Nonlinear dispersive equations, solitary waves and noise

A. de Bouard

CMAP, Ecole Polytechnique, France
joint works with A. Debussche, E. Gautier, R. Fukuizumi

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Nonlinear dispersive equations

▶ model equations for propagation of waves in nonlinear dispersive media

▶ “Universal models” : water waves, nonlinear optics, plasma physics, energy transfer in molecular systems, Bose-Einstein condensation ...

▶ conservative equations

▶ In general

$$i\partial_t u + P(D)u + F(u) = 0$$

where $$u = u(t, x)$$ is the unknown and

$$\widehat{P(D)u}(\xi) = p(\xi)\hat{u}(\xi)$$

where $$p$$ is real valued; dispersion relation : $$\omega(k) = -p(k)$$
Examples

Korteweg - de Vries equations

\[(KdV)\]
\[\partial_t u + \partial_x^3 u + \partial_x(u^2) = 0\]

- Boussinesq, 1872, Korteweg- de Vries, 1895 after observation by J. Scott Russel, 1834
  http://www.ma.hw.ac.uk/solitons

- asymptotic model for long waves at the surface of water (small amplitude, shallow water, unidirectional propagation)
  rigorous derivation (from free surface Euler equations) : W. Craig, CPDE, 1985

- here, \(p(k) = k^3\) (strong dispersion)

Examples

Nonlinear Schrödinger equations

\[(NLS) \quad i \partial_t u + \Delta u + F(u) = 0\]

- here, \(p(k) = |k|^2\)
- describes envelope of wave packets in water waves (deep water) Zakharov, 1968, \(F(u) = \kappa |u|^2 u\)
- propagation of light in dispersive and nonlinear media (e.g. optic fibers) \(F(u) = \kappa |u|^2 u\)
  - rigorous derivation (as envelope, from Maxwell equations) : Donnat, Joly, Metivier, Rauch, 1996
  - in optic fibers : special case of the Manakov system
- Bose-Einstein condensation : Gross-Pitaevskii equation \(F(u) = V(x)u + \kappa |u|^2 u\)
Integrable equations

Some of those equations are integrable by inverse scattering

\[ \leadsto \] infinity of invariant functionals

\[ \leadsto \] existence of solitons: localized solutions propagating without change of form, elastic interactions

\[ \leadsto \] resolution into solitons + dispersive tail

KdV: Gardner, Green, Kruskal, Miura, 1967

cubic NLS equations in dimension one \((x \in \mathbb{R})\): Zakharov-Shabat, 1972

Non integrable equations: higher dimension, higher order nonlinearity, external (possibly random) forces... which of those properties still hold? sometimes still localized solutions: solitary waves
Why adding noise?

KdV-type equations (no rigorous mathematical derivation)

- **random pressure field** (water surface waves): e.g., turbulent wind on the surface
  - additive noise, white in time $\dot{\xi}(t, x)$

- **variations of the bottom topography** modeled by a stationary (in $x$) random process:
  - add a term $(\partial_x u)\dot{\xi}(t)$, white in time

Craig, de Bouard, Diaz-Espinoza, Guyenne, Sulem, Nonlinearity, 2008

  - random potentials: $u\dot{\xi}(t, x)$ or $(\partial_x u)\dot{\xi}(t, x)$

- model for weak turbulence (Kuksin et al.), random perturbations of nonlinearity, velocity or dispersion (Garnier)
Why adding noise?

NLS-type equations

- **temperature effects** (light propagation in molecular systems): \( u\xi(t, x) \)
  Bang, Christiansen, If, Rasmussen, Gaididei, Phys. Rev. E, 1994

- **inhomogeneities in the medium** (e.g. optic fibers): \( \xi(t, x) \)
  Falkovich, Kolokolov, Lebedev, Turitsyn, Phys. Rev. E, 1994
  amplifiers: \( \xi(t, x) \)
  dispersion management \( \xi(t)\Delta u \)

- **fluctuations of the laser frequency** (Bose-Einstein condensation): induces fluctuations of the confining potential
  Abdullaev, Baizakov, Konotop, Nonlinearity and Disorder, 2001 \( \xi(t)V(x) \)
Solitary waves

