Dynamics of Stochastic Systems and their Approximation

MFO, 26 August 2011

Hunting French Ducks in a Noisy Environment

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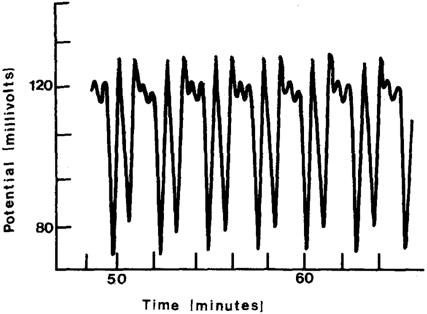
Mixed-Mode Oscillations (MMOs)

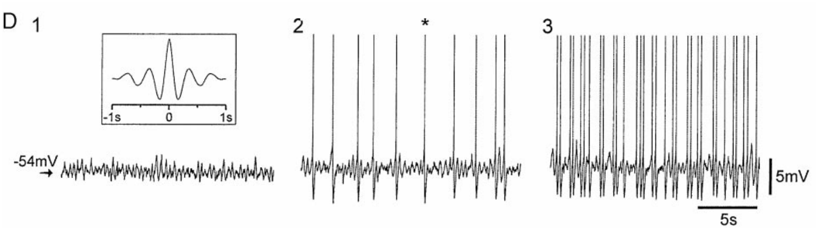
Belousov–Zhabotinsky reaction



Recording from bromide ion electrode; T=25° C; flow rate = 3.99 ml/min; Ce^{+3} catalyst [Hudson, Hart, Marinko '79]

MMOs in Biology

Layer II Stellate Cells



D: subthreshold membrane potential oscillations (1 and 2) and spike clustering (3) develop at increasingly depolarized membrane potential levels positive to about –55 mV. Autocorrelation function (*inset* in 1) demonstrates the rhythmicity of the subthreshold oscillations [Dickson *et al* '00]

Questions: Origin of small-amplitude oscillations?
Source of irregularity in pattern?

Mechanisms for MMOs

- In ODEs and PDEs with bifurcations
- Through the canard phenomenon
- Noise-induced [Muratov, Vanden Eijnden '08]

A few references

- Numerical studies: [Borowski, Kuske, Li, Cabrera '11], ...
- Approximation of FPE: [Lindner, Schimansky-Geier '99], [Simpson, Kuske '11]
- Large deviations: [Doss, Thieullen '09]
- Sample-path behaviour for canards: [Sowers '08]

MMOs & Slow–Fast Systems

MMOs can be observed in slow–fast systems undergoing a folded-node bifurcation (1 fast, 2 slow variables)

Normal form of folded-node [Benoît, Lobry '82; Szmolyan, Wechselberger '01]

$$\begin{aligned} \epsilon \dot{x}_t &= y_t - x_t^2 \\ \dot{y}_t &= -(\mu + 1)x_t - z_t \\ \dot{z}_t &= \frac{\mu}{2} \end{aligned}$$

Timescale separation: $\varepsilon \ll 1$

Questions: Dynamics for small $\varepsilon > 0$?
Effect of noise?

Approach: General results for deterministic slow–fast systems; canards
Random perturbations of slow–fast systems; application to MMOs

General Slow–Fast Systems: Singular Limits

$$\begin{tikzcd}
\boxed{\begin{array}{l}\text{In slow time } t\\ \varepsilon\dot{x} = f(x,y)\\ \ \dot{y} = g(x,y)\end{array}} \arrow[rr, Leftrightarrow, "t\mapsto s"] \arrow[d, "\varepsilon\to0"] & & \boxed{\begin{array}{l}\text{In fast time } s = t/\varepsilon\\ x' = \ \ f(x,y)\\ y' = \varepsilon g(x,y)\end{array}} \arrow[d, "\varepsilon\to0"] \\
\boxed{\begin{array}{l}\text{Slow subsystem}\\ \ 0 = f(x,y)\\ \ \dot{y} = g(x,y)\end{array}} \arrow[rr, phantom, "\not\Leftrightarrow"] & & \boxed{\begin{array}{l}\text{Fast subsystem}\\ x' = \ \ f(x,y)\\ y' = 0\end{array}}
\end{tikzcd}$$

Study slow variable y on *slow* or *critical* manifold $f(x,y) = 0$

Study fast variable x for frozen slow variable y

Slow (or Critical) Manifolds

$$\mathcal{C}_0 = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \colon f(x,y) = 0\}$$

Definition

- $\mathcal{C}_0$ is *normally hyperbolic* at $(x,y) \in \mathcal{C}_0$ if

$$\frac{\partial}{\partial x} f(x,y) \text{ has only eigenvalues } \lambda_j = \lambda_j(x,y) \text{ with } \operatorname{Re}\lambda_j \neq 0$$

