Nonlinear dispersive equations, solitary waves and noise

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Nonlinear dispersive equations

- model equations for propagation of waves in nonlinear dispersive media
- "Universal models" : water waves, nonlinear optics, plasma physics, energy transfer in molecular systems, Bose-Einstein condensation ...
- conservative equations
- In general

$$i\partial_t u + P(D)u + F(u) = 0$$

where $u = u(t, x)$ is the unknown and

$$\widehat{P(D)u}(\xi) = p(\xi)\hat{u}(\xi)$$

where p is **real valued** ; dispersion relation : $\omega(k) = -p(k)$

Examples

Korteweg - de Vries equations

$$(KdV) \qquad \partial_t u + \partial_x^3 u + \partial_x(u^2) = 0$$

- Boussinesq, 1872, Korteweg- de Vries, 1895 after observation by J. Scott Russel, 1834
  http ://www.ma.hw.ac.uk/solitons
- asymptotic model for long waves at the surface of water (small amplitude, shallow water, unidirectional propagation)

  rigorous derivation (from free surface Euler equations) :
  W. Craig, CPDE, 1985
- here, $p(k) = k^3$ (strong dispersion)
- Model for plasma physics : Herman, J. Phys. A, 1990

Examples

Nonlinear Schrödinger equations

$$(NLS) \qquad i\partial_t u + \Delta u + F(u) = 0$$

- here, $p(k) = |k|^2$
- describes enveloppe of wave packets in water waves (deep water) Zakharov, 1968, $F(u) = \kappa |u|^2 u$
- propagation of light in dispersive and nonlinear media (e.g. optic fibers) $F(u) = \kappa |u|^2 u$
  rigorous derivation (as enveloppe, from Maxwell equations) : Donnat, Joly, Metivier, Rauch, 1996
  in optic fibers : special case of the Manakov system
- Bose-Einstein condensation : Gross-Pitaevskii equation $F(u) = V(x)u + \kappa |u|^2 u$

Integrable equations

Some of those equations are integrable by inverse scattering

$\rightsquigarrow$ infinity of invariant functionals

$\rightsquigarrow$ existence of solitons : localized solutions propagating without change of form, elastic interactions

$\rightsquigarrow$ resolution into solitons + dispersive tail

KdV : Gardner, Green, Kruskal, Miura, 1967

cubic NLS equations in dimension one ($x \in \mathbf{R}$) : Zakharov-Shabat, 1972

Non integrable equations : higher dimension, higher order nonlinearity, external (possibly random) forces... which of those properties still hold ? sometimes still localized solutions : solitary waves

Why adding noise ?

KdV-type equations (no rigorous mathematical derivation)

- **random pressure field** (water surface waves) : e.g., turbulent wind on the surface
  $\rightsquigarrow$ additive noise, white in time $\dot{\xi}(t,x)$
- **variations of the bottom topography** modeled by a stationary (in x) random process :
  $\rightsquigarrow$ add a term $(\partial_x u)\dot{\xi}(t)$, white in time
  Craig, de Bouard, Diaz-Espinoza, Guyenne, Sulem, Nonlinearity, 2008
- **temperature effects** (plasma physics) : Herman, J. Phys. A, 1990
  random potentials : $u\dot{\xi}(t,x)$ or $(\partial_x u)\dot{\xi}(t,x)$
- model for weak turbulence (Kuksin et ; al.), random perturbations of nonlinearity, velocity or dispersion (Garnier)

Why adding noise ?

NLS-type equations

- **temperature effects** (light propagation in molecular systems) : $\rightsquigarrow$ potential terms $u\dot{\xi}(t,x)$
  Bang, Christiansen, If, Rasmussen, Gaididei, Phys. Rev. E, 1994
- **inhomogeneities in the medium** (e.g. optic fibers) :
  Falkovich, Kolokolov, Lebedev, Turitsyn, Phys. Rev. E, 1994
  amplifiers : $\rightsquigarrow$ additive noise $\dot{\xi}(t,x)$
  dispersion management $\rightsquigarrow$ perturbation of dispersion $\dot{\xi}(t)\Delta u$
- **fluctuations of the laser frequency** (Bose-Einstein condensation) : induces fuctuations of the confining potential
  Abdullaev, Baizakov, Konotop, Nonlinearity and Disorder, 2001 $\rightsquigarrow$ $\dot{\xi}(t)V(x)$

