the homotopy type. Its sole purpose is to keep the whole diagram strictly commutative.

Everything we have done so far is natural with respect to n and k, so we may pass to the direct limit in those variables (recall that stabilizing with respect to k involves wedge with an identity map on the $H(..)$ part, but wedge with a trivial

map on the $Map(..)$ part).

(4.5). The stabilization procedure. This must be adjusted to the needs of the preceding subsection. Namely the two factors S^m in $S^m \wedge S^m \wedge |X_+|$ play rather different roles, so we must stabilize in both of these factors. To do this we just alternate in stabilizing either the first or the second.

In order to stabilize in the first S^m, say, we must write down (or better, contemplate) a large diagram involving four versions of the diagram of (4.4), one for each of the terms in

$$\begin{array}{ccc} S^{m-1} & \longrightarrow & D_1^m \\ \downarrow & & \downarrow \\ D_2^m & \longrightarrow & S^m \end{array} .$$

Nothing new appears in this diagram except for fancy notations of contractible spaces as for example the factor in

$$H(V^k S^n)^P \times (Map(V^k S^n, S^n \wedge D_1^m) \wedge Map(S^n \wedge D_1^m, V^k S^n \wedge D_1^m \wedge S^m |X_+|)) .$$

These fancy terms simply ensure that the whole diagram is strictly commutative.

Taking homotopy fibres we thus do obtain from the diagram a well defined map representing

$$\text{fibre}(|C(S^{2m-1} \wedge X_+)|^+ \to A(*)) \longrightarrow \Omega \text{ fibre}(|C(S^{2m} \wedge X_+)|^+ \to A(*))$$

(where $C(..)$ is the short hand notation used before for the cyclic bar construction), namely

$$\text{fibre}(|C(S^{m-1} \wedge S^m \wedge X_+)|^+ \to |C(D_1^m \wedge S^m \wedge X_+)|^+) \longrightarrow$$
$$\text{fibre}(|C(D_2^m \wedge S^m \wedge X_+)|^+ \to |C(S^m \wedge S^m \wedge X_+)|^+) ,$$

together with a chain of two transformations (one of these in the wrong direction but highly connected) to a map representing the homotopy equivalence

$$\Omega^\infty S^\infty |S^{2m-1} \wedge X_+| \longrightarrow \Omega \, \Omega^\infty S^\infty |S^{2m} \wedge X_+| .$$

We apply Ω^{2m-1} to all this. Then we may splice, for varying m, to obtain a chain of transformations of direct systems. Passing to the limit we obtain what we are after; the appropriate concluding remarks here are similar to the proof of lemma 4.2. This completes the argument.

§5. The splitting of $A(X)$.

Let $\Omega^\infty S^\infty|X_+| \to A(X)$ be the map given by the Barratt-Priddy-Quillen-Segal theorem (section 1), and let $A(X) \to A^S(X)$ be the stabilization map (it will be defined in lemma 5.2 below).

Theorem 5.1. There is a map $A^S(X) \to \Omega^\infty S^\infty|X_+|$ so that the diagram

$$\begin{array}{ccccc} \Omega^\infty S^\infty|X_+| & & \overset{=}{\longrightarrow} & & \\ \downarrow & & & & \\ A(X) & \longrightarrow & A^S(X) & \longrightarrow & \Omega^\infty S^\infty|X_+| \end{array}$$

is weakly homotopy commutative.

Proof. This results from theorem 4.1 in view of the following lemma. □

Lemma 5.2. There is a natural stabilization map $A(X) \to A^S(X)$. Its composition with $\Omega^\infty S^\infty|X_+| \to A(X)$ is weakly homotopic to the map used in theorem 4.1.

