Amplitude Equations

natural slow-fast systems for SPDEs

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Introduction

- Complicated models near a *change of stability*
  $\Rightarrow$ Natural slow-fast system
- Dominant pattern evolve on a *slow time-scale*
- Stable pattern decay/disappear on a *fast time-scale*
- Evolution of dominant modes given by simplified model
  $\Rightarrow$ Amplitude equations

Many examples of formal derivation, e.g. [Cross,Hohenberg,'93]

Known Deterministic Results for PDEs

- PDEs on bounded domains:
  Invariant manifolds – center manifold
  Solutions approximated by an ODE on the manifold
- PDEs on unbounded domains:
  Many rigorous results (cf. Schneider, Uecker, et.al)
  Validation of Amplitude or Modulation equations
  as reduced models

Our Problem:

- What is the impact of noise on the dominant pattern?
- No center manifold theory for SPDEs available

Our Aim

AIM:

- Validation of amplitude equations for SPDEs
- Understand effect of noise transported by the nonlinearity
- Here only one example: The Swift-Hohenberg model
- noise constant in space or space-time white

The Swift-Hohenberg model is a celebrated model in pattern formation. The first change of stability is a toy model for the convective instability in Rayleigh-Benard convection.

Slow-Fast System I

SPDE close to bifurcation generate naturally slow-fast system

A slow SDE coupled to a fast SPDE.

Consider

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \epsilon^2 \partial_t W \qquad \text{(SPDE)}$$

- $\mathcal{L}$ non-positive operator - kernel $\mathcal{N}$ (dominant modes)
- $\nu\epsilon^2$ the distance from bifurcation
- ϵ^2 noise strength
- $\{W(t)\}_{t\geq 0}$ some Wiener process

Slow-Fast System II

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \epsilon^2 \partial_t W \qquad \text{(SPDE)}$$

Split

$$u(t) = \epsilon v_c(\epsilon^2 t) + \epsilon v_s(\epsilon^2 t)$$

with $v_c \in \mathcal{N}$ and $v_s \perp \mathcal{N}$

$$\partial_T v_c = \qquad \nu v_c - P_c(v_c + v_s)^3 + \partial_T \tilde{W}_c \qquad \text{(SLOW)}$$

$$\partial_T v_s = \epsilon^{-2}\mathcal{L} v_s + \nu v_s - P_s(v_c + v_s)^3 + \partial_T \tilde{W}_s \qquad \text{(FAST)}$$

where

P_c projects onto $\mathcal{N}$ and $P_s = I - P_c$

$\tilde{W}(T) = \epsilon W(T\epsilon^{-2})$ rescaled Wiener process

Outline

Consider Swift-Hohenberg equation only.

- A (very) brief review for full noise on bounded domains

  [DB, Maier-Paape, Schneider 01], [DB, Hairer 04,05]

- Stabilization effects arising from degenerate noise

  [Hutt, et.al., 07,08], [DB, Mohammed 10], [Roberts 03], [DB, Hairer, Pavliotis 07]

- work in progress on unbounded domains

  [DB, Klepel, Mohammed]

- Outlook / Open Problems

Consider the Swift-Hohenberg equation

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \epsilon^2 \partial_t W \qquad \text{(SH)}$$

with

- $\mathcal{L} = -(1+\partial_x^2)^2$
- periodic boundary conditions on $[0, 2\pi]$
- dominant modes $\mathcal{N} = \text{span}\{\sin, \cos\}$
- space-time white noise $\partial_t W$

Theorem (Attractivity) [DB, Hairer 04]

For all small $T > 0$: $u_c(T\epsilon^{-2}) = \mathcal{O}(\epsilon)$ and $u_s(T\epsilon^{-2}) = \mathcal{O}(\epsilon^2)$

Theorem (Approximation) [DB, Hairer 04]

$u(0) = \epsilon a(0) + \epsilon^2 \psi(0)$ with $a(0)$ and $\psi(0)$ both $\mathcal{O}(1)$.
Let $a(T) \in \mathcal{N}$ solve

$$\partial_T a = \nu a - P_c a^3 + \partial_T \tilde{W}_c$$

and $\psi(t) \perp \mathcal{N}$ solves $\quad \partial_t \psi = \mathcal{L}\psi + \partial_t W_s$