- $\mathcal{C}_0$ is *asymptotically stable* or *attracting* at $(x,y) \in \mathcal{C}_0$ if

$$\operatorname{Re}\lambda_j(x,y) < 0 \quad \text{for all } j$$

- $\mathcal{C}_0$ is *unstable* at $(x,y) \in \mathcal{C}_0$ if

$$\operatorname{Re}\lambda_j(x,y) > 0 \quad \text{for at least one } j$$

Fenichel's Theorem: Adiabatic Manifolds

Theorem [Tihonov '52; Fenichel '79]

Assume $\mathcal{C}_0$ is normally hyperbolic.

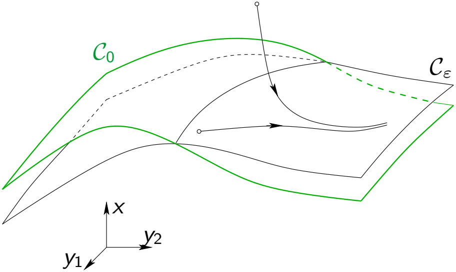
$\exists$ *adiabatic manifold* $\mathcal{C}_\varepsilon$ s.t.

- $\mathcal{C}_\varepsilon$ is locally invariant
- $\mathcal{C}_\varepsilon = \mathcal{C}_0 + \mathcal{O}(\varepsilon)$

If $\mathcal{C}_0$ is *uniformly attracting*, i.e.,

$$\operatorname{Re}(\lambda_j(x,y)) \leqslant -\delta_0 < 0 \quad \forall (x,y)$$

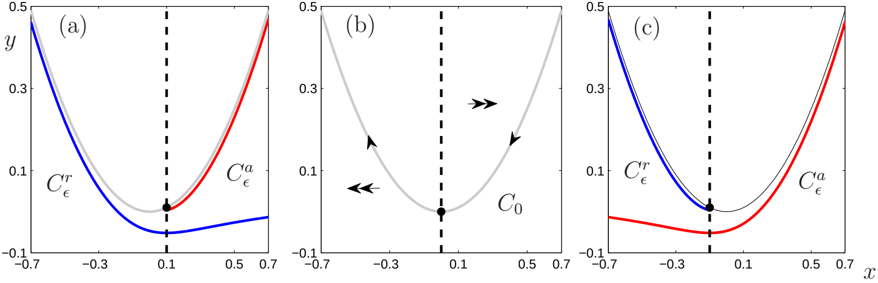
then $\mathcal{C}_\varepsilon$ attracts nearby solutions exponentially fast



Example: The Planar Fold

Normal form near fold point

$$\begin{aligned} \epsilon \dot{x} &= y - x^2 \\ \dot{y} &= \mu - x \end{aligned} \qquad (+ \text{ higher-order terms})$$

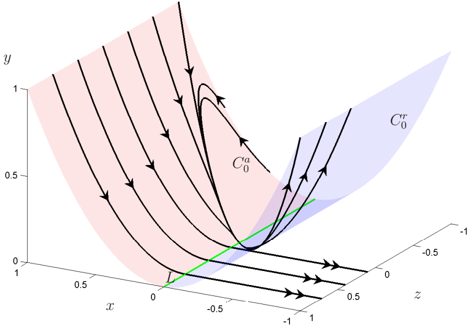


$\varepsilon = 0.05$

(a) $\mu = 0.1$ (b) $\mu = 0.0$ (c) $\mu = -0.1$

Folded-Node Bifurcation: Slow Manifold

$$
\begin{aligned}
\epsilon\dot{x} &= y - x^2\\
\dot{y} &= -(\mu+1)x - z\\
\dot{z} &= \frac{\mu}{2}
\end{aligned}
$$

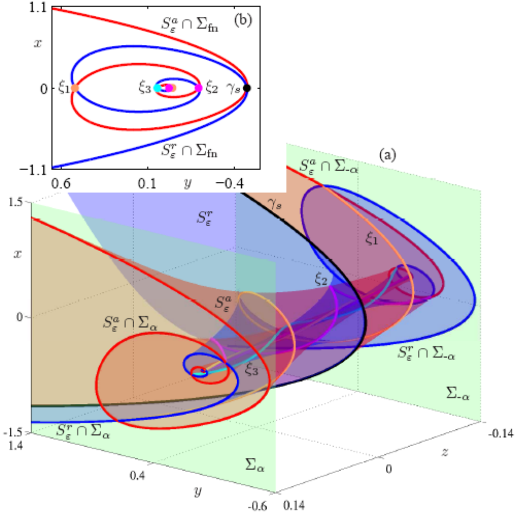


(Arrows show slow flow)