Proof. Letting $\widetilde{A}(X)$ denote the factor in the natural splitting (section 1)

$$A(X) \simeq \widetilde{A}(X) \times A(*)$$

we define a direct system

$$A(X) \longrightarrow \Omega\, \widetilde{A}(S^1 \wedge X_+) \longrightarrow \Omega^2\, \widetilde{A}(S^2 \wedge X_+) \longrightarrow$$

in which the first map is provided by lemma 1.4, and the other maps are given by the maps of vertical homotopy fibres in the appropriate stabilization diagrams (as described in the beginning of section 4). The map from the initial term of the system to its telescope gives the required map $A(X) \to A^S(X)$.

To make the asserted comparison we consider the map of direct systems

$$\begin{array}{ccccccc} \Omega^\infty S^\infty|X_+| & \longrightarrow & \Omega\, \Omega^\infty S^\infty|S^1 \wedge X_+| & \longrightarrow & \Omega^2\, \Omega^\infty S^\infty|S^2 \wedge X_+| & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ A(X) & \longrightarrow & \Omega\, \widetilde{A}(S^1 \wedge X_+) & \longrightarrow & \Omega^2\, \widetilde{A}(S^2 \wedge X_+) & \longrightarrow \end{array}$$

where the vertical maps are the natural ones (the weak homotopy commutativity of the first square is due to lemma 1.4). The maps in the upper direct system are homotopy equivalences: the first map by lemma 1.4, and the other maps by the excision property of stable homotopy. The maps in the direct system defining $A^S(X)$ are eventually highly connected (cf. the beginning of section 4). So it will suffice to compare the vertical maps in the diagram with the map used in theorem 4.1, and to show these coincide in a stable range.

The diagram of inclusions (section 1)

$$\begin{array}{ccc} H_1^0(G) & \longrightarrow & H_\infty^0(G) \\ & \searrow & \downarrow \\ & & H_\infty^\infty(G) \end{array}$$

with $G = |G(S^m \wedge X_+)|$, induces the left part of the following diagram.

$$\begin{array}{ccccc} |S^m \wedge X_+| & \longrightarrow & \Omega^\infty S^\infty |(S^m \wedge X_+)_+| & \longrightarrow & \Omega^\infty S^\infty |S^m \wedge X_+| \\ & \searrow & \downarrow & & \downarrow \\ & & A(S^m \wedge X_+) & \longrightarrow & \tilde{A}(S^m \wedge X_+) \end{array}$$

The vertical map on the right is, up to de-looping, the same as the m-th vertical map in the diagram above, and the composite map on the bottom is an approximation to the map used in theorem 4.1. The composite map on top is the Hurewicz map from homotopy to stable homotopy (lemma 1.1), hence it is (2m-1)-connected. So the two maps in question do agree in a stable range, and the proof is complete. □

Remark 5.3. The maps in theorem 5.1 are maps of infinite loop spaces, and the diagram is weakly homotopy commutative as a diagram of infinite loop spaces.

Here is an indication of proof for the first assertion, the second involves similar considerations. Two of the maps are clearly infinite loop maps, namely the map $\Omega^\infty S^\infty |X_+| \to A(X)$ as it is the map of underlying spaces of a map of Γ-spaces, and the map $A^S(X) \to \Omega^\infty S^\infty |X_+|$ of theorem 4.1 as it was defined as the telescope of a map of spectra.

The remaining map $A(X) \to A^S(X)$ is also a map of infinite loop spaces provided that we use a possibly different infinite loop structure on $A^S(X)$. For the stabilization diagram

$$\begin{array}{ccc} A(S^{m-1} \wedge X_+) & \longrightarrow & A(D_1^m \wedge X_+) \\ \downarrow & & \downarrow \\ A(D_2^m \wedge X_+) & \longrightarrow & A(S^m \wedge X_+) \end{array}$$

is in fact the diagram of underlying spaces of a diagram of Γ-spaces. Therefore there is a Γ-space of which $A^S(X)$ is the underlying space, and the map

$$\Omega \text{ fibre}(A(S^1 \wedge X_+) \to A(*)) \longrightarrow A^S(X)$$

is a map of underlying spaces of Γ-spaces. The map

$$A(X) \longrightarrow \Omega \text{ fibre}(A(S^1 \wedge X_+) \to A(*))$$

of lemma 1.4, too, is a map of underlying spaces of Γ-spaces. Hence so is the composite map $A(X) \to A^S(X)$.