Solitary waves

Localized solutions propagating without change of form (due to nonlinearity + dispersion)

- travelling waves : $u(x,t) = \varphi_v(x - vt)$, $v \in \mathbf{R}^d$
- standing waves : $u(x,t) = e^{i\omega t}\varphi_\omega(x)$, $\omega \in \mathbf{R}$

Symmetries of equations lead to more general families of solutions

KdV-type equations : two parameter family of solutions

$$u_{c,x_0}(x,t) = \varphi_c(x - ct - x_0), c > 0, x_0 \in \mathbf{R}$$

NLS-type equations : $2d + 2$-parameter family of solutions

$$u_{\omega,\theta,v,x_0}(x,t) = \varphi_\omega(x - 2vt - x_0)e^{i(\omega t - v.x + v^2 t + \theta)}$$

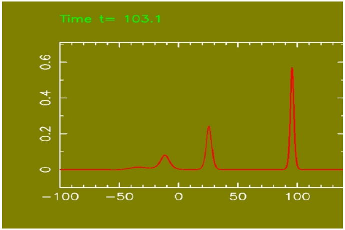
(translation, Galilean and Gauge transformations)

Solitary waves

Integrable cases : resolution into soliton for any localized solution

Theorem : (Eckaus-Schuur, 1986, Deft-Zhou, 1993)
Let $u(x,t)$ be a regular and localized solutions of the KdV equation ; then there exist $N \in \mathbf{N}$, $c_1, \cdots, c_N > 0$, $\gamma_1, \cdots, \gamma_N \in \mathbf{R}$, $\nu > 0$ and $\mu > 0$ such that

$$\lim_{t \to \infty} \sup_{x \geq -\mu - \nu t^{1/3}} |u(x,t) - \sum_{k=1}^{N} \varphi_{c_k}(x - c_k t - \gamma_k)| = 0$$



Solitary waves

Symplectic formulation of the equation

$$\partial_t u = JH'(u)$$

where $J = \partial_x$ for (KdV), $J = i$ for (NLS) and

$$(KdV) \qquad H(u) = \frac{1}{2}\int (\partial_x u)^2 dx - \frac{1}{3}\int u^3 dx$$

energy or Hamiltonian

$$(NLS) \qquad H(u) = \frac{1}{2}\int |\nabla u|^2 dx - \frac{1}{4}\int |u|^4 dx$$

the mass or charge si also conserved :

$$m(u) = \frac{1}{2}\int |u|^2 dx$$

Solitary wave = critical point of the action functional

$$\mathcal{E}_\omega(u) = H(u) + \omega m(u)$$

Ground state = critical point minimizing $\mathcal{E}_\omega$

Solitary waves

Numerical methods for computations of solitary waves

- Shooting methods (radial solutions)

$$\begin{cases} \varphi''(r) + \frac{d-1}{r}\varphi'(r) - \omega\varphi(r) + \kappa|\varphi(r)|^{2\sigma}\varphi(r) = 0 \\ \varphi(0) = \beta, \ \varphi'(0) = 0 \end{cases}$$

  $+$ dichotomy argument to find β^* such that $\lim_{r\to\infty}\varphi(r) = 0$

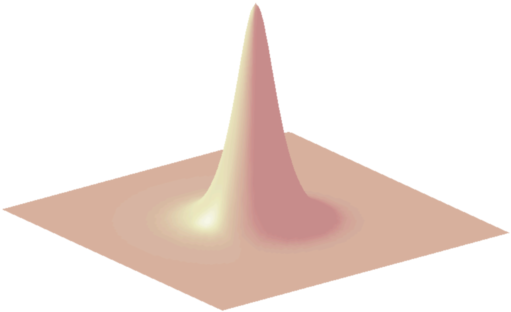
  Excited states, central vortices, systems of equations
  Di Menza, M2AN, 2009