$\varepsilon = 0$: Slow manifold has a decomposition

$$\mathcal{C}_0 = \{(x,y,z) \in \mathbb{R}^3 \colon y = x^2\} = \mathcal{C}_0^{\mathsf{a}} \cup L \cup \mathcal{C}_0^{\mathsf{r}}$$

Folded-Node: Adiabatic Manifolds and Canard Solutions



[Desroches *et al* '11 (to appear)]

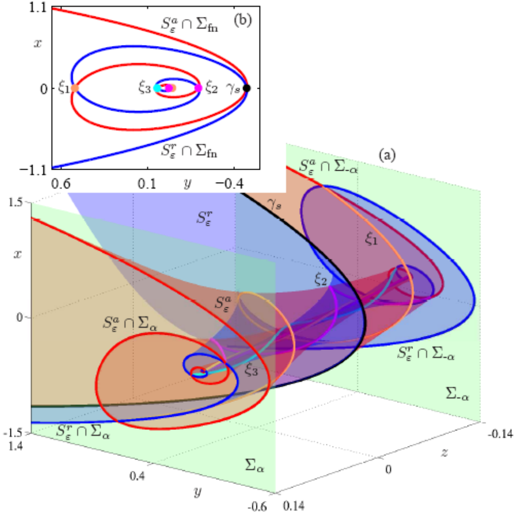
Assume

- ε sufficiently small
- $\mu \in (0, 1)$, $\mu^{-1} \notin \mathbb{N}$

Theorem

[Benoît, Lobry '82;
Szmolyan, Wechselberger '01;
Wechselberger '05;
Brøns, Krupa, Wechselberger '06]

Folded-Node: Adiabatic Manifolds and Canard Solutions



[Desroches *et al* '11 (to appear)]

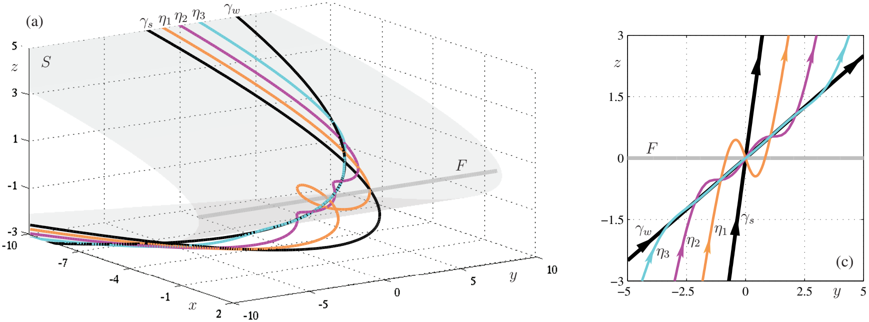
Assume

- ε sufficiently small
- $\mu \in (0,1)$, $\mu^{-1} \notin \mathbb{N}$

Theorem

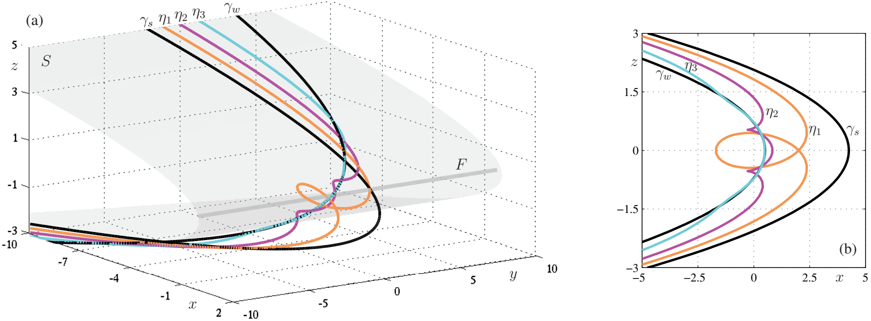
- Existence of *strong* and *weak* (maximal) canards $\gamma_\varepsilon^{\text{s,w}}$
- $\gamma_\varepsilon^{\text{s}}$ makes 1/2 oscillation (or 1 *twist*) around $\gamma_\varepsilon^{\text{w}}$
- $2k+1 < \mu^{-1} < 2k+3$: $\exists$ k *secondary* canards γ_ε^j
- γ_ε^j makes $(2j+1)/2$ oscillations around $\gamma_\varepsilon^{\text{w}}$

Folded-Node: Canard Spacing



[Desroches, Krauskopf, Osinga '08]

Folded-Node: Canard Spacing



[Desroches, Krauskopf, Osinga '08]