It remains to be seen that the two infinite loop structures on $A^S(X)$ are equi-

valent. In view of their definitions these infinite loop structures are compatible in the following sense. They are definable in terms of spectra (obtainable from the Γ-structure, resp. from stabilization in the X-variable) and the two spectra can be combined into a *double spectrum*. Further both spectra are connective. But this implies they are equivalent (the argument is probably well known; cf. [13, section 16] for a detailed account in a particular case).

Remark 5.4. The maps in theorem 5.1 are compatible with pairings.

Here is an indication of why this is so. In the case of $\Omega^\infty S^\infty |X_+| \to A(X)$ it is immediate from the definition of the pairings.

To treat the case of the map $A(X) \to A^S(X)$ one shows that the stabilization map

$$\text{fibre}(\, A(S^m \wedge X_+) \to A(*)\,) \longrightarrow \Omega\ \text{fibre}(\, A(S^{m+1} \wedge X_+) \to A(*)\,)$$

is the same, up to homotopy, as the adjoint of the composite map (cf. section 1)

$$S^1 \wedge \widetilde{A}(S^m \wedge X_+) \longrightarrow \widetilde{Q}(S^1) \wedge \widetilde{A}(S^m \wedge X_+) \longrightarrow \widetilde{A}(S^{m+1} \wedge X_+)$$

where $S^1 \to \widetilde{Q}(S^1)$ is the Hurewicz map (to prove this one has to use that $A(X)$ is definable in a more general context than we are using here, i.e., for X which are not necessarily pointed nor connected - cf. a similar point in the proof of lemma 1.4). Thus stabilization itself is definable in terms of the pairing, and so the pairing on $\widetilde{A}$ induces one on A^S and the required compatibility holds.

To treat the case of the map $A^S(X) \to \Omega^\infty S^\infty |X_+|$ one redefines $A^S(X)$ in terms of the cyclic bar construction (section 4). One notes that the smash product also induces a pairing in terms of the cyclic bar construction, and that this pairing is (obviously) compatible with the one on stable homotopy via the two maps of theorem 4.1 of which the map in question is one. To finish one has to chase the pairing through the chain of maps of theorem 3.1 in order to compare with the pairing formerly used. This ends the indication.

§6. Appendix: The stabilization of K-theory.

The stabilization of $A(X)$ to $A^S(X)$ may be mimicked with K-theory provided that one works with a suitably extended notion of K-theory in the framework of simplicial rings [14, section 1]. The extended notion of K-theory is needed even in the treatment of the stabilized K-theory of an ordinary ring.

We need some notation. If A is an abelian group and X a set we denote $A[X]$ the direct sum of A with itself indexed by the elements of X. Similarly $A[X]$ is defined if A is a simplicial abelian group and X a simplicial set, and is a bisimplicial abelian group (which we may diagonalize if we wish to a simplicial abelian group). If R is a (simplicial) ring and G a (simplicial) group then $R[G]$ may be equipped with a multiplication in the usual way so that it is a 'group ring'. For pointed X we let $\tilde{A}[X] = A[X]/A[*]$. If A has an R-module structure then so have $A[X]$ and $\tilde{A}[X]$, respectively.

The set of connected components $\pi_0 R$ is a ring in a natural way (the exotic case $1 = 0$ in $\pi_0 R$ may be ignored, for in this case R is contractible (multiply by a path from 1 to 0) and such an R is without interest to us); we let $K_0(\pi_0 R)$ denote its projective class group, as usual.