- Complex time integration : solve the heat flow

$$\partial_t u = H'(u)$$

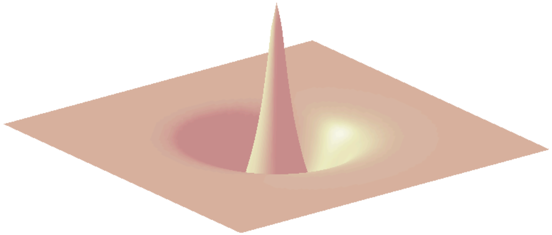
  with renormalization of the L^2-norm at each time step $\rightsquigarrow$ convergence to a solitary wave

Solitary waves



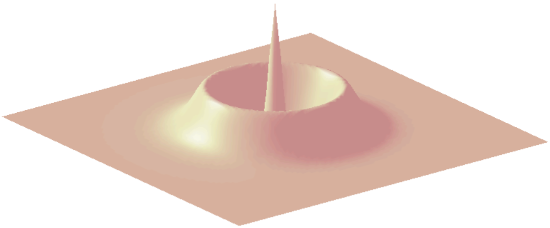
computation of the ground state of NLS by shooting method
(Di Menza, 2009)

Solitary waves



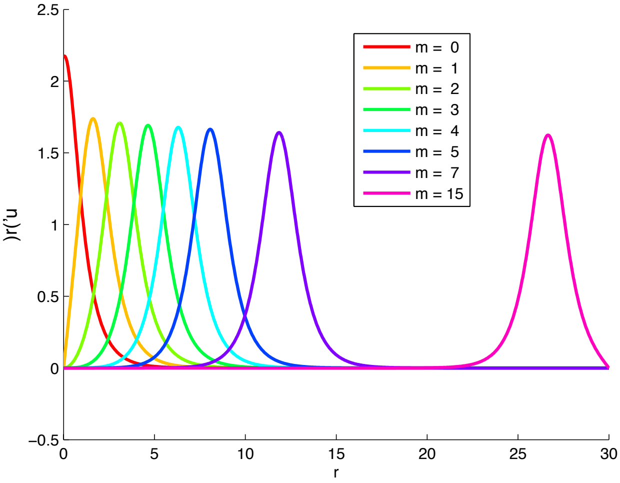
computation of excited state of NLS by shooting method
(Di Menza, 2009)

Solitary waves



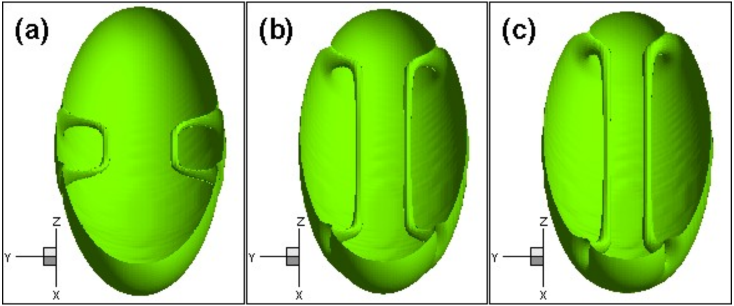
computation of excited state of NLS by shooting method
(Di Menza, 2009)

Solitary waves



computation of vortices of NLS by shooting method
(Belaouar, 2011)

Solitary waves



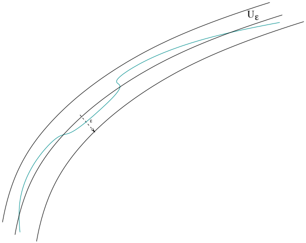
computation of vortices of 3-D NLS by complex time integration (Danaïla, 2006)

Solitary waves : orbital stability

Define

$$(GKdV) \qquad U_\varepsilon = \left\{ u \in H^1,\ \inf_{s \in \mathbf{R}} \|u - \varphi_\omega(. - s)\|_{H^1} \leq \varepsilon \right\}$$

φ_ω is orbitally stable if, given $\varepsilon > 0$, any solution starting sufficiently close to φ_ω stays in U_ε for all time