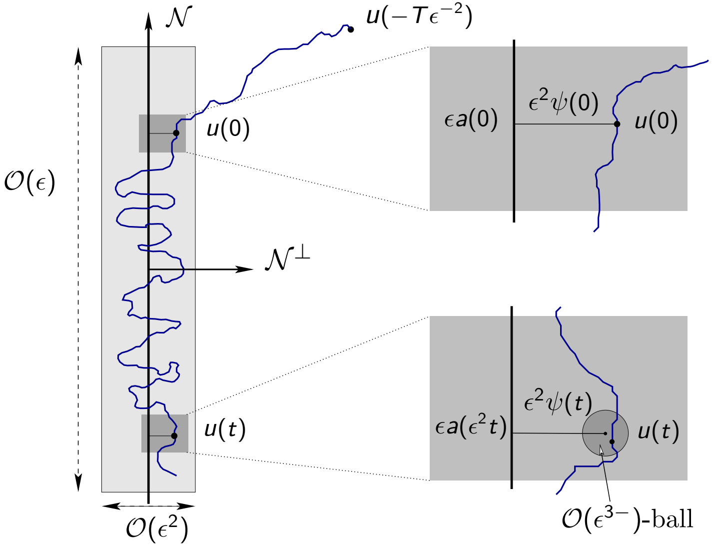
Then for $t \in [0, T_0\epsilon^{-2}]$

$$u(t) = \epsilon a(\epsilon^2 t) + \epsilon^2 \psi(t) + \mathcal{O}(\epsilon^{3-})$$

Remark: Approximation remains true for invariant measures.

Approximative Center Manifold

[DB, Hairer 05]



Stabilization due to Noise

Interesting result:

Additive noise may lead to stabilization (or a shift of bifurcation) of dominant modes (pattern disappears)

Stabilization due to Noise

Well known phenomenon due to multiplicative Noise. For example:

By Itô noise, due to Itô-Stratonovic correction, or Stratonovic noise due to averaging over stable and unstable directions

- **For SDE:** [Arnold, Crauel, Wihstutz '83], [Pardoux, Wihstutz '88 '92].....
- **For SPDE:** [Kwiecinska '99], [Caraballo, Mao et.al. '01], [Cerrai '05], [Caraballo, Kloeden, Schmallfuß '06]....

By Rotation: [Baxendale et.al.'93], [Crauel et.al.'07].....

Only very few examples due to additive noise:

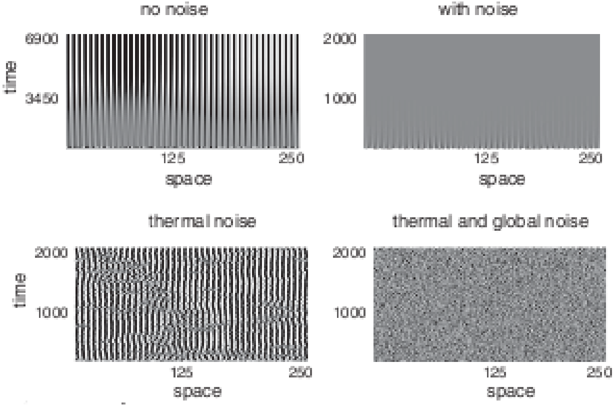
Blow up through a small tube: e.g. [Scheutzow et.al.'93]...

Numerical Examples

Swift-Hohenberg-model

[Hutt, Schimansky-Geier et.al. 07, 08]

Space-independent (gobal) noise destroys modulated pattern in 1D-Swift-Hohenberg-Eq.



Swift-Hohenberg Equation

Example: Swift-Hohenberg Equation

$$\partial_t u = -(1+\partial_x^2)^2 u + \tfrac{1}{20} u - u^3 + \tfrac{\sigma}{10}\partial_t \beta(t) \qquad \text{(SH)}$$

- $u(t,x) \in \mathbb{R}, \ t > 0, \ x \in [0, 2\pi]$
- periodic boundary conditions

