SPDES driven by Poisson Random Measures and their numerical Approximation

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September 7, 2011

Outline

- Lévy processes - Poisson Random Measure
- Stochastic Integration in Banach spaces
- SPDEs driven by Poisson Random Measure
- Their Numerical Approximation

The equation we would like to approximate

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary.

The Equation:

$$
(\star) \quad \begin{cases} \frac{du(t,\xi)}{dt} &= \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \alpha \nabla u(t,\xi) + g(u(t,\xi)) \dot{L}(t) \\ & \quad + f(u(t,\xi)), \quad \xi \in \mathcal{O},\ t > 0; \\ u(0,\xi) &= u_0(\xi) \quad \xi \in \mathcal{O}; \\ u(t,\xi) &= u(t,\xi) = 0, \quad t \geq 0, \xi \in \partial\mathcal{O}; \end{cases}
$$

where $u_0 \in L^p(\mathcal{O})$, $p \geq 1$, g a certain mapping and $L(t)$ is a space time Lévy noise specified later.

Problem: To find an approximation of

$$u : [0,\infty) \times \mathcal{O} \longrightarrow \mathbb{R}$$

solving Equation $(\star)$.

A Lévy Process

Definition

Let E be a Banach space. An E–valued stochastic process $L = \{L(t), 0 \leq t < \infty\}$ is a Lévy process over $(\Omega; \mathcal{F}; \mathbb{P})$ iff

- $L(0) = 0$;
- L has independent and stationary increments;
- L is stochastically continuous, i.e. for any $A \in \mathcal{B}(E)$ the function $t \mapsto \mathbb{E}1_A(L(t))$ is continuous on $\mathbb{R}^+$;
- L has a.s. càdlàg [a] paths;

[a] càdlàg = continue à droite, limitée à gauche.

Lévy - Khinchin - Formula

E denotes a separable Banach space and E' the dual on E.
If L is an E-valued Lévy process, then there exist

- $a \in E'$,
- $Q : E' \to E$ is a positive operator,
- and $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ is a Lévy measure
  (called usually the characteristic measure of L).

such that following formula holds for all $y \in E'$

$$\mathbb{E}\, e^{i\langle L(1),y\rangle} = \exp\left\{ i\langle a, y\rangle\lambda - \frac{1}{2}\langle Qy, y\rangle + \int_E \left(e^{i\lambda\langle y,a\rangle} - 1 - i\lambda y 1_{\{|y|\le 1\}} \right) \nu(dy) \right\}.$$

A Lévy Process

In what follows E denotes a separable Banach space, $\mathcal{B}(E)$ denotes the Borel-σ algebra on E and E' the dual on E.

Definition

(see Linde (1986), Section 5.4) A symmetric[a] σ–finite, Borel-measure $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ is called a Lévy measure if $\nu(\{0\}) = 0$ and the function

$$E' \ni a \mapsto \exp\left(\int_E (\cos(\langle x, a \rangle) - 1)\, \nu(dx) \right) \in \mathbb{C}$$

is a characteristic function of a certain Radon measure on E.

[a] $\nu(A) = \nu(-A)$ for all $A \in \mathcal{B}(E)$

An arbitrary σ-finite Borel measure ν is a Lèvy measure if its symmetrization $\nu + \nu^-$ is a symmetric Lévy measure.

Poisson Random Measure

Remark Let L be a Lévy process over $(\Omega, \mathcal{F}, \mathbb{P})$. Defining the so–called counting measure for $A \in \mathcal{B}(E)$

$$N(t,A) = \sharp\,\{s \in (0,t] : \Delta L(s) = L(s) - L(s-) \in A\} \in \mathbb{N} \cup \{\infty\}$$

one can show that

- $N(t,A)$ is a random variable over $(\Omega; \mathcal{F}; \mathbb{P})$;
- $N(t,A) \sim \text{Poisson}\,(t\nu(A))$ and $N(t,\emptyset) = 0$;
- For any disjoint sets $A_1, \ldots, A_n$, the random variables $N(t,A_1), \ldots, N(t,A_n)$ are independent; (independently scattered)