Lemma

For $z = 0$: Distance between canards γ_ε^k and γ_ε^{k+1} is $\mathcal{O}(\mathrm{e}^{-c_0(2k+1)^2\mu})$

Folded-Node: Proof of Canard-Spacing Lemma

Lemma

For $z = 0$: Distance between canards γ_ε^k and γ_ε^{k+1} is $\mathcal{O}(\mathrm{e}^{-c_0(2k+1)^2\mu})$

Proof

- Let $z_0 \leqslant z \leqslant 0$ and consider z as "time"
- Blow-up transformation *removes* ε-dependence (see below)
- Explicit expressions for strong and weak maximal canards [Benoît '90]
- Deviation u of arbitrary solution from weak canard satisfies

$$\mu \frac{\mathrm{d}u}{\mathrm{d}z} = \begin{pmatrix} 4z & 2 \\ -2(\mu+1) & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -2u_1^2 \\ 0 \end{pmatrix}$$

- Eigenvalues $2z \pm \mathrm{i}\,\omega(z)$ for $z < 0$: Rotation $+$ contraction
- Suffices to calculate contraction rate

Random Perturbations of General Slow–Fast Systems

$$
\begin{aligned}
\mathrm{d}x_t &= \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t \\
\mathrm{d}y_t &= \quad g(x_t, y_t)\,\mathrm{d}t + \ \sigma' \ G(x_t, y_t)\,\mathrm{d}W_t
\end{aligned}
$$

- ▷ $\{W_t\}_{t\geqslant 0}$ k-dimensional (standard) Brownian motion
- ▷ adiabatic parameter $\varepsilon > 0$ (*no quasistatic* approach)
- ▷ noise intensities $\sigma = \sigma(\varepsilon) > 0$, $\sigma' = \sigma'(\varepsilon) \geqslant 0$ with $\sigma'(\varepsilon)/\sigma(\varepsilon) = \varrho(\varepsilon) \leqslant 1$

Timescales: We are interested in the regime

$$T_{\text{relax}} = \mathcal{O}(\varepsilon) \ll T_{\text{driving}} = 1 \ll T_{\text{Kramers}} = \varepsilon\, \mathrm{e}^{\overline{V}/\sigma^2} \qquad \text{(in slow time)}$$

Assumption: $\mathcal{C}_0$ is uniformly attracting (for the deterministic system)

Deviation from the Adiabatic Manifold due to Noise

Main idea

- Consider deterministic process $(x_t^{\text{det}}, y_t^{\text{det}}) \in \mathcal{C}_\varepsilon$ (using invariance of $\mathcal{C}_\varepsilon$)
- Linearize SDE for deviation $\xi_t := x_t - x_t^{\text{det}}$ from adiabatic manifold

$$\mathrm{d}\xi_t^0 = \frac{1}{\varepsilon} A(y_t^{\text{det}})\xi_t^0 \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F_0(y_t^{\text{det}}) \,\mathrm{d}W_t$$

where $A(y_t^{\text{det}}) = \partial_x f(x_t^{\text{det}}, y_t^{\text{det}})$ and F_0 is 0*th*-order approximation to F

Key observation

- Resulting process ξ_t^0 is a (non-autononous) Gaussian process
- $\dfrac{1}{\sigma^2} \operatorname{Cov} \xi_t^0$ is a particular solution of the deterministic slow–fast system

$$\begin{aligned} \varepsilon \dot{X}(t) &= A(y_t^{\text{det}})X(t) + X(t)A(y_t^{\text{det}})^{\mathrm{T}} + F_0(y_t^{\text{det}})F_0(y_t^{\text{det}})^{\mathrm{T}} \\ \dot{y}_t^{\text{det}} &= g(x_t^{\text{det}}, y_t^{\text{det}}) \end{aligned}$$

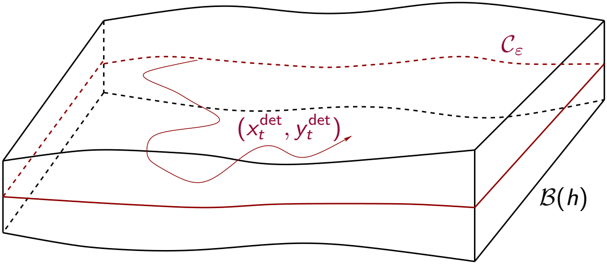
- System admits an adiabatic manifold $\{(\bar{X}(y,\varepsilon), y) \colon y \in \mathcal{D}_0\}$

Defining Typical Neighbourhoods of Adiabatic Manifolds

Typical neighbourhoods

$$\mathcal{B}(h) = \Big\{(x,y)\colon \Big\langle \big[x-\bar{x}(y,\varepsilon)\big], \bar{X}(y,\varepsilon)^{-1}\big[x-\bar{x}(y,\varepsilon)\big]\Big\rangle < h^2\Big\}$$

where $\mathcal{C}_\varepsilon = \{(\bar{x}(y,\varepsilon),y)\colon y \in \mathcal{D}_0\}$