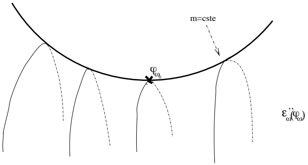
Theorem : φ_{ω_0} is stable iff $\mathcal{E}_\omega(\varphi_\omega)$ is a convex function of ω near ω_0

Benjamin, Bona-Souganidis-Strauss, Weinstein, Grillakis-Shatah-Strauss

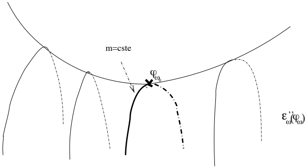
Solitary waves : orbital stability

Key point : study of $L_\omega = \mathcal{E}''_\omega(\varphi_\omega)$ (not positive definite)

- if $\frac{d}{d\omega} m(\varphi_{\omega_0}) > 0$ then φ_{ω_0} is a minimum of H for constant m



- if $\frac{d}{d\omega} m(\varphi_{\omega_0}) < 0$ then there is a curve ψ_α with $m(\psi_\alpha) = m(\varphi_{\omega_0})$ on which H is maximized at φ_{ω_0}



Solitary waves : asymptotic stability

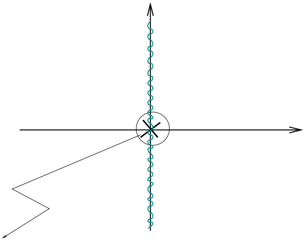
linearization : $u(t,x) = \varphi_\omega(x-\omega t) + v(t, x-\omega t)$ (GKdV)

$$\rightsquigarrow \quad \frac{dv}{dt} = JL_\omega v$$

- $\sigma_e(JL_\omega) = i\mathbf{R}$
- nullspace generated by invariances of equation

  ex (GKdV) :

  $$\begin{cases} \partial_x L_\omega(\partial_x \varphi_\omega) = 0 \\ \partial_x L_\omega(\partial_\omega \varphi_\omega) = -\partial_x \varphi_\omega \end{cases}$$



Solitary waves : asymptotic stability

GKdV : Pego-Weinstein, 1994, Martel-Merle, 2001
NLS : Buslaev-Perelman, 1992, Buslaev-Sulem, 2003

Principal part written as a modulated solitary wave :

$$u(t,x) = \varphi_{\omega(t)}(x - x(t)) + v(t, x - x(t))$$

and $\omega(t)$, $x(t)$ chosen in order to eliminate secular modes

- cvgence to a solitary wave with shifted parameters
  $\omega(t) \to \omega^+, \quad x(t) \sim \omega^+ t + x^+$ as $t \to +\infty$
- cvgence of v in a space with exponential weight (or locally in space)
- also true for n-solitary wave solutions
  Martel-Merle-Tsai, 2003
- recent results on non elastic collision of solitons for GKdV
  Martel-Merle, 2009

Stochastic case : KdV

$$du + (\partial_x^3 u + \partial_x(u^2))dt = \begin{cases} \varepsilon dW \\ \varepsilon u dW \end{cases}$$

$W(t)$ infinite dimensional Wiener process i.e.

$$W(t,x) = \sum_j \Phi(e_j)(x) W_j(t)$$

W_j indep. 1-D BM, (e_j) c.o.s. in $L^2(\mathbf{R})$,

- Φ Hilbert-Schmidt operator from $L^2(\mathbf{R})$ into $H^1(\mathbf{R})$, if additive noise
- $\Phi(e_j) = k * e_j$, with $k \in H^1(\mathbf{R}) \cap L^1(\mathbf{R})$ if multiplicative noise ; equivalently

$$W(t,x) = \int_0^t \int_{\mathbf{R}} k(x-y) B(ds,dy)$$

where B is a Brownian sheet on $\mathbf{R}^+ \times \mathbf{R}$

Stochastic case : KdV

Let $u^\varepsilon(0,x) = \varphi_{\omega_0}(x)$; write the solution u^ε of the stochastic equation as