Poisson Random Measure

Definition

Let $(Z, \mathcal{Z})$ be a measurable space and $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. A Poisson random measure on $(Z, \mathcal{Z})$ is a family

$$\eta = \{\eta(\omega, \centerdot), \omega \in \Omega\}$$

of non-negative integer valued measures $\eta(\omega, \centerdot) : \mathcal{S} \to \mathbb{N}$, such that

- $\eta(\centerdot, \emptyset) = 0$ a.s.
- η is a.s. σ–additive.
- η is a.s. independently scattered.
- for each $A \in \mathcal{Z}$ such that $\mathbb{E}\,\eta(\cdot, A)$ is finite, $\eta(\cdot, A)$ is a Poisson random variable with parameter $\mathbb{E}\,\eta(\cdot, A)$.

Poisson Random Measure

Let $(S, \mathcal{S})$ be a measurable space and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space.

Definition

(see Ikeda Watanabe - 1981) A time homogeneous Poisson random measure η on $(S, \mathcal{S})$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, is a measurable function

$$\eta : (\Omega, \mathcal{F}) \to (M_I(S \times \mathbb{R}_+), \mathcal{M}_I(S \times \mathbb{R}_+)),$$

such that

(i) for each $B \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+)$, $\eta(B) := i_B \circ \eta : \Omega \to \bar{\mathbb{N}}$ is a Poisson random variable with parameter[a] $\mathbb{E}\eta(B)$;

(ii) η is independently scattered;

(iii) for each $U \in \mathcal{S}$, the $\bar{\mathbb{N}}$-valued process $(N(t, U))_{t\geq 0}$ defined by

$$N(t, U) := \eta(U \times (0, t]), \quad t \geq 0$$

is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta(U \times (s, t])$ is independent of $\mathcal{F}_s$.

[a] If $\mathbb{E}\eta(B) = \infty$, then obviously $\eta(B) = \infty$ a.s..

Poisson Random Measure

Example

Let η be a time homogeneous Poisson random measure on E with intensity ν, where ν is a Lévy measure. Then, the stochastic process

$$[0,\infty) \ni t \mapsto \hat{L}(t) := \int_0^t \int_E z\, \tilde{\eta}^a(ds, dt)$$

is a Lévy process with characteristic measure ν.

[a] Give a Poisson random measure $\eta : \mathcal{B}(E) \times \mathcal{B}([0,\infty)) \to \mathbb{N}_0$ we denote the compensated Poisson random measure by $\tilde{\eta}$.

Poisson Random Measure

Definition

Let

$$\eta : \Omega \times \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$$

be a Poisson random measure on E over $(\Omega; \mathcal{F}; \mathbb{P})$ and $\{\mathcal{F}_t, 0 \leq t < \infty\}$ the filtration induced by η. Then the predictable measure

$$\gamma : \Omega \times \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$$

is called compensator of η, if for any $A \in \mathcal{B}(E)$ the process

$$\tilde{\eta}(A \times (0, t]) := \eta(A \times (0, t]) - \gamma(A \times [0, t])$$

is a local martingale over $(\Omega; \mathcal{F}; \mathbb{P})$.

Remark The compensator is unique up to a $\mathbb{P}$-zero set and in case of a time homogeneous Poisson random measure given by

$$\gamma(A \times [0, t]) = t\, \nu(A), \quad A \in \mathcal{B}(E).$$

Space - Time - White - Noise

Let us recall the Definition of a Gaussian white noise (Dalang 2003):

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(S, \mathcal{S}, \sigma)$ a measure space. Then a Gaussian white noise on S based on σ is a measurable mapping

$$W : (\Omega, \mathcal{F}) \to (M(E), \mathcal{M}(E))^a$$

- For $A \in \mathcal{S}$, $W(A)$ is a real valued Gaussian random variable with mean 0 and variance $\sigma(A)$, provided $\sigma(A) < \infty$;
- if A and $B \in \mathcal{S}$ are disjoint, then the random variables $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.