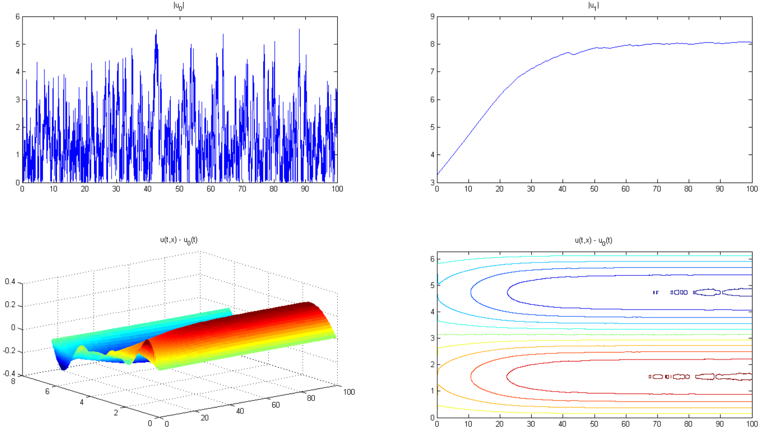
Observation:

- 0 is is stabilized by large noise
- Noise only in time, constant in space

Swift-Hohenberg

(Nolde/Wöhrl)

$\sigma = 0.5$

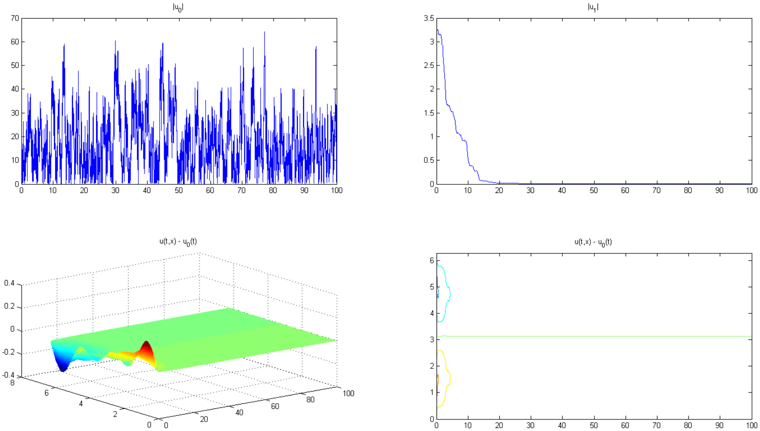


Semi-implicit spectral Galerkin-method using fft in Matlab

Swift-Hohenberg

(Nolde/Wöhrl)

$\sigma = 5$



Semi-implicit spectral Galerkin-method using fft in Matlab

Results for Swift-Hohenberg Equation

The SPDE – Swift-Hohenberg

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \sigma\epsilon\,\partial_t\beta \tag{SH}$$

- $\mathcal{L} = -(1+\partial_x^2)^2$
- periodic boundary conditions on $[0, 2\pi]$
- $\mathrm{span}\{\sin, \cos\}$ – dominant pattern
- β is a real-valued Brownian motion
- $\sigma\epsilon \ll 1$ – noise strength
- $|\nu|\epsilon^2 \ll 1$ – distance from bifurcation

The Ansatz

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta \qquad \text{(SH)}$$

Ansatz:

$$u(t,x) = \epsilon A(\epsilon^2 t)\mathrm{e}^{ix} + c.c. + \epsilon Z(t) + \mathcal{O}(\epsilon^2)$$

fast process – $Z(t) = \int_0^t e^{-(t-\tau)} d\beta(\tau)$

complex-valued amplitude – A

Result: Amplitude Equation [DB, Mohammed, '10]

$$\partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \qquad \text{(A)}$$

Interesting Facts

$$\partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \qquad \text{(A)}$$

- Amplitude equation is deterministic
- Noise leads to a stabilizing deterministic correction

New contribution to the amplitude equation:

$$3A \cdot \sigma^2 (\epsilon \partial_T \tilde{\beta})^2 \ ,$$

where $\tilde{\beta}(T) = \epsilon\beta(T\epsilon^{-2})$
is a Brownian motion on the slow time-scale $T = \epsilon^2 t$

Noise^2?

What is noise^2?