Localized solutions propagating without change of form (due to nonlinearity + dispersion)

- **travelling waves**: \( u(x, t) = \varphi_v(x - vt), \ v \in \mathbb{R}^d \)
- **standing waves**: \( u(x, t) = e^{i\omega t} \varphi_\omega(x), \ \omega \in \mathbb{R} \)

Symmetries of equations lead to more general families of solutions

**KdV-type equations**: two parameter family of solutions

\[
 u_{c,x_0}(x, t) = \varphi_c(x - ct - x_0), \ c > 0, x_0 \in \mathbb{R}
\]

**NLS-type equations**: \(2d + 2\)-parameter family of solutions

\[
 u_{\omega,\theta,v,x_0}(x, t) = \varphi_\omega(x - 2vt - x_0)e^{i(\omega t - v \cdot x + v^2 t + \theta)}
\]

(translation, Galilean and Gauge transformations)
Solitary waves

Integrable cases: resolution into soliton for any localized solution

**Theorem:** (Eckaus-Schuur, 1986, Deft-Zhou, 1993)

Let $u(x, t)$ be a regular and localized solutions of the KdV equation; then there exist $N \in \mathbb{N}$, $c_1, \ldots, c_N > 0$, $\gamma_1, \ldots, \gamma_N \in \mathbb{R}$, $\nu > 0$ and $\mu > 0$ such that

$$\lim_{t \to \infty} \sup_{x \geq -\mu - \nu t^{1/3}} |u(x, t) - \sum_{k=1}^{N} \varphi_{c_k}(x - c_k t - \gamma_k)| = 0$$
Solitary waves

Symplectic formulation of the equation

$$\partial_t u = JH'(u)$$

where $J = \partial_x$ for (KdV), $J = i$ for (NLS) and

\[(KdV) \quad H(u) = \frac{1}{2} \int (\partial_x u)^2 dx - \frac{1}{3} \int u^3 dx\]

energy or Hamiltonian

\[(NLS) \quad H(u) = \frac{1}{2} \int |\nabla u|^2 dx - \frac{1}{4} \int |u|^4 dx\]

the mass or charge is also conserved:

$$m(u) = \frac{1}{2} \int |u|^2 dx$$

Solitary wave = critical point of the action functional

$$\mathcal{E}_\omega(u) = H(u) + \omega m(u)$$

Ground state = critical point minimizing $\mathcal{E}_\omega$
Solitary waves

Numerical methods for computations of solitary waves

- Shooting methods (radial solutions)

\[ \begin{cases} 
\varphi''(r) + \frac{d-1}{r}\varphi'(r) - \omega \varphi(r) + \kappa |\varphi(r)|^{2\sigma} \varphi(r) = 0 \\
\varphi(0) = \beta, \ \varphi'(0) = 0 
\end{cases} \]

+ dichotomy argument to find \( \beta^* \) such that \( \lim_{r \to \infty} \varphi(r) = 0 \)

Excited states, central vortices, systems of equations
Di Menza, M2AN, 2009

- Complex time integration: solve the heat flow

\[ \partial_t u = H'(u) \]

with renormalization of the \( L^2 \)-norm at each time step \( \leadsto \) convergence to a solitary wave
Solitary waves

computation of the ground state of NLS by shooting method
(Di Menza, 2009)
Solitary waves

computation of excited state of NLS by shooting method
(Di Menza, 2009)
Solitary waves

computation of excited state of NLS by shooting method
(Di Menza, 2009)
Solitary waves

computation of vortices of NLS by shooting method
(Belaouar, 2011)
Solitary waves

computation of vortices of 3-D NLS by complex time integration
(Danaïla, 2006)
Solitary waves : orbital stability

Define

\[(GKdV) \quad U_\varepsilon = \left\{ u \in H^1, \inf_{s \in \mathbb{R}} \| u - \varphi_\omega (\cdot - s) \|_{H^1} \leq \varepsilon \right\}\]

\(\varphi_\omega\) is orbitally stable if, given \(\varepsilon > 0\), any solution starting sufficiently close to \(\varphi_\omega\) stays in \(U_\varepsilon\) for all time.