First-exit times

$$\tau_{\mathcal{D}_0} = \inf\{s>0\colon y_s \notin \mathcal{D}_0\}$$
$$\tau_{\mathcal{B}(h)} = \inf\{s>0\colon (x_s,y_s) \notin \mathcal{B}(h)\}$$

Concentration of Sample Paths near Adiabatic Manifolds

Theorem [Berglund & G '03]

- Assume *non-degeneracy of noise term*:

$$\|\bar{X}(y,\varepsilon)\| \text{ and } \|\bar{X}(y,\varepsilon)^{-1}\| \text{ uniformly bounded in } \mathcal{D}_0$$

- Then $\exists\, \varepsilon_0 > 0 \;\exists\, h_0 > 0 \;\forall\, \varepsilon \leqslant \varepsilon_0 \;\forall\, h \leqslant h_0$

$$\mathbb{P}\bigl\{\tau_{\mathcal{B}(h)} < \min(t, \tau_{\mathcal{D}_0})\bigr\} \leqslant C_{n,m}(t)\, \exp\biggl\{-\frac{h^2}{2\sigma^2}\bigl[1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)\bigr]\biggr\}$$

where $C_{n,m}(t) = \bigl[C^m + h^{-n}\bigr]\biggl(1 + \dfrac{t}{\varepsilon^2}\biggr)$

Remarks

- Bound is sharp: Similar lower bound
- If initial condition not on $\mathcal{C}_\varepsilon$: additional transitional phase
- On longer time scales: Behaviour of slow variables becomes crucial ($\rightarrow$ Assumptions on g)

Stochastic Folded Nodes: Rescaling

$$\begin{aligned}
\mathrm{d}x_t &= \frac{1}{\varepsilon}(y_t - x_t^2)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)} \\
\mathrm{d}y_t &= [-(\mu+1)x_t - z_t]\,\mathrm{d}t + \sigma'\,\mathrm{d}W_t^{(2)} \\
\mathrm{d}z_t &= \frac{\mu}{2}\,\mathrm{d}t
\end{aligned}$$

Rescaling (blow-up transformation): $(x, y, z, t) = (\sqrt{\varepsilon}\bar{x}, \varepsilon\bar{y}, \sqrt{\varepsilon}\bar{z}, \sqrt{\varepsilon}\,\bar{t})$

$$\begin{aligned}
\mathrm{d}x_t &= (y_t - x_t^2)\,\mathrm{d}t + \frac{\sigma}{\varepsilon^{3/4}}\,\mathrm{d}W_t^{(1)} \\
\mathrm{d}y_t &= [-(\mu+1)x_t - z_t]\,\mathrm{d}t + \frac{\sigma'}{\varepsilon^{3/4}}\,\mathrm{d}W_t^{(2)} \\
\mathrm{d}z_t &= \frac{\mu}{2}\,\mathrm{d}t
\end{aligned}$$

Rescale noise intensities: $(\sigma, \sigma') = (\varepsilon^{3/4}\bar{\sigma}, \varepsilon^{3/4}\bar{\sigma}')$ and consider z as “time”

Stochastic Folded Nodes: Final Reduction Step

Deviation $(\xi_z, \eta_z) = (x_z - x_z^{\text{det}}, y_z - y_z^{\text{det}})$ satisfies

$$\begin{aligned} \mathrm{d}\xi_z &= \frac{2}{\mu}(\eta_z - \xi_z^2 - 2x_z^{\text{det}}\xi_z)\,\mathrm{d}z + \frac{\sqrt{2}\sigma}{\sqrt{\mu}}\,\mathrm{d}W_z^{(1)} \\ \mathrm{d}\eta_z &= -\frac{2}{\mu}(\mu+1)\xi_z\,\mathrm{d}z + \frac{\sqrt{2}\sigma'}{\sqrt{\mu}}\,\mathrm{d}W_z^{(2)} \end{aligned}$$

We're in business . . . (almost)

- ▹ For small μ: Slowly driven system with two fast variables
- ▹ Calculate asymptotic covariance matrix
- ▹ Use Neishtadt's theorem on delayed Hopf bifurcations to obtain the correct asymptotic behaviour of the size of the covariance tube (see next slide)
- ▹ Use general result on concentration of sample paths

