CLUSTER INTERACTIONS FOR QUASIPERIODIC TILINGS

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A cluster for the octagonal square-rhombus tiling is presented, which has the property that among all tilings completely covered by the cluster the perfectly quasiperiodic and eightfold symmetric ones have the highest cluster density. Since on these eightfold symmetric tilings there is considerable overlap of clusters, it seems likely that these tiling have the highest cluster density even among all square-rhombus tilings. An interaction favouring the cluster therefore will have ground states which are perfectly quasiperiodic and eightfold symmetric.

1 Introduction

Many quasiperiodic tilings have characteristic clusters of tiles that occur very frequently, cover the whole structure, and therefore frequently overlap with neighboring such clusters. Cluster overlaps introduce restrictions in the relative positions and orientations of clusters. This suggests that such clusters can play a primary role in the stabilization of quasicrystalline structures. Giving such clusters a low energy, their density will be maximized, which requires frequent overlaps and therefore creates order. A model exploiting this mechanism had first been proposed by Jeong and Steinhardt.[1] They showed that giving low energy to a *few selected clusters* could indeed generate quasiperiodic structures which are perfectly ordered. This was a considerable improvement over matching rule type models, in which very many clusters had to be preferred against all other clusters not allowed to occur in the structure. A similar cluster interaction approach was found to work even for a tiling which does not admit any local matching rules.[2]

Independently of such cluster interactions, the possibility of a *single* cluster completely covering a quasiperiodic tiling had been investigated.[3,4] Gummelt[4,5] found a cluster with the property that every tiling completely covered by it is a perfect Penrose tiling. It is again the large overlaps between neighboring such clusters which is responsible for this enforcement of quasiperiodic order. Jeong and Steinhardt[6] could show recently that the class of tilings having the highest density of Gummelt's cluster is precisely the class of all Penrose tilings. This shows that a cluster interaction favouring Gummelt's cluster has precisely the set of Penrose tilings as its ground states.

2 Cluster Interactions for the Octagonal Tiling

In the present paper we shall concentrate on cluster interactions for the octagonal tiling. The (undecorated) octagonal tiling is somewhat special in that it does not admit any local matching rules. Despite of this it does admit a simple cluster interaction.[2] If the two clusters shown in Fig. 1 are given low energy, where the ratio of the energies of the two clusters may vary in a large interval, the ground state consists of perfectly quasiperiodic, octagonal tilings. The reason why this is possible is that the undecorated tiling does admit local matching rules which enforce at least perfectly ordered tilings, albeit not necessarily octagonal ones. The tilings admitted by these matching rules, among them the octagonal tilings, all are quasiperiodic, and have at least fourfold symmetry.[7] The clusters shown in Fig. 1 are in fact selected in such a way that they favour these matching rules being satisfied, and their relative energies are chosen such that among the tilings admitted by the matching rules the ones with octagonal symmetry have lowest energy.

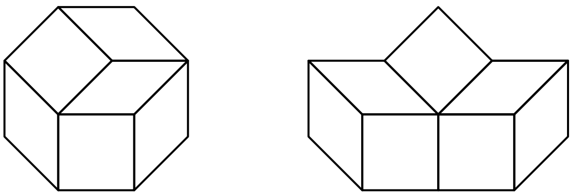

Figure 1: The octagon cluster (left) and the ship cluster (right). Favouring these two clusters leads to a quasiperiodic, octagonal ground state.

The matching rules enforcing ordered tilings of at least fourfold symmetry are given by the alternation condition, which requires that along any lane of tiles the two kinds of rhombi alternate.[2] The alternation condition can be enforced by arrowing the edges of all tiles, and requiring that the arrowing on shared edges agrees.

It is easy to see that favouring only one of the two clusters in Fig. 1 is not enough to enforce the octagonal tiling. The ground state would simply be a periodic approximant.[2] Even though the octagon cluster alone covers the whole octagonal tiling with considerable overlaps, this does not enforce anything. An octagonal tiling can be decomposed into square and rhombic supertiles of edge length $3 + 2\sqrt{2}$, which can be reassembled in many different ways without affecting the octagon density. Since the rhombic supertile has a slightly higher octagon density than the square supertile, the periodic tiling with only rhombic supertiles clearly has higher octagon cluster density than the octagonal tiling.