If A is a simplicial abelian group we denote $M_k(A)$ the simplicial abelian group of $k\times k$ matrices in A. If R is a simplicial ring then so is $M_k(R)$ and we denote $\hat{GL}_k(R)$ the multiplicative simplicial monoid of homotopy units in $M_k(R)$ (the matrices in the connected components indexed by the elements of $GL_k(\pi_0 R) \subset M_k(\pi_0 R)$). The K-theory of the simplicial ring R is, by definition,

$$K(R) = K_0(\pi_0 R) \times \lim_{\rightarrow} B\hat{GL}_k(R)^+$$

(in [14] the factor $K_0(\pi_0 R)$ was replaced by Z in order to simplify the comparison with $A(X)$).

The functor $R \mapsto K(R)$ is a homotopy functor in a suitable sense (cf. the properties of $A(X)$ stated in the beginning of section 4). In particular if $R \to R'$ is a weak homotopy equivalence then so is $K(R) \to K(R')$. It extends the K-theory of Quillen in the sense that it reduces to the latter in the case of a ring considered as a simplicial ring in a trivial way.

The *stabilized K-theory* of R is defined to be

$$K^S(R) = \lim_{\rightarrow m} \Omega^m \text{ fibre}(K(R[G(S^m)]) \to K(R))$$

where S^m denotes a simplicial set representing the m-sphere and $G(..)$ is Kan's loop group functor; the maps in the direct system are defined as in section 4. It is natural, in fact, to consider a slight generalization, the functor of two variables R and X (a simplicial set)

$$K^S(X,R) \;=\; \varinjlim_m \Omega^m \,\mathrm{fibre}(\, K(R[G(S^m\wedge X_+)]) \to K(R)\,)$$

In detail, the terms in the direct system are defined for $m > 0$, and the maps are (loops of) the maps of vertical homotopy fibres of $K(R[G(\ ?\ )])$ applied to the stabilization diagram

$$\begin{array}{ccc} S^{m-1}\wedge X_+ & \longrightarrow & D_1^m\wedge X_+ \\ \downarrow & & \downarrow \\ D_2^m\wedge X_+ & \longrightarrow & S^m\wedge X_+ \end{array}$$

This $K^S(X,R)$ is a *homology theory* in the X variable [14], the *coefficients* of the homology theory are given by $K^S(*,R) \approx K^S(R)$.

Here are some remarks about the numerical significance of stabilized K-theory. Let R be a ring (not simplicial ring), let $K_i^S(R) = \pi_i K^S(R)$. There is a spectral sequence (with trivial action in the E^2 term)

$$H_p(GL(R),K_q^S(R)) \;\Rightarrow\; H_{p+q}(GL(R),M(R))\ ,$$

with abutment the homology of $GL(R)$ acting by conjugation on $M(R)$, the essentially finite matrices in R. This is proved by the method of [14, lemma 1.5]: to deduce the existence of the spectral sequence in a stable range, one compares the spectral sequence for stable homotopy of the map $B\hat{GL}(R[G(S^m)]) \to BGL(R)$ with that of the corresponding map after the + construction. After a suitable dimension shift the latter spectral sequence has the desired E^2 term, while the former one collapses and gives the desired abutment (everything in a stable range).

Stabilized K-theory may be 'computed' in the following way. Let again R be a ring (not simplicial ring). Let $F(R)$ be the homotopy fibre

$$F(R) \;=\; \mathrm{fibre}(\, BGL(R) \to BGL(R)^+\,)\ .$$

Then $F(R)$ is an acyclic space with $\pi_1 F(R) \approx St(R)$ (the Steinberg group), and $\pi_i F(R) \approx K_{i+1}(R)$ if $i > 1$.

