Self-Affine Quasiperiodic Tilings

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1 Introduction

Many quasiperiodic tilings are constructed as sections through special periodic tilings, so-called klotz tilings [1], which have tiles with boundaries either parallel or perpendicular to the cut space. It is therefore manifest that such sections are quasiperiodic tilings. Many of these quasiperiodic tilings are self-similar or self-affine, a property which can also be expressed in terms of the klotz tilings.

On the other hand, self-similar and self-affine tilings are usually defined by substitution rules. The conditions under which such substitution tilings have a pure point component in their Fourier spectrum have been analyzed in detail [2]. If the expansion map of a substitution tiling moreover induces an automorphism of the translation module of the tiling, then the tiling is quasiperiodic in the sense that the pure point component of its Fourier spectrum is supported on a finite rank Z-module [2].

We shall show that under the same conditions one can always construct a klotz tiling, whose tiles have non-empty interiors and boundaries of measure zero. This makes manifest the quasiperiodicity of such substitution tilings in a more strict sense. In particular, it implies that their Fourier spectrum is pure point and quasiperiodic.

2 Self-affinity of quasiperiodic tilings

Let $\mathcal{T}$ be a quasiperiodic tiling which is a section of a klotz tiling $\mathcal{K}$ with periodicity lattice L, and let $g = g_1 \oplus g_2$ be a lattice automorphism of L, where g_1 is expanding on the space E_1 parallel to the cut space and g_2 is contracting on its orthogonal complement E_2. Denote by π_1 and π_2 the projectors on E_1 and E_2, respectively. $\mathcal{T}$ is called self-affine if such a lattice automorphism g exists and the tiles of the klotz tiling satisfy:

$$\pi_1(gK_i) = \bigcup_{j=1}^{m} \bigcup_{\ell=1}^{S_{ij}} \pi_1(K_j - t_{ij\ell}) \qquad \text{and} \qquad \pi_2(K_j) = \bigcup_{i=1}^{m} \bigcup_{\ell=1}^{S_{ij}} \pi_2(gK_i + t_{ij\ell}). \tag{1}$$

S_{ij} indicates how many copies of tile $\pi_1 K_j$ are contained in the expansion of tile $\pi_1 K_i$. The matrix S, which is called the *substitution matrix*, is assumed to be primitive. If we multiply the relations (1) by g^{-1}, the spaces E_1 and E_2 exchange their role, and it becomes evident that the sections of $\mathcal{K}$ parallel to E_2 yield quasiperiodic tilings which are self-affine too, but this time with respect to g^{-1}. These two kinds of quasiperiodic tilings are said to be Galois-dual to each other [3,4].