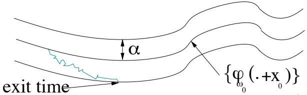
$$u^\varepsilon(t,x) = \varphi_{\omega^\varepsilon(t)}(x - x^\varepsilon(t)) + \varepsilon\eta^\varepsilon(t, x - x^\varepsilon(t))$$

where the parameters $x^\varepsilon(t)$ and $\omega^\varepsilon(t)$ are random modulation parameters, chosen such that for all t,
$(\eta^\varepsilon(t), \varphi_{\omega_0}) = (\eta^\varepsilon(t), \partial_x\varphi_{\omega_0}) = 0$

This decomposition holds as long as $\|\varepsilon\eta^\varepsilon(t)\|_{H^1} \leq \alpha$ and $|\omega^\varepsilon(t) - \omega_0| \leq \alpha$ for $\alpha > 0$ sufficiently small.

Question : Can we estimate the time τ_α^ε with

$$\tau_\alpha^\varepsilon = \inf\{t > 0,\ \|\varepsilon\eta^\varepsilon(t)\|_{H^1} \geq \alpha \text{ or } |\omega^\varepsilon(t) - \omega_0| \geq \alpha\}?$$



Stochastic case : KdV

joint works with A. Debussche, E. Gautier, 2007-2010

For all $T > 0$, and $\varepsilon > 0$ with $\varepsilon^2 T$ sufficiently small,

$$\mathbf{P}(\tau_\alpha^\varepsilon \leq T) \leq \exp(-\frac{C(\alpha)}{\varepsilon^2 T})$$

Moreover, in the additive case, if W replaced by W_n approximating space-time white noise, then there exists a constant $C(\alpha, \omega_0)$ such that for all $T > 0$,

$$\liminf_{n \to +\infty} \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbf{P}(\tau_\alpha^{n,\varepsilon} \leq T) \geq -\frac{C(\alpha, \omega_0)}{T}$$

Remark :

- LDP holds in multiplicative case ; however, have to solve a controllability problem by a potential $\rightsquigarrow$ open problem
- lower bounds on the exit time without modulation : $-\frac{C(\alpha,\omega_0)}{T^3}$

Stochastic case : KdV

Multiplicative homogeneous case :

η^ε converges to η as ε goes to zero, on fixed time intervals, in the mean square sense ; η solution of

$$d\eta = \partial_x L_{\omega_0} \eta dt + Q(\varphi_{\omega_0} d\tilde{W})$$

with $\eta(0) = 0$ and

- $\tilde{W}(t,x) = W(t, x + \omega_0 t)$
- Q : projector on the "stable manifold"
- η is a centered Gaussian process ("Ornstein-Uhlenbeck" if $\partial_x L_{\omega_0}$ dissipative operator)

Pego, Weinstein, CMP, 1994 : $Q\partial_x L_{\omega_0}$ dissipative in spaces with exponential weights

$\rightsquigarrow$ η converges weakly to a Gaussian stationary measure as t goes to infinity

Stochastic case : KdV

- The modulation equations are given by

$$\begin{cases} dx^\varepsilon = \omega_0 + \varepsilon B_1 dt + \varepsilon dB_2 + o(\varepsilon) \\ d\omega^\varepsilon = \varepsilon dB_1 + o(\varepsilon) \end{cases}$$

with (B_1, B_2) a $\mathbf{R}^2$-valued brownian motion, corresponding to projection of the noise on the center manifold

- Keeping only first order terms in ε, we obtain

$$\max_{x \in \mathbf{R}} \mathbf{E}\left(\varphi_{\omega^\varepsilon(t)}(x - x^\varepsilon(t))\right) \leq K_{\omega_0} \varepsilon^{-1/2} t^{-5/4}$$

for large t

Wadati, J. Phys. Soc. Japan, 1983 : additive pure time white noise, $Ct^{-3/2}$

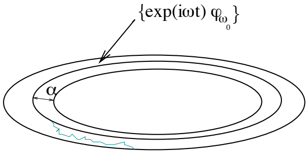
Stochastic case : NLS

Same kind of results for NLS equation with

- confining potential : $V(x) = |x|^2$
- small noise : $\varepsilon V(x) u \circ dW(t)$