**Theorem:** \(\varphi_{\omega_0}\) is stable iff \(\mathcal{E}_\omega(\varphi_\omega)\) is a convex function of \(\omega\) near \(\omega_0\)

Benjamin, Bona-Souganidis-Strauss, Weinstein, Grillakis-Shatah-Strauss
**Key point:** study of \( L_\omega = \mathcal{E}_\omega''(\varphi_\omega) \) (not positive definite)

- If \( \frac{d}{d\omega} m(\varphi_{\omega_0}) > 0 \) then \( \varphi_{\omega_0} \) is a minimum of \( H \) for constant \( m \).

- If \( \frac{d}{d\omega} m(\varphi_{\omega_0}) < 0 \) then there is a curve \( \psi_\alpha \) with \( m(\psi_\alpha) = m(\varphi_{\omega_0}) \) on which \( H \) is maximized at \( \varphi_{\omega_0} \).
Solitary waves: asymptotic stability

Linearization: 
\[ u(t, x) = \varphi_\omega(x - \omega t) + v(t, x - \omega t) \quad \text{(GKdV)} \]

\[ \frac{dv}{dt} = JL_\omega v \]

\[ \sigma_e(JL_\omega) = i\mathbb{R} \]

- Nullspace generated by invariances of equation \( \text{ex (GKdV)} \):
\[
\begin{align*}
\partial_x L_\omega(\partial_x \varphi_\omega) &= 0 \\
\partial_x L_\omega(\partial_\omega \varphi_\omega) &= -\partial_x \varphi_\omega
\end{align*}
\]
Solitary waves : asymptotic stability

NLS : Buslaev-Perelman, 1992, Buslaev-Sulem, 2003

Principal part written as a modulated solitary wave :

\[ u(t, x) = \varphi_{\omega(t)}(x - x(t)) + v(t, x - x(t)) \]

and \( \omega(t), x(t) \) chosen in order to eliminate secular modes

- Convergence to a solitary wave with shifted parameters
  \( \omega(t) \to \omega^+ \), \( x(t) \sim \omega^+ t + x^+ \) as \( t \to +\infty \)

- Convergence of \( v \) in a space with exponential weight (or locally in space)

- Also true for \( n \)-solitary wave solutions
  Martel-Merle-Tsai, 2003

- Recent results on non elastic collision of solitons for GKdV
  Martel-Merle. 2009
Stochastic case: KdV

\[ du + (\partial_x^3 u + \partial_x (u^2)) dt = \begin{cases} \varepsilon dW \\ \varepsilon udW \end{cases} \]

\(W(t)\) infinite dimensional Wiener process i.e.

\[ W(t, x) = \sum_j \Phi(e_j)(x) W_j(t) \]

\(W_j\) indep. 1-D BM, \((e_j)\) c.o.s. in \(L^2(\mathbb{R})\),

- \(\Phi\) Hilbert-Schmidt operator from \(L^2(\mathbb{R})\) into \(H^1(\mathbb{R})\), if additive noise
- \(\Phi(e_j) = k * e_j\), with \(k \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})\) if multiplicative noise; equivalently

\[ W(t, x) = \int_0^t \int_{\mathbb{R}} k(x - y) B(ds, dy) \]

where \(B\) is a Brownian sheet on \(\mathbb{R}^+ \times \mathbb{R}\).
Stochastic case : KdV

Let \( u^\varepsilon(0, x) = \varphi_{\omega_0}(x) \); write the solution \( u^\varepsilon \) of the stochastic equation as

\[
    u^\varepsilon(t, x) = \varphi_{\omega^\varepsilon(t)}(x - x^\varepsilon(t)) + \varepsilon \eta^\varepsilon(t, x - x^\varepsilon(t))
\]

where the parameters \( x^\varepsilon(t) \) and \( \omega^\varepsilon(t) \) are random modulation parameters, chosen such that for all \( t \),

\[
(\eta^\varepsilon(t), \varphi_{\omega_0}) = (\eta^\varepsilon(t), \partial_x \varphi_{\omega_0}) = 0
\]

This decomposition holds as long as \( \|\varepsilon \eta^\varepsilon(t)\|_{H^1} \leq \alpha \) and \( |\omega^\varepsilon(t) - \omega_0| \leq \alpha \) for \( \alpha > 0 \) sufficiently small.