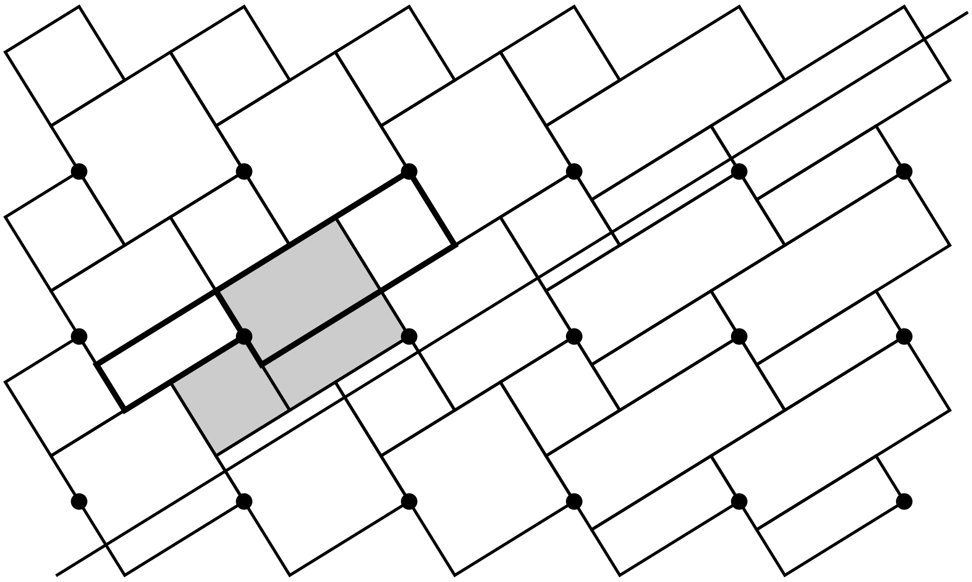

Figure 1: Klotz tiling for the Fibonacci tiling, with a typical section space. The lattice of the klotz tiling is indicated with dots. Under substitution, the two shaded tiles, which form a unit cell of the klotz tiling, are mapped to the two tiles drawn with thick lines. On the right, a part of the klotz tiling is drawn in terms of such substituted tiles.

3 Primitive substitution tilings

We now assume only a self-affine substitution tiling $\mathcal{T}$, with linear expansion map g_1, whose tiles satisfy:

$$g_1 k_i = \bigcup_{j=1}^{m} \bigcup_{\ell=1}^{S_{ij}} (k_j - t_{ij\ell}). \tag{2}$$

The substitution matrix S is again assumed to be primitive. The *translation module* T of the tiling is generated by all distances between translates of like tiles. T is finitely generated. The reference points for the different kinds of tile can be chosen such that the vectors $t_{ij\ell}$ generate the translation module T [2].

The expansion map g_1 induces a homomorphism of T into itself. With respect to a basis of T, g_1 acts by an $n \times n$ matrix M on T, where n is the rank of T. The expansion map g_1 can be complemented by an essentially unique companion map g_2 such that the map $g = g_1 \oplus g_2$ is equivalent to M. g_2 has as eigenvalues all algebraic conjugates of the eigenvalues of g_1 (with the same multiplicity), as long as these are not themselves among the eigenvalues of g_1. If all eigenvalues of g_1 are equal, the tiling is called *self-similar*, otherwise it is called *self-affine*.

The translation module T can be lifted to a lattice L such that $\pi_1 L = T$, and the mapping g then becomes a lattice homomorphism of L, whose expanding and contracting eigenspaces of are orthogonal.

4 The Fourier transform of substitution tilings

It can be shown [2] that the Fourier transform of a local density on a substitution tiling (more presicely: the Fourier transform of the autocorrelation of that density) has a pure point component if and only if the companion map g_2 of g_1 is strictly contractive. The support of this pure point component (the *Fourier module*) then is given by:

$$\mathcal{F} = \bigcup_{n=1}^{\infty} g_1^{-n} T^*, \tag{3}$$

where T^* is the *dual* or *reciprocal module* of T. If T is not a lattice, the dual module is defined as $T^* = \pi_1 L^*$. Two cases have to be distinguished [2]:

- g_1 is an automorphism of T, i.e., g_1 is surjective. The Fourier module then is identical with T^*, and thus has finite rank. Such a tiling is *quasiperiodic*.
- g_1 is not surjective on T. The Fourier module then is not of finite rank. There are two subcases: If T is a lattice, the tiling is *limit-periodic*. Otherwise it is *limit-quasiperiodic*.

The theory leading to these assertions cannot exclude the presence of a continuous component in the Fourier spectrum. There exist, in fact, examples with a continuous component. Solomyak [5,6] has given a criterion which allows to exclude a such continuous component, but in practice this criterion is often difficult to use.

5 The quasiperiodic case

We now concentrate on the quasiperiodic case, i.e., g_1 is an automorphism of T, and g_2 is contractive. We can translate the substitution rule for the tiles into a relation for their acceptance domains k_i':

$$k_j' = \bigcup_{i=1}^{m} \bigcup_{\ell=1}^{S_{ij}} (g_2 k_i' + t_{ij\ell}'), \tag{4}$$

where the vectors $t_{ij\ell}'$ are the "conjugates" of $t_{ij\ell}$, namely, the complements of the $t_{ij\ell}$ in their lift to the lattice L.