Instead of $\epsilon\partial_T\tilde{\beta}$ consider a fast Ornstein-Uhlenbeck-process

$$Z(T\epsilon^{-2}) = Z_\epsilon(T) = \epsilon^{-1}\int_0^T e^{-(T-s)\epsilon^{-2}} d\tilde{\beta}(s) \approx \epsilon\partial_T\tilde{\beta}(T)\ ,$$

where $\tilde{\beta}(T) = \epsilon\sigma\beta(\epsilon^{-2}T)$ is a rescaled Brownian motion

[DB, Hairer, Pavliotis, '07] [DB, Mohammed '08] Averaging with error bounds

$dX = \mathcal{O}(\epsilon^{-r})dt + \mathcal{O}(\epsilon^{-r})d\tilde{\beta}$ and $X(0) = \mathcal{O}(\epsilon^{-r})$, $r > 0$,
then

$$\int_0^T X(s)Z_\epsilon(s)^2 ds = \frac{1}{2}\int_0^T X(s)ds + \mathcal{O}(\epsilon^{1-2r})$$

other averaging results

For odd powers:

$$\int_0^T X(s)Z_\epsilon(s)ds = \mathcal{O}(\epsilon^{1-r})$$

$$\int_0^T X(s)Z_\epsilon(s)^3 ds = \mathcal{O}(\epsilon^{1-3r})$$

and so on ...

Note: $X = \mathcal{O}(f_\epsilon)$
if $\forall\ p > 1,\ T > 0$ there is a $C > 0$ such that

$$\mathbb{E} \sup_{[0,T]} |X|^p \leq C f_\epsilon^p$$

The Theorem

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta \qquad \text{(SH)}$$

$$\partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \qquad \text{(A)}$$

Theorem – Approximation [DB, Mohammed '10]

u is solution of (SH) in H^1 – A is solution of (A)
$u(0) = \epsilon\, A(0)\mathrm{e}^{ix} + c.c. + \epsilon\psi_0$ with $\psi_0 \perp \mathrm{e}^{\pm ix}$.

Then for κ, T_0, $p > 0$ there is $C > 0$ such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0/\epsilon^2]} \|u(t) - \epsilon v(\epsilon^2 t)\|_{H^1} > \epsilon^{2-\kappa}\Big) < C\epsilon^p + \mathbb{P}(\|u(0)\|_{H^1} > \epsilon^{1-\kappa/2})$$

with $v(T) = A(T)\mathrm{e}^{ix} + c.c. + Z_\epsilon(T) + e^{T\mathcal{L}\epsilon^{-2}}\psi_0$.

Remark: Theorem holds in a much more general setting.

Numerical justification

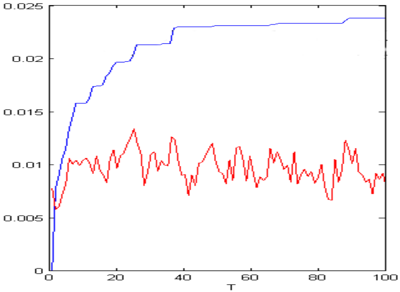
(Wöhrl)

Numerical approximation of

$$r(T) = \|\epsilon^{-1}u(T\epsilon^{-2}) - \left[Z_\epsilon(T) + A(T)e^{ix} + c.c.\right]\|_\infty$$

$$\text{and} \quad \sup_{s\in[0,T]} r(s)$$

$\nu = 1$, $\sigma = \sqrt{3/5}$, $\epsilon = 1/100$,
mean over a few (< 100) realizations



Open Problem

If we consider higher order corrections [DB, Mohammed, '10]

or Burgers equation & degenerate noise [DB, Hairer, Pavliotis, 07]

we need to treat terms like

$$\int_0^T X(s) Z_\epsilon(s)^2 d\tilde{\beta}(s)$$

Problem:

In order to derive *error estimates* we rely on Martingal representation/approximation results.

Thus $\dim(\mathcal{N}) = 1$ is necessary ϟ

For $\dim(\mathcal{N}) > 1$ weak convergence results available

$\Longrightarrow$ **Averaging** (well known)

Martingale Approximation Result

For $\int_0^T f(a, Z_\epsilon) d\tilde{\beta}$ with $Z_\epsilon(t) = \frac{1}{\epsilon} \int_0^t e^{-(t-s)\lambda\epsilon^{-2}} d\tilde{\beta}(s)$.