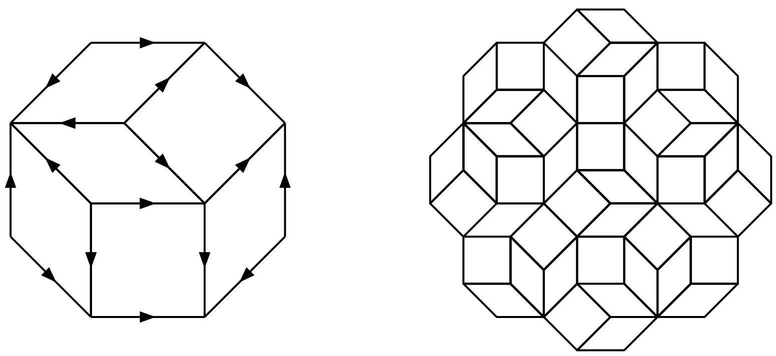

Figure 2: Arrowed octagon cluster (left), and its inflation at a smaller scale (right). The two have the same asymmetry, and enforce the same matching conditions.

If one takes *arrowed* octagon clusters, however, the situation is different. Such an arrowed octagon cluster is shown in Fig. 2 (left). A tiling completely covered by arrowed octagon clusters has to satisfy the alternation condition, and the periodic tiling with the highest density of unarrowed octagon clusters does not fall into this category. The tilings satisfying the alternation condition can again be decomposed into square and rhombic supertiles. The tiling with the highest octagon cluster density is still the one with the highest density of rhombic supertiles, but this time subject to the alternation condition. Among the tilings satisfying the alternation condition it is the octagonal tiling which has the highest density of rhombic supertiles, and therefore the highest density of arrowed octagon clusters.

If one prefers clusters without decoration, it is sufficient to inflate the arrowed cluster once (Fig. 2, left). The unarrowed cluster so obtained has precisely the same asymmetry as the arrowed octagon cluster, and therefore imposes the same matching conditions. In particular, each arrowed edge is replaced by a hexagon formed by a square and two rhombi, which has the same asymmetry as the arrowed edge. Therefore, among all tilings satisfying the alternation condition it is the octagonal tiling which has the highest density of this cluster.

The question now arises whether the octagonal tiling has the highest cluster density among *all* square-rhombus tilings, not only among those completely covered by the cluster (which implies that they satisfy the alternation condition). Since there are large overlaps of clusters in the octagonal tiling, it is very hard to imagine that there exists a tiling which is not completely covered by the cluster, but still has a higher cluster density. We do not have a formal proof that this is impossible, however. Unfortunately, the method of proof used by Jeong and Steinhardt[6] for the Penrose tiling does not seem to be applicable in our case. The reason is that in our case, and unlike to the Penrose tiling, there is not only the octagonal tiling completely covered by the cluster, but also a whole variety of other tilings with lower cluster density. Since these other tilings are locally indistinguishable from the octagonal tiling, there is little hope that one can show by *local* reasoning that the octagonal tiling has higher cluster density than any tiling not completely covered by the cluster.

3 Conclusions

We have presented a cluster for the octagonal square-rhombus tiling, which has the property that every tiling completely covered by that cluster must satisfy the alternation condition, and is therefore perfectly ordered and quasiperiodic. Among the tilings completely covered by the cluster, the octagonal tiling is the unique tiling with the highest cluster density. Although we have no rigorous proof, there are strong indications that the octagonal tiling has the highest cluster density even among all square-rhombus tilings.

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References

1. H.-C. Jeong and P. J. Steinhardt, *Phys. Rev. Lett.* **73**, 1943 (1994).
2. F. Gähler and H.-C. Jeong, *J. Phys. A* **28**, 1807 (1995).
3. M. Duneau, *Proceedings of the 5th International Conference on Quasicrystals*, eds. C. Janot and R. Mosseri (World Scientific, Singapore, 1995), p. 116.
4. P. Gummelt, *Proceedings of the 5th International Conference on Quasicrystals*, eds. C. Janot and R. Mosseri (World Scientific, Singapore, 1995), p. 84.
5. P. Gummelt, *Geometriae Dedicata* **62**, 1 (1996).
6. H.-C. Jeong and P. J. Steinhardt, *Phys. Rev.* B **55**, 3520 (1997).
7. A. Katz, in *Beyond Quasicrystals*, eds. F. Axel and D. Gratias (Les Editions de Physique and Springer Verlag, 1995), p. 141.