[a] $M(S)$ denotes the set of all measures from $\mathcal{S}$ into $\mathbb{R}$,
i.e. $M(S) := \{\mu : \mathcal{S} \to \mathbb{R}\}$ and $\mathcal{M}(S)$ is the σ-field on $M(S)$ generated by functions $i_B : M(S) \ni \mu \mapsto \mu(B) \in \mathbb{R}$, $B \in \mathcal{S}$.

Space - Time - White - Noise

Put

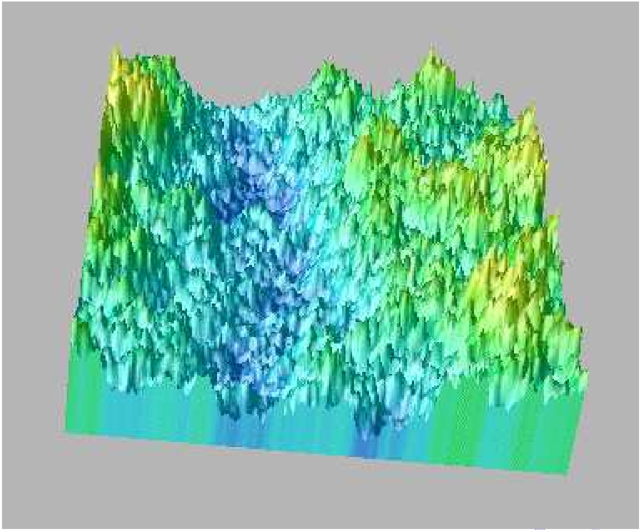
- $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.
- $S = \mathcal{O} \times [0, \infty)$,
- $\mathcal{S} = \mathcal{B}(\mathcal{O}) \times \mathcal{B}([0, \infty))$
- $\sigma = \lambda_{d+1}$[1].

Then, by definition, the space time Gaussian white noise is the measure valued process process

$$t \mapsto W(\,\cdot\, \times [0, t)).$$

[1] λ_{d+1} denotes the Lebesgue measure in $\mathbb{R}^d$.

Space - Time - White - Noise



Space - Time - White - Noise

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $(S, \mathcal{S}, \sigma)$ be a measurable space. Then the Lévy white noise on S based on σ with characteristic jump size measure $\nu \in \mathcal{L}(\mathbb{R})$ is a measurable mapping

$$L : (\Omega, \mathcal{F}) \to (M(S), \mathcal{M}(S))^a$$

such that

- For $A \in \mathcal{S}$, $L(A)$ is a real valued infinite divisible random variables with characteristic exponent
$$e^{i\theta L(A)} = \exp\left(\sigma(A) \int_{\mathbb{R}} \left(1 - e^{i\theta x} - i\sin(\theta x)\right) \nu(dx)\right),$$
provided $\sigma(A) < \infty$.
- if A and $B \in \mathcal{S}$ are disjoint, then the random variables $L(A)$ and $L(B)$ are independent and $L(A \cup B) = L(A) + L(B)$.

[a] $M(E)$ denotes the set of all measures from $\mathcal{E}$ into $\mathbb{R}$, i.e. $M(S) := \{\mu : \mathcal{S} \to \mathbb{R}\}$ and $\mathcal{M}(S)$ is the σ-field on $M(S)$ generated by functions $i_B : M(S) \ni \mu \mapsto \mu(B) \in \mathbb{R}$, $B \in \mathcal{S}$.