Denoting the homotopy fibre of the map $B\hat{GL}(R[G(S^m)]) \to BGL(R)^+$ by U, one shows that after the + construction one obtains a homotopy equivalence

$$U^+ \;\simeq\; \mathrm{fibre}(\, B\hat{GL}(R[G(S^m)])^+ \to BGL(R)^+\,)\ .$$

On the other hand U may be identified to the homotopy pullback of the diagram

$$F(R) \longrightarrow BGL(R) \longleftarrow B\hat{GL}(R[G(S^m)])\ .$$

As $U \to U^+$ is an acyclic map, the spectral sequence of a generalized homology theory h_* for the map $U \to F(R)$ therefore gives a spectral sequence

$$H_p(F(R),\, h_q\mathrm{fibre}(\, B\hat{GL}(R[G(S^m)]) \to BGL(R)\,)) \;\Rightarrow\; h_{p+q}\mathrm{fibre}(\, B\hat{GL}(R[G(S^m)])^+ \to BGL(R)^+\,)\ .$$

The fibre involved in the E^2 term may be identified, in a stable range, with the Eilenberg-Mac Lane space $B\mathcal{M}(\widetilde{R}[S^{m-1}])$. Taking h_* to be the stable homotopy groups one obtains hence that, in a stable range, the stable groups can be identified to the actual ones and the spectral sequence collapses. Whence the isomorphism

$$K_i^S(R) \approx H_i(F(R), M(R))$$

where, as one checks, the homology involves the action of $\pi_1 F(R)$ on $M(R)$ pulled back from the conjugation action of $GL(R)$. In particular,

$$K_0^S(R) \approx H_0(St(R), M(R)) \approx R/[R,R] , \qquad K_1^S(R) \approx H_1(St(R), M(R)) .$$

It will be indicated now how the results on $A(X)$ described in the earlier sections can be adapted to K-theory.

The heart of the matter is to recast the definition of stabilized K-theory in terms of the cyclic bar construction. Let Y be an m-connected simplicial set, $m \geq 0$, and let SY be its suspension. As in section 3 one constructs a natural chain of maps (five of them, just as in theorem 3.1) between $N\hat{GL}_k(R[G(SY)])$ and $N^{cy}(\hat{GL}_k(R), M_k(\widetilde{R}[SY]))$ satisfying that each of the maps in the chain is $(2m+1)$-connected. One deduces from this a homotopy equivalence

$$K^S(X,R) \simeq \varinjlim_{k,m} \Omega^m \text{ fibre}(|N^{cy}(\hat{GL}_k(R), M_k(\widetilde{R}[S^m \wedge X_+]))|^+ \to |N\hat{GL}_k(R)|^+) .$$

Let us insert here as a parenthesis how to go from this homotopy equivalence to an interesting new definition of stabilized K-theory which we do not have occasion to use, though. If R is a ring and A an R-bimodule (resp. simplicial ring and simplicial bimodule) then $R \oplus A$ can be considered as a ring (resp. simplicial ring) by giving A trivial multiplication. Now suppose that A is connected. Then there is a natural isomorphism

$$\hat{GL}_k(R \oplus A) \approx \hat{GL}_k(R) \ltimes M_k(A)$$

where the term on the right is the semi-direct product in the sense of (2.2). Hence lemma 3.1 gives a homotopy equivalence

$$\text{diag } N^{cy}(\hat{GL}_k(R), NM_k(A)) \xrightarrow{\simeq} N\hat{GL}_k(R \oplus A) .$$

On the other hand, $NM_k(A) \approx M_k(NA) \approx M_k(\widetilde{A}[S^1])$, and so we can conclude

$$K^S(X,R) \simeq \varinjlim_{m} \Omega^m \text{ fibre}(K(R \oplus \widetilde{R}[S^{m-1} \wedge X_+]) \to K(R)) .$$

Notice in particular that $\widetilde{R}[S^{m-1}]$ is just an Eilenberg-Mac Lane group, and

$$K^S(R) \simeq \varinjlim_{m} \Omega^m \text{ fibre}(K(R \oplus \widetilde{R}[S^{m-1}]) \to K(R)) .$$