Stochastic Folded Nodes: Covariance Matrix

Lemma

For $z \leqslant \sqrt{\mu}$, the covariance matrix $\bar{X}(z,\mu) = \begin{pmatrix} \bar{X}_{11} & \bar{X}_{12} \\ \bar{X}_{21} & \bar{X}_{22} \end{pmatrix}$ satisfies

$$\bar{X}_{11}, \bar{X}_{22} \asymp \begin{cases} \frac{1}{|z|} & \text{for } z \leqslant -\sqrt{\mu} \;, \\ \frac{1}{\sqrt{\mu}} & \text{for } |z| \leqslant \sqrt{\mu} \;, \end{cases} \qquad |\bar{X}_{11} - \bar{X}_{22}| = \mathcal{O}(1) \;, \qquad |\bar{X}_{12}| = |\bar{X}_{21}| = \mathcal{O}(1)$$

Proof

- Coordinate change → canonical form
- Slow–fast system undergoing dynamic Hopf bifurcation
  (pair of complex eigenvalues crosses the imaginary axis at $z = 0$)
- Result follows from [Neishtadt '87]

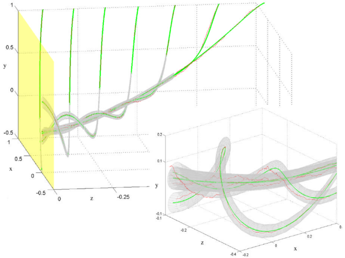
Stochastic Folded Nodes: Concentration of Sample Paths

Theorem [Berglund, G & Kuehn '10 (submitted to JDE)]

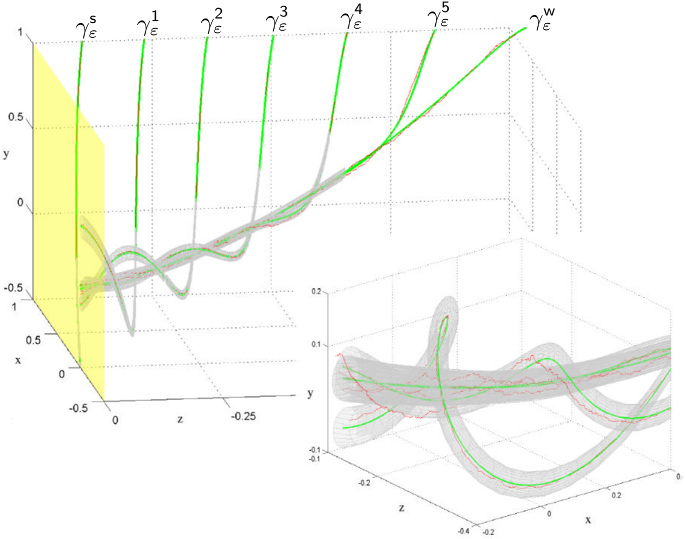
$$\mathbb{P}\Bigl\{\tau_{\mathcal{B}(h)} < z\Bigr\} \leqslant C(z_0, z)\, \exp\biggl\{-\kappa\frac{h^2}{2\sigma^2}\biggr\} \qquad \forall z \in [z_0, \sqrt{\mu}]$$

Recall: For $z = 0$

- Distance between canards γ_ε^k and γ_ε^{k+1} is $\mathcal{O}(\mathrm{e}^{-c_0(2k+1)^2\mu})$
- Section of $\mathcal{B}(h)$ is close to circular with radius $\mu^{-1/4}h$
- Noisy canards become indistinguishable when typical radius $\mu^{-1/4}\sigma \approx$ distance



Ducks or Pasta . . . ?

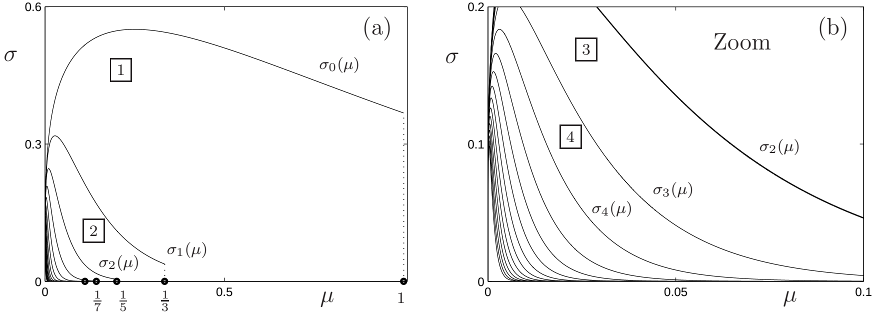


Noisy Small-Amplitude Oscillations

Theorem [Berglund, G & Kuehn '10 (submitted)]

Canards with $\frac{2k+1}{2}$ oscillations become indistinguishable from noisy fluctuations for

$$\sigma > \sigma_k(\mu) = \mu^{1/4}\,\mathrm{e}^{-(2k+1)^2\mu}$$



Early Escape and Global Returns

Early escape

- Consider $z > \sqrt{\mu}$
- $\mathcal{S}_0 =$ neighbourhood of γ^{w}, growing like $\sqrt{z}$

Theorem [Berglund, G & Kuehn ’10]

$\exists \kappa, \kappa_1, \kappa_2, C > 0$

s.t.