**Question** : Can we estimate the time \( \tau^\varepsilon_\alpha \) with

\[
    \tau^\varepsilon_\alpha = \inf \{ t > 0, \|\varepsilon \eta^\varepsilon(t)\|_{H^1} \geq \alpha \text{ or } |\omega^\varepsilon(t) - \omega_0| \geq \alpha \}\?
Stochastic case : KdV

joint works with A. Debussche, E. Gautier, 2007-2010

For all $T > 0$, and $\varepsilon > 0$ with $\varepsilon^2 T$ sufficiently small,

$$P(\tau^\varepsilon_\alpha \leq T) \leq \exp\left(-\frac{C(\alpha)}{\varepsilon^2 T}\right)$$

Moreover, in the additive case, if $W$ replaced by $W_n$ approximating space-time white noise, then there exists a constant $C(\alpha, \omega_0)$ such that for all $T > 0$,

$$\liminf_{n \to +\infty} \liminf_{\varepsilon \to 0} \varepsilon^2 \log P(\tau^{n,\varepsilon}_\alpha \leq T) \geq -\frac{C(\alpha, \omega_0)}{T}$$

Remark :

- LDP holds in multiplicative case; however, have to solve a controllability problem by a potential $\rightsquigarrow$ open problem
- lower bounds on the exit time without modulation : $-\frac{C(\alpha, \omega_0)}{T^3}$
Stochastic case : KdV

Multiplicative homogeneous case :
\( \eta^\varepsilon \) converges to \( \eta \) as \( \varepsilon \) goes to zero, on fixed time intervals, in the mean square sense; \( \eta \) solution of

\[
d\eta = \partial_x L_0 \eta dt + Q(\varphi_0 d\tilde{W})
\]

with \( \eta(0) = 0 \) and

- \( \tilde{W}(t, x) = W(t, x + \omega_0 t) \)
- \( Q \) : projector on the “stable manifold”
- \( \eta \) is a centered Gaussian process (“Ornstein-Uhlenbeck” if \( \partial_x L_0 \) dissipative operator)

Pego, Weinstein, CMP, 1994 : \( Q\partial_x L_0 \) dissipative in spaces with exponential weights

\( \rightsquigarrow \) \( \eta \) converges weakly to a Gaussian stationary measure as \( t \) goes to infinity
The modulation equations are given by

\[
\begin{aligned}
    dx^\varepsilon &= \omega_0 + \varepsilon B_1 dt + \varepsilon dB_2 + o(\varepsilon) \\
    d\omega^\varepsilon &= \varepsilon dB_1 + o(\varepsilon)
\end{aligned}
\]

with \((B_1, B_2)\) a \(\mathbb{R}^2\)-valued brownian motion, corresponding to projection of the noise on the center manifold

Keeping only first order terms in \(\varepsilon\), we obtain

\[
\max_{x \in \mathbb{R}} E \left( \varphi_{\omega^\varepsilon(t)}(x - x^\varepsilon(t)) \right) \leq K\omega_0 \varepsilon^{-1/2} t^{-5/4}
\]

for large \(t\)

Stochastic case : NLS

Same kind of results for NLS equation with

- confining potential : \( V(x) = |x|^2 \)
- small noise : \( \varepsilon V(x) u \circ dW(t) \)

AdB, Fuikuizumi, 2009

\[
\psi^\varepsilon(t, x) = e^{i\theta^\varepsilon(t)}(\phi_{\omega^\varepsilon}(t)(x) + \varepsilon\eta^\varepsilon(t, x))
\]

No asymptotic stability (bounds states) but asymptotics on \( \eta \) for \( \omega \) close to \( \omega_{\text{min}} = -d \)
Stochastic case : numerical approximations

**Space-time NLS** : Crank-Nicolson in time + finite differences or finite elements in space + semi-implicit discretization of the noise $u \circ dW(t, x)$

Debussche-Di Menza, 2002, Barton-Smith-Debussche-Di Menza, 2005

convergence of the semi-discretization in time + order estimates dB, Debussche, 2004, 2006