Lemma [DB, Hairer, Pavliotis, '07]

$M(t)$ continuous martingale with quadratic variation f

g arbitrary adapted increasing process with $g(0) = 0$

Then (with respect to an enlarged filtration) there exists a continuous martingale $\tilde{M}(t)$ with quadratic variation g

such that $\forall\, \gamma < 1/2 \;\; \exists\, C > 0$ with

$$\begin{aligned} \mathbb{E} \sup_{[0,T]} |M - \tilde{M}|^p \quad &\leq \quad C\, \mathbb{E} \sup_{[0,T]} |f - g|^{p/2} \\ &\quad + C \big(\mathbb{E} g(T)^{2p}\big)^{1/4} \big(\mathbb{E} \sup_{[0,T]} |f - g|^p\big)^\gamma . \end{aligned}$$

Unbounded or Large Domains

Modulated Pattern

Problem:

A full band of eigenvalues changes stability

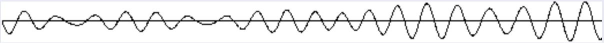
Modulation of dominant pattern

Periodic Pattern

$u(x) \approx \epsilon A e^{ix} + c.c.$ where $A \in \mathbb{C}$

Modulated Pattern (many modes near ± 1 contribute)

$u(x) \approx \epsilon A(\epsilon x) e^{ix} + c.c.$ where $A : \mathbb{R} \mapsto \mathbb{C}$



Results on Large Domains

$$\partial_t u = \mathcal{L}u + \epsilon^2 \nu u - u^3 + \epsilon^{\frac{3}{2}} \xi \qquad (SH)$$

Results for deterministic PDE (on $\mathbb{R}$):

[Kirrmann, Mielke, Schneider, 92] and many more ...

Result for SPDE (on $[-L/\epsilon, L/\epsilon]$):

Only on large domains [DB, Hairer, Pavliotis '05]

$$u(t,x) = \epsilon A(\epsilon^2 t, \epsilon x) \cdot \mathrm{e}^{ix} + c.c. + \mathcal{O}(\epsilon^2)$$

$A(T,X) \in \mathbb{C}$ solves stochastic Ginzburg-Landau equation:

$$\partial_T A = 4\partial_X^2 A + \nu A - 3|A|^2 A + \sigma\eta \qquad \text{(GL)}$$

space-time white noise $\eta(T,X) \in \mathbb{C}$, even if ξ colored in space!

Degenerate Noise on $\mathbb{R}$

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta \qquad \text{on } \mathbb{R} \qquad \text{(SH)}$$

Ansatz:

$$u(t,x) = \epsilon A(\epsilon^2 t, \epsilon x)\mathrm{e}^{ix} + c.c. + \epsilon Z(t) + \mathcal{O}(\epsilon^2)$$

fast OU-process $Z(t) = \int_0^t e^{-(t-\tau)} d\beta(\tau)$

Result: Amplitude Equation [DB, Mohammed, '11]

$$\partial_T A = 4\partial_X^2 A + (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \qquad \text{(GL)}$$

Open Problems – Regularity of A

- Kirrmann-Mielke-Schneider needed $A \in C_b^{1,4}([0,T] \times \mathbb{R})$
- DB-Hairer-Pavliotis needed $A = B + Z$
  with $B \in C^0([0,T], H^1(-L,L))$
  and $Z \in C^0([0,T] \times [-L,L])$ Gaussian
- DB-Mohammed needed $A \in C^0([0,T], H^\alpha(\mathbb{R}))$, $\alpha > 1/2$

But: Amplitude A is for space-time white noise only Hölder continuous and $\|A(T,\cdot)\|_\infty = \infty$ for all $T > 0$.

And: Bounds for terms like

$$e^{t\mathcal{L}}[A(\epsilon x)e^{ix}] \approx [e^{4T\partial_X^2} A](\epsilon x) \cdot e^{ix}$$

require some regularity of A.

Outlook

Further results / Work in Progress

- Attractivity results — usually straightforward
- Higher order corrections – Martingale terms [Roberts, Wei Wang, '09], [DB, Mohammed '11]
- Large Domains – Modulated Pattern [DB, Klepel, Mohammed]
- Levy noise [DB, Hausenblas]
- Local shape of random invariant manifolds [DB, Wei Wang, '09]
- Approximation of invariant measures [DB, Hairer, '04]

Summary

- SPDEs near a change of stability
- Transient dynamics via amplitude equations
- Stabilisation due to additive noise
- Effect of noise on dominant modes
- Noise transported by nonlinearity between Fourier-modes