This ends the parenthesis. □

Let $h(X,R)$ denote the (unreduced) homology of X with coefficients in R, it is represented by $|R[X]|$. There is a natural map $h(X,R) \to K^S(X,R)$. It arises from the homotopy equivalence $h(X,R) \simeq \varinjlim \Omega^m |\widetilde{R}[S^m \wedge X_+]|$ together with the identification of $\widetilde{R}[S^m \wedge X_+]$ with the part in degree 0 of $N^{cy}(\hat{GL}_1(R), M_1(\widetilde{R}[S^m \wedge X_+]))$.

Proposition 6.1. If R is commutative then $h(X,R) \to K^S(X,R)$ is a coretraction, up to weak homotopy.

This is the analogue of theorem 4.1. Concerning the proof, if A is an R-module (resp. simplicial R-module) considered as a bimodule in a trivial way (both the left and the right structure are given by the original module structure) then the trace map

$$\hat{GL}_k(R)^p \times M_k(A) \longrightarrow A$$
$$(g_1, .. g_p, a) \longmapsto tr(g_1 .. g_p a)$$

is insensitive to cyclic rearrangement of the factors. Therefore it is compatible with the face maps of the cyclic bar construction and defines a map

$$N^{cy}(\hat{GL}_k(R), M_k(A)) \longrightarrow A$$

which is a retraction with section as described. To complete the proof one has to check naturality with regard to stabilization, as in section 4.

One constructs a natural transformation $K(R[G(X)]) \to K^S(X,R)$ by producing artificially a map $K(R[G(X)]) \to \Omega$ fibre$(K(R[G(S^1 \wedge X_+)]) \to K(R))$ as in lemma 1.4, using pairings.

The inclusion of the 'monomial matrices', $\Sigma_k \int G(X) \to \hat{GL}_k(R[G(X)])$, induces a map, as usual, $\Omega^\infty S^\infty |X_+| \to K(R[G(X)])$.

Let $\Omega^\infty S^\infty |X_+| \to h(X,R)$ be the Hurewicz map from stable homotopy to R-homology.

Proposition 6.2. The diagram of the above maps commutes up to weak homotopy,

$$\begin{array}{ccc} \Omega^\infty S^\infty |X_+| & \longrightarrow & h(X,R) \\ \downarrow & & \downarrow \\ K(R[G(X)]) & \longrightarrow & K^S(X,R) \end{array} .$$

Putting this together with the preceding result we obtain for commutative R an analogue of the splitting theorem 5.1, a diagram

$$\begin{array}{ccccc} \Omega^\infty S^\infty |X_+| & \longrightarrow & h(X,R) & & \\ \downarrow & & \downarrow & \searrow^{=} & \\ K(R[G(X)]) & \longrightarrow & K^S(X,R) & \longrightarrow & h(X,R) \end{array}$$

that commutes up to weak homotopy and whose maps have the naturality properties indicated in section 5: they are infinite loop maps and compatible with the respective pairings.

Proposition 6.2 is the analogue of lemma 5.2, and the proof of the latter may be adapted. One can also deduce it from lemma 5.2 because of the following naturality property: there is a natural transformation

$$A(X) \longrightarrow K(R[G(X)]) ,$$

it induces a corresponding transformation of the stabilized theories, and

$$\begin{array}{ccccc} & A(X) & \longrightarrow & A^S(X) & \longleftarrow \Omega^\infty S^\infty |X_+| \\ \Omega^\infty S^\infty |X_+| \nearrow & \downarrow & & \downarrow & \downarrow \\ \Omega^\infty S^\infty |X_+| \searrow & K(R[G(X)]) & \longrightarrow & K^S(X,R) & \longleftarrow h(X,R) \end{array}$$

commutes up to (weak) homotopy, and finally in the case of commutative R so does

$$\begin{array}{ccc} A^S(X) & \longrightarrow & \Omega^\infty S^\infty |X_+| \\ \downarrow & & \downarrow \\ K^S(X,R) & \longrightarrow & h(X,R) . \end{array}$$

Using the notion of 'Hochschild homology' one can give a variant of the map $K^S(X,R) \to h(X,R)$ which is more generally defined. We no longer assume that R is commutative, but we do assume that R is given as an algebra (resp. simplicial algebra) over some commutative ring (resp. simplicial ring) k, and that it is flat over k (resp. degreewise flat).