AdB, Fuikuizumi, 2009

$$\psi^\varepsilon(t,x) = e^{i\theta^\varepsilon(t)}(\phi_{\omega^\varepsilon(t)}(x) + \varepsilon\eta^\varepsilon(t,x))$$



No asymptotic stability (bounds states) but asymptotics on η for ω close to $\omega_{\min} = -d$

Stochastic case : numerical approximations

Space-time NLS : Crank-Nicolson in time + finite differences or finite elements in space + semi-implicit discretization of the noise $u \circ dW(t,x)$

Debussche-Di Menza, 2002, Barton-Smith-Debussche-Di Menza, 2005

convergence of the semi-discretization in time + order estimates
dB, Debussche, 2004, 2006

Additive KdV : Crank-Nicolson in time + finite elements in space

Debussche-Printems, 1999, 2001

convergence of the semi-discretization in time
Debussche-Printems, 2006

Stochastic case : numerical approximations

NLS with time-dependent noise :
Splitting methods :

- $i\partial_t u + \Delta u = 0$: FFT
- nonlinearity + noise : explicit integration

M. Gazeau, 2011, work in progress with R. Belaouar

Alternative to Crank-Nicolson : relaxation scheme (avoids nonlinear implicitness)

C. Besse, deterministic NLS, 1998

Stochastic case : numerical approximations (KdV)

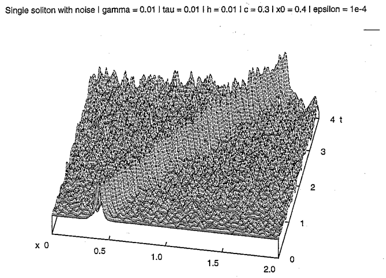
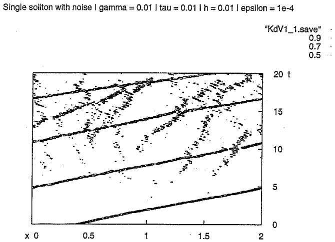
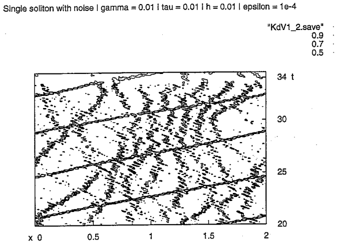


FIG. 3.13: *Single soliton with noise,* $\gamma = 10^{-2}$, $c = 0.3$, $\varepsilon = 10^{-4}$, $x_0 = 0.4$ *on* $[0, 3]$.

Stochastic case : numerical approximations (KdV)

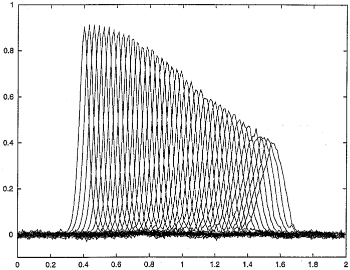


Stochastic case : numerical approximations (KdV)

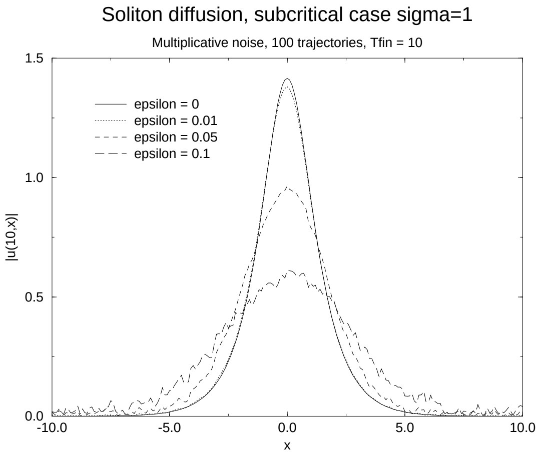


Stochastic case : numerical approximations (KdV)