**Additive KdV** : Crank-Nicolson in time + finite elements in space

Debussche-Printems, 1999, 2001

convergence of the semi-discretization in time

Debussche-Printems, 2006
Stochastic case: numerical approximations

**NLS with time-dependent noise:**
Splitting methods:
- $i\partial_t u + \Delta u = 0$: FFT
- nonlinearity + noise: explicit integration

M. Gazeau, 2011, work in progress with R. Belaouar

Alternative to Crank-Nicolson: relaxation scheme (avoids nonlinear implicitness)

C. Besse, deterministic NLS, 1998
Stochastic case: numerical approximations (KdV)

Fig. 3.13: Single soliton with noise, $\gamma = 10^{-2}$, $c = 0.3$, $\epsilon = 10^{-4}$, $x_0 = 0.4$ on $[0,3]$. 
Stochastic case: numerical approximations (KdV)
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Stochastic case: numerical approximations (NLS)

Soliton diffusion, subcritical case $\sigma=1$

Multiplicative noise, 100 trajectories, $T_{\text{fin}} = 10$

- $\epsilon = 0$
- $\epsilon = 0.01$
- $\epsilon = 0.05$
- $\epsilon = 0.1$
Stochastic case: numerical approximations (NLS)

Soliton diffusion, critical case $\sigma=2$

Multiplicative noise, 100 trajectories, $T_{\text{fin}} = 2$

- $\epsilon = 0$
- $\epsilon = 0.05$
- $\epsilon = 0.1$
- $\epsilon = 0.2$

Graph showing the behavior of $|u(2,x)|$ with different values of $\epsilon$. The curves represent the solutions for different values of multiplicative noise intensity.
Stochastic case: numerical approximations (NLS)

Soliton diffusion, supercritical case \( \sigma = 3 \)

Multiplicative noise, 100 trajectories, \( T_{\text{fin}} = 0.3 \)

- \( \epsilon = 0 \)
- \( \epsilon = 0.1 \)
- \( \epsilon = 0.2 \)
- \( \epsilon = 0.3 \)
Conclusion and open problems

Dynamics of stochastic nonlinear dispersive equations still needs investigations: lots of applications

Open problems in asymptotic dynamics:

- Take account of asymptotic stability for KdV in additive case?
- Explain diffusion NLS soliton with space-time noise?
- KdV in periodic case $x \in \mathbb{T}$: Y. Tsutsumi, 2009

Assume $|k \ast |v|^2|^2_{L^2} \geq \delta^2 |v|^4_{L^2}$ with $2\delta^2 \geq |k|^2_{L^2}$

Then the solution $u$ of

$$
du + (\partial^3_x u + \partial_x (u^2))dt = udW
$$

satisfies:

$$
|u(t)|_{L^2} \to 0, \text{ a.s. as } t \to +\infty
$$
Conclusion and open problems

- Dynamics at time $1/\varepsilon^2$ : averaging ?

  Inverse scattering : Garnier, 1998 (NLS), 2001 (KdV)
  Noise representing perturbations of velocity, dispersion or nonlinearity ; Propagation of solitons : equations on the scattering data ; no estimate on the remaining terms for the original solution

  Action-angle variables : Kuksin, Piatnitski, 2008
  Hasminski-Whitham averaging for

  \[ \partial_t u + \partial_x^3 u - \nu \partial_x^2 u + \partial_x (u^2) = \sqrt{\nu} \xi; \]

  modeling of weak turbulence.
Rigorous derivation of stochastic models

Needs stochastic homogenization (and more...) e.g.

- Dirichlet to Neumann operator in domain with random boundary (water wave problem)

- Propagation in molecular systems: stochastic NLS equation formally obtained from interacting particle system

\[\begin{align*}
i\hbar \frac{d\phi_n}{dt} + \sum_{p \neq n} J_{pn} \phi_p + \chi u_n \phi_n &= 0 \\
M \frac{d^2 u_n}{dt^2} + M\lambda \frac{du_n}{dt} + M\omega_0^2 u_n &= \chi |\phi_n|^2 + \eta_n
\end{align*}\]

Last step: diffusion approximation (current work with A. Debussche, M. Gazeau)