Let A be a (simplicial) R-bimodule, over k. Following K. Dennis, one defines the *Hochschild homology*

$$H(R/k,A)$$

as the additive version of the cyclic bar construction, the simplicial object

$$[p] \longmapsto R \otimes_k \dots \otimes_k R \otimes_k A$$

(degreewise tensor product) with face and degeneracy maps as in the cyclic bar construction. We will need the fact, due to Dennis [talk at Evanston conference, January 1976, unpublished], that the Hochschild homology is *Morita invariant* in the sense of the following lemma.

Recall that two rings are called Morita equivalent if their module categories are equivalent categories. This relation is equivalent [2, chapter II] to the following property which in our present more general situation we will take as the definition.

We say that R is Morita equivalent over k with a (simplicial) k-algebra R' if there exist (simplicial) bimodules ${}_RE_{R'}$, ${}_{R'}F_R$ over k which are (degreewise) projective both from the left and the right, so that

$$E \otimes_{R'} F \approx R , \quad F \otimes_R E \approx R'$$

as (simplicial) R-bimodules, resp. R'-bimodules.

Lemma (K. Dennis). In this situation there is a natural homotopy equivalence

$$H(R/k, A) \simeq H(R'/k, F\otimes_R A\otimes_R E) .$$

Proof. Letting $B = F\otimes_R A$ we may reformulate the assertion as a homotopy equivalence

$$H(R/k, E\otimes_{R'} B) \simeq H(R'/k, B\otimes_R E) .$$

To prove this it suffices to consider the case of rings rather than simplicial rings and establish the homotopy equivalence by a chain of two natural maps. The general case then follows in view of the realization lemma. So we assume R, R' are rings, not simplicial rings.

The common source of the two maps to be constructed will be the following bisimplicial object. The object in bidegree (p,q) is given by

$$\begin{array}{ccc} & \longleftarrow p \longrightarrow & \\ & R \otimes \ldots\ldots \otimes R & \\ \otimes & & \otimes \\ E & & B \\ \otimes & & \otimes \\ & R' \otimes \ldots \otimes R' & \\ & \longleftarrow q \longrightarrow & \end{array} \qquad (H)_{p,q}$$

(tensor products over k), and the way this has been written as a circle is to suggest in which way the various face maps are given by multiplication at the appropriate tensor product signs.

Let $H(E,R'/k,B)$ be the simplicial object

$$[q] \longmapsto E \otimes R' \otimes \ldots \otimes R' \otimes B$$
$$\longleftarrow q \longrightarrow .$$

(a 'two-sided bar construction'). It maps to the trivial simplicial object $E\otimes_{R'} B$ by the map which in degree q multiplies together all the factors. This map is a homotopy equivalence. Indeed, using the right projectivity of E over R' we can reduce the assertion to the case where $E = R'$. But this case is clear (the simplicial object is a 'cone').

The bisimplicial object (H) may be identified to one

$$H(R/k, H(E,R'/k,B))$$

(a combination of the cyclic bar construction and the two-sided bar construction) and the map described just before, induces a map from this bisimplicial object to the simplicial object $H(R/k, E\otimes_R, B)$. The latter map is a homotopy equivalence degreewise in the p-direction. Indeed this follows from the homotopy equivalence established just before in view of the flatness of R over k. In view of the realization lemma it therefore follows that (H) maps by homotopy equivalence to $H(R/k, E\otimes_R, B)$.