Space - Time - White - Noise

Again put

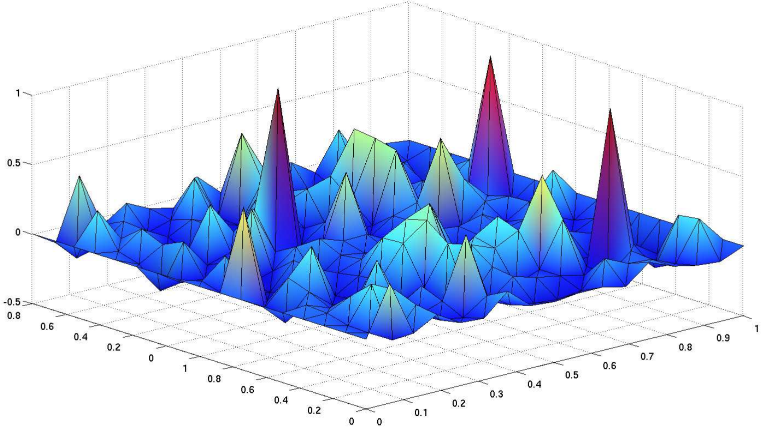
- $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.
- $S = \mathcal{O} \times [0, \infty)$,
- $\mathcal{S} = \mathcal{B}(\mathcal{O}) \times \mathcal{B}([0, \infty))$
- $\sigma = \lambda_{d+1}$.

Then, by definition, the space time Lévy white noise is the measure valued process process

$$t \mapsto L(\,\cdot\, \times [0, t));$$

(for more details we refer to Breźniak and Hausenblas (2009) or Peszat and Zabczyk (2007))

Space - Time - White - Noise



Space - Time - White - Noise

The Numerical Approximation

The Numerical Approximation of a Lévy walk

Let μ be a Poisson random measure on $\mathbb{R}$ with intensity ν, i.e. let $L = \{L(t) : 0 \le t < \infty\}$ be the corresponding Lévy process with characteristic measure ν defined by

$$[0,\infty) \ni t \mapsto L(t) := \int_0^t \int_{\mathbb{R}} z\,\tilde{\mu}(dz, ds) \in \mathbb{R}.$$

Question:

Given the characteristic measure of a Lévy process,
how to simulate the Lévy walk ?

That means, how to simulate

$$\left(\Delta_\tau^0 L, \Delta_\tau^1 L, \Delta_\tau^2 L, \ldots, \Delta_\tau^k L, \ldots\right),$$

where

$$\Delta_\tau^k L := L((k+1)\tau) - L(k\tau), \quad k \in \mathbb{N}.$$

The Numerical Approximation of a Lévy walk

Different approaches:

- Direct generation: in case of stable, exponential, gamma, geometric and negative binomial distribution.
- Generation from compound Poisson Process (Rubenthaler (2003), Amussen and Rósinski, Fornier (2010))
- generation with shot noise representation (Bondesson (1980), Rosinski (2001), Kawai and Imai (2007))
- Approximation of the density function by series (Eberlein, ...)
- Via Fouriertransform (Taqqu ...)

The Numerical Approximation of a Lévy walk

Generation with shot noise representation

Assume U is a random variable on $[0, T]$ the Lévy measure can be written as

$$\nu(B) = \int_0^\infty \mathbb{P}\left(H(r, U) \in B\right)\, dr, \quad B \in \mathcal{B}(\mathbb{R}).$$

- $\{E_k : k \in \mathbb{N}\}$ sequence of iid exponential random variables with unit mean;
- $\{\Gamma_k : k \in \mathbb{N}\}$ sequence of standard Poisson arrivials time, in particular

$\{\Gamma_k - \Gamma_{k-1} : k \in \mathbb{N}\}$ are exponential distributed with certain parameter;

- $\{U_k : k \in \mathbb{N}\}$ sequence of iid uniform distributed random variables on $[0, 1]$

The Numerical Approximation

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}_\tau^k L_\epsilon$ which approximates $\Delta_\tau^k L_\epsilon$;
- Approximating the small jumps by a Wiener process (optional);