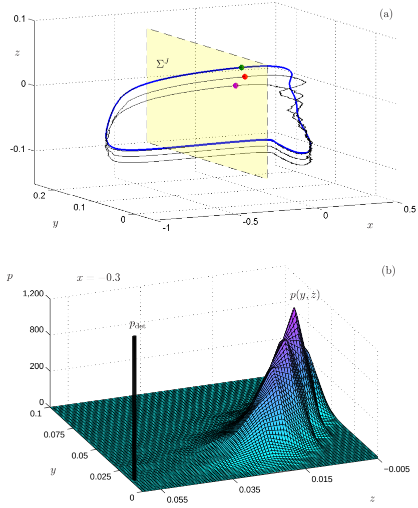
for $\sigma|\log\sigma|^{\kappa_1} \leqslant \mu^{3/4}$

$$\mathbb{P}\bigl\{\tau_{\mathcal{S}_0} > z\bigr\} \leqslant C|\log\sigma|^{\kappa_2}\,\mathrm{e}^{-\kappa(z^2-\mu)/(\mu|\log\sigma|)}$$

Remark

r.h.s. small for $z \gg \sqrt{\mu|\log\sigma|/\kappa}$

Global returns



Early Escape: Proof

Theorem

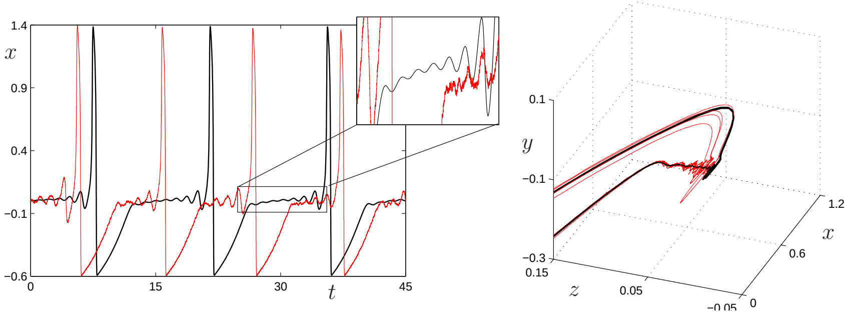
For $\sigma|\log\sigma|^{\kappa_1} \leqslant \mu^{3/4}$

$$\mathbb{P}\bigl\{\tau_{\mathcal{S}_0} > z\bigr\} \leqslant C|\log\sigma|^{\kappa_2}\,\mathrm{e}^{-\kappa(z^2-\mu)/(\mu|\log\sigma|)}$$

Proof

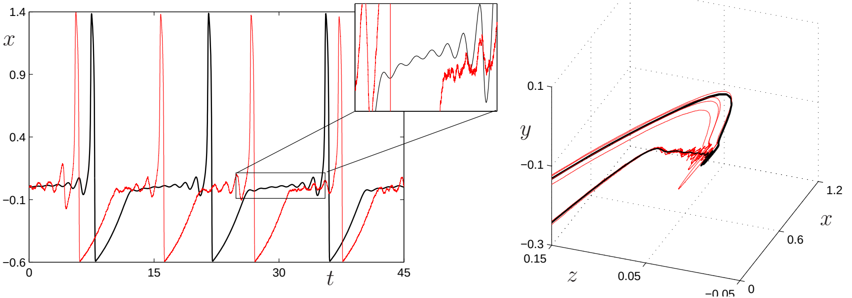
- Diffusion-dominated escape from smaller set around $\gamma_\varepsilon^{\mathrm{w}}$
  - Estimate distance covered in short time intervals
  - Use Markov property to restart
- Drift-dominated escape from $\mathcal{S}_0$
  - Use polar coordinates and averaging
  - Consider radius only
  - Show that drift dominates diffusion

A Model Allowing for Global Returns



$$
\begin{aligned}
\mathrm{d}x_t &= \frac{1}{\varepsilon}(y_t - x_t^2 - x_t^3)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)} \\
\mathrm{d}y_t &= [-(\mu+1)x_t - z_t]\,\mathrm{d}t + \sigma'\,\mathrm{d}W_t^{(2)} \\
\mathrm{d}z_t &= \left[\frac{\mu}{2} + ax_t + bx_t^2\right]\mathrm{d}t
\end{aligned}
$$