By identifying (H) to a bisimplicial object $H(R'/k, H(B,R/k,E))$ one similarly sees that (H) maps by homotopy equivalence to $H(R'/k, B\otimes_R E)$. This completes the proof of the lemma. □

The lemma applies to the case where $R' = M_k(R)$, the $k\times k$ matrices in R. The required (simplicial) bimodules are given in this case by the 'row vectors' and 'column vectors', respectively. Hence we have a homotopy equivalence

$$H(R/k, A) \simeq H(M_k(R)/k, F\otimes_R A\otimes_R E) \simeq H(M_k(R)/k, M_k(A)) .$$

This homotopy equivalence is compatible with stabilization (stabilization is given on $M_k(R)$, resp. $M_k(A)$, by adding 1, resp. 0, in the lower right corner), one sees this by comparing stabilization with the maps involved in the lemma.

The map from $N^{cy}(\hat{GL}_k(R), M_k(A))$ to $H(M_k(R)/k, M_k(A))$ given by

$$\hat{GL}_k(R) \times \ldots \times \hat{GL}_k(R) \times M_k(A) \longrightarrow M_k(R) \otimes_k \ldots \otimes_k M_k(R) \otimes_k M_k(A)$$
$$\longleftarrow p \longrightarrow \qquad\qquad \longleftarrow p \longrightarrow$$

therefore induces a map

$$K^S(X,R) \longrightarrow H(R/k, R[X]) .$$

This map is the promised generalization of the map $K^S(X,R) \to h(X,R)$ constructed earlier. For, as one may check, it reduces to the latter in the case where R is commutative and $k = R$.

Remark. Maps like the ones here, from (unstabilized) K-theory to group homology, resp. Hochschild homology, have been constructed earlier by K. Dennis [talk at Evanston conference, January 1976, unpublished]. Dennis' constructions are somewhat different from the ones here. It remains to be seen if the maps are equivalent.

Concluding remark. It has been stressed that the material on K-theory described in this appendix is an analogue of the splitting theorem for $A(X)$. However the connection is more than just an analogy, both of these results may be considered as special cases of one and the same general result. To formulate this result one needs a common framework for $A(X)$ and K-theory.

One such common framework is a K-theory of 'rings up to homotopy'. This was indicated in [14] as a means of how to think about $A(X)$ in terms of what one is accustomed to from K-theory. In fact, it is a useful way to think about $A(X)$, occasionally: the splitting theorem for $A(X)$ was found that way (and for a while it even required the K-theory of rings up to homotopy in its proof - the only result about $A(X)$ so far which ever did that). In the long run the K-theory of rings up to homotopy may hopefully turn out to be useful as a computational tool.

The K-theory of rings up to homotopy does involve serious technical problems. The prime one is to give sense to the classifying space of the homotopy monoid of homotopy invertible matrices. May [8] has made a start in dealing with these problems, in particular he has given a definition and verified a few of the elementary properties. However as May states, there is difficulty in showing his definition is the correct one in the sense that it produces $A(X)$ from the appropriate ring up to homotopy. (There is an alternative framework in which to handle those technical problems, a notion of ring up to homotopy elaborating on one proposed by Segal [11, section 5]. Here that particular difficulty does not arise).

In this framework of rings up to homotopy and their K-theory, the general result referred to is simply propositions 6.1 and 6.2, with the abuse of allowing R to be a 'ring up to homotopy', resp. 'commutative ring up to homotopy'. Note how this explains the difference of why we get a splitting theorem in the case of $A(X)$ but not in the case of K-theory. We get a splitting theorem only if the map $\Omega^\infty S^\infty |X_+| \to h(X,R)$ is a homotopy equivalence. For this to hold, 'R-homology' must be stable homotopy, so R must be $\Omega^\infty S^\infty$ and we must be dealing with $A(X)$.

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