The Numerical Approximation

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}_\tau^k L_\epsilon$ which approximates $\Delta_\tau^k L_\epsilon$;
- Approximating the small jumps by a Wiener process (optional);

Cutting off the small jumps

- fix a cut off parameter $\epsilon > 0$;
- define the new characteristic measure

$$\nu^{c\epsilon} : \mathcal{B}(\mathbb{R}) \ni B \mapsto \nu\left(B \cap ((-\infty, -\epsilon] \cup [\epsilon, \infty))\right).$$

- Let $\mu^{c\epsilon}$ be a Poisson random measure corresponding where the supports does not include $[-\epsilon, \epsilon]$, i.e.

$$\mu^{c\epsilon} : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^+) \ni B \times I \mapsto \int_{(-\infty,-\epsilon]\cup[\epsilon,\infty)} \int_I 1_B(z)\, \mu(dz, dt) \in \mathbb{N} \cup \{\infty\}.$$

- Let $L_\epsilon = \{L(t) : 0 \leq t < \infty\}$ the Lévy process which arises by cutting off the small jumps, i.e.

$$L_\epsilon(t) := \int_0^t \int_{\mathbb{R}} \zeta\, \tilde{\mu}^{c\epsilon}(d\zeta, ds), \quad t \geq 0.$$

The Numerical Approximation

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}_\tau^k L_\epsilon$ which approximates $\Delta_\tau^k L_\epsilon$;
- Approximating the small jumps by a Wiener process (optional);

Approximating the characteristic measure by a discrete measure

Approximate the characteristic measure $\nu^{c\epsilon}$ by a characteristic measure $\hat{\nu}^{c\epsilon}$ having the following form:

- Let $B_1, \dots, B_J$ be disjoint sets with $\cup_{j=1}^J B_j \subset (-\infty, -\epsilon] \cup [\epsilon, \infty)$;
- let $c_1, \cdots, c_J$ be points belonging to $\mathbb{R}$ such that $c_j \in D_j$, $j = 1 \dots J$;
- $\nu^{c\epsilon}$ can be written as

$$\hat{\nu}^{c\epsilon} := \sum_{j=1}^{J} \nu(B_j)\delta_{c_j}.$$

Example

- Put $B_j = [x_j, x_{j+1})$, $j = 1, \dots, J/2 - 1$, $B_{J/2} = [x_{J/2}, \infty)$, $B_{J/2+j} = [-x_j, -x_{j+1})$, $j = 1, \dots, J/2 - 1$, such that $\nu(B_j) < \gamma$;
- Put $B_J = [-x_{J-1}, -\infty)$, where $x_0 = \epsilon < x_1 < \cdots < x_{J/2} < \infty$;
- Put $c_j := x_j$.

The Numerical Approximation

Different Discretization:

- equally weighted mesh: Let $D_1 = [\epsilon, x_2), D_J = [x_J, \infty)$, and $D_i = [x_i, x_{i+1})$ such that $\nu(D_i) = \nu(D_j)$ for $1 \le i, j \le J \sim O(\epsilon^{-1})$.
- equally spaced mesh: Let $D_1 = [\epsilon, x_2), D_J = [x_J, \infty)$, and $D_i = [x_i, x_{i+1})$ such that $\lambda(D_i) = \lambda(D_j)$ for $1 \le i, j \le J \sim O(\epsilon^{-1})$.
- strechted mesh: Let $D_1 = [\epsilon^2, x_2), D_J = [x_J, \infty)$, and choose $\epsilon_j := \lambda(D_j)$ according to

$$\epsilon_j \quad := \quad \begin{cases} j\epsilon^2 & \text{if } x_j < 1, \\ j^\gamma \epsilon & \text{if } x_j \ge 1, \end{cases}$$

  for some $0 < \gamma < 1$.
  (Here, one can fit 2 and γ according to α and β)

The Numerical Approximation

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}_\tau^k L_\epsilon$ which approximates $\