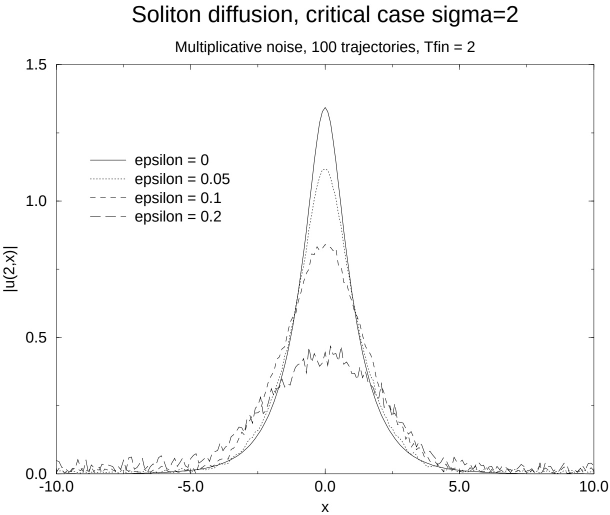
a)



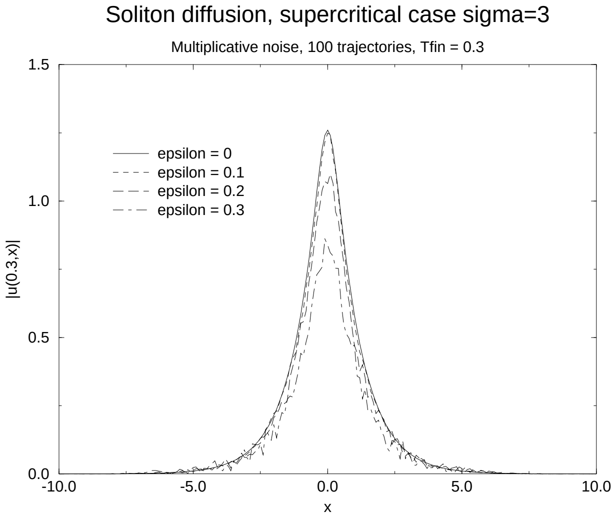
Stochastic case : numerical approximations (NLS)



Stochastic case : numerical approximations (NLS)



Stochastic case : numerical approximations (NLS)



Conclusion and open problems

Dynamics of stochastic nonlinear dispersive equations still needs investigations : lots of applications

Open problems in asymptotic dynamics :

- Take account of asymptotic stability for KdV in additive case ?
- Explain diffusion NLS soliton with space-time noise ?
- KdV in periodic case $x \in \mathbb{T}$: Y. Tsutsumi, 2009

Assume $|k * |v|^2|_{L^2}^2 \geq \delta^2 |v|_{L^2}^4$ with $2\delta^2 \geq |k|_{L^2}^2$

Then the solution u of

$$du + (\partial_x^3 u + \partial_x(u^2))dt = udW$$

satisfies :

$$|u(t)|_{L^2} \to 0, \ a.s. \text{ as } t \to +\infty$$

Conclusion and open problems

- Dynamics at time $1/\varepsilon^2$: averaging ?

  Inverse scattering : Garnier, 1998 (NLS), 2001 (KdV)
  Noise representing perturbations of velocity, dispersion or nonlinearity ; Propagation of solitons : equations on the scattering data ; no estimate on the remaining terms for the original solution

  Action-angle variables : Kuksin, Piatnitski, 2008
  Hasminski-Whitham averaging for

$$\partial_t u + \partial_x^3 u - \nu \partial_x^2 u + \partial_x(u^2) = \sqrt{\nu}\dot{\xi};$$

  modeling of weak turbulence.

Conclusion and open problems

Rigorous derivation of stochastic models

Needs stochastic homogenization (and more...) e.g.

- Dirichlet to Neumann operator in domain with random boundary (water wave problem)
- Propagation in molecular systems : stochastic NLS equation formally obtained from interacting particle system

$$\begin{cases} i\hbar \frac{d\phi_n}{dt} + \sum_{p \neq n} J_{pn}\phi_p + \chi u_n \phi_n = 0 \\ M\frac{d^2 u_n}{dt^2} + M\lambda \frac{du_n}{dt} + M\omega_0^2 u_n = \chi|\phi_n|^2 + \eta_n \end{cases}$$

Last step : diffusion approximation (current work with A. Debussche, M. Gazeau)