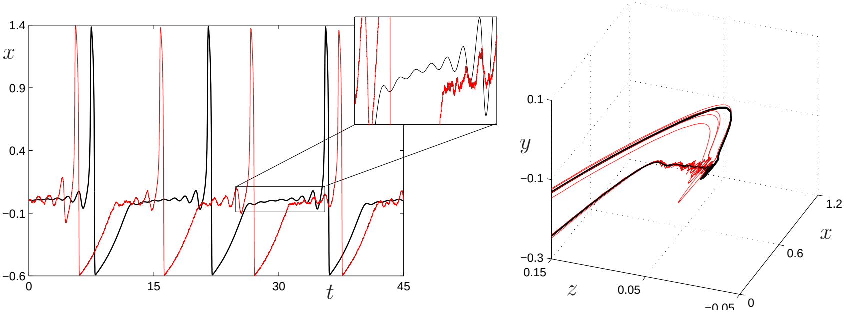
Mixed-Mode Oscillations in the Presence of Noise



Observations

- Noise smears out small-amplitude oscillations
- Early transitions modify the mixed-mode pattern

Outlook: Investigate MMO Patterns in Noisy Systems



Open Problems

- Study return mechanism via return map in the presence of noise
- Can preselected MMO patterns be achieved by tuning of parameters?

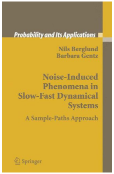
References

MMOs with Noise

- Nils Berglund, Barbara Gentz and Christian Kuehn, *Hunting French ducks in a noisy environment*, preprint, submitted to J. Differential Equations (2010)

Slow–Fast Systems with Noise

- Nils Berglund, Barbara Gentz, *Geometric singular perturbation theory for stochastic differential equations*, J. Differential Equations 191, 1–54 (2003)
- ———, *Noise-Induced Phenomena in Slow–Fast Dynamical Systems. A Sample-Paths Approach,* Springer, London (2005)



Probability and Its Applications

Nils Berglund
Barbara Gentz

Noise-Induced Phenomena in Slow-Fast Dynamical Systems

A Sample-Paths Approach

Springer

Introduction to Noise in Slowly-Driven Systems

- ———, *Beyond the Fokker–Planck equation: pathwise control of noisy bistable systems*, J. Phys. A 35, 2057–2091 (2002)
- ———, *Metastability in simple climate models: Pathwise analysis of slowly driven Langevin equations*, Stoch. Dyn. 2, 327–356 (2002)

Folded-Node: The Slow Subsystem

Slow subsystem

$$0 = y - x^2 \quad \Longrightarrow \quad \dot{y} = 2x\dot{x}$$

implies

$$\begin{aligned} 2x\dot{x} &= -(\mu+1)x - z \\ \dot{z} &= \frac{\mu}{2} \end{aligned}$$

Desingularized slow subsystem

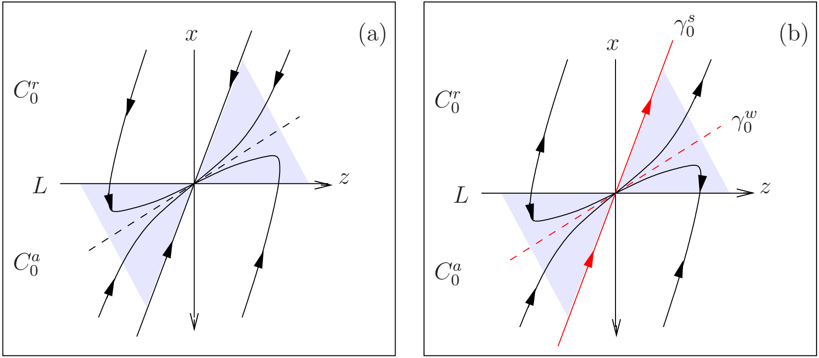
(obtained by setting $t = 2x\bar{t}$)

$$\begin{aligned} \dot{x} &= -(\mu+1)x - z \\ \dot{z} &= \mu x \end{aligned}$$

Properties of the desingularized slow flow

- Equilibrium $(x, z) = (0, 0)$
- Eigenvalues $(\lambda_s, \lambda_w) = (-1, -\mu)$
- Origin $(0, 0)$ is a stable node for $\mu \in (0, 1)$

Folded-Node: Singular Maximal Canards $\gamma_0^{s,w}$



(a) desingularized slow flow (b) slow flow

Definition

A *maximal singular canard* is an orbit in $\mathcal{C}_0^a \cap \mathcal{C}_0^r$

A *maximal canard* is an orbit in $\mathcal{C}_\varepsilon^a \cap \mathcal{C}_\varepsilon^r$