The relations (4) are the fixed-point equations of a contractive mapping in the space of all compact, non-empty subsets of E_2, endowed with the Hausdorff metric. There exists a unique solution for the k_i'. We therefore can define klotz tiles $K_i = k_i \times k_i'$, and with these we can build an L-periodic klotz tiling $\mathcal{K}$. The relative placements of the tiles is fixed by the lifts of the $t_{ij\ell}$ to the lattice L.

It still remains to be shown, however, that the periodic structure $\mathcal{K}$ obtained in this way indeed is a tiling. We first note that the solution of the fixed-point equations consists of nice sets, which are equal to the closure of their interiors. In particular, their interiors are non-empty.

To see this we suppose for the moment that the interior of k_i' is empty. Then there are sections of $\mathcal{K}$ which do not touch K_i or any of its L-translates (Baire). On the other hand, since the set of the k_i in a primitive substitution tiling is relatively dense, there exist also sections whose intersection with the translates of K_i is relatively dense. Such sections are even dense, and so there exists a sequence of relatively dense sections converging to an empty section. Since k_i' is compact, this leads to a contradiction.

Moreover, the iterates of a contractive mapping applied to any starting point always converge to the unique fixed-point. In particular, the iterates of the mapping applied to any compact subset of the *interior* of the fixed-point (which here is a set) converges to the fixed-point, which therefore must agree with the closure of its interior. The acceptance domains are therefore nice sets. They typically have a fractal boundary, but they are not fractal themselves.

Next we must show that the klotz tiles indeed fill space without overlaps. For this we assume in addition that the substitution mapping acting on tilings is injective. In other words, we assume that every tiling generated by the substitution can be composed to a unique predecessor. This is called the *unique composition property* [5], a property which for most examples is satisfied, and which is not difficult to be checked. The unique composition property implies that the interiors of the klotz tiles cannot overlap, for otherwise a section through the interior of the intersection would yield a tiling which has more than one predecessor.

On the other hand, the klotz tiles also cover space entirely. To see this we start with an infinite substitution tiling, and consider the set of klotz tiles corresponding to it. These klotz tiles cover a slab of finite width. Each tile in this slab is now covered in a unique way by "inflated" klotz tiles, which are the images of the klotz tiles under the mapping g. Such inflated klotz tiles in general intersect with more than one klotz tile. However, due to the unique composition property the coverings of different klotz tiles must be compatible, i.e., they must agree on their overlap. We therefore obtain a consistent covering of the whole slab with inflated klotz tiles.

If we now apply g^{-1} to this slab of inflated klotz tiles, we obtain a thicker slab of klotz tiles of the original size. Repeating this procedure, we arrive at an infinite, periodic klotz tiling without holes or overlaps. The generic sections of this tiling are precisely the substitution tilings we started with. Sections of such a periodic klotz tiling are manifestly quasiperiodic, and their Fourier transform must be pure point. Of course, the same holds true also for the Galois-duals of these tilings.

6 Conclusions

We have shown that *quasiperiodic* self-similar or self-affine substitution tilings can always be obtained as sections of a periodic klotz tiling. The tiles of this klotz tiling have non-empty interior, and agree with the closure of their interior. The klotz tiles fill the whole space, and overlap only on their boundaries.

The existence of such a klotz tiling makes the quasiperiodicity of the substitution tiling manifest. It implies that its Fourier transform is a pure point measure supported on a finite rank module. For the quasiperiodicity of the substitution tiling it was necessary that the expansion map g_1 induces an automorphism of the translation module T of the tiling, and that the companion map g_2 is strictly contractive. We also had to assumed the unique composition property.

The klotz tiling also provides us with the Galois dual [3,4] of the substitution tiling, which is quasiperiodic as well and has a Fourier transform which is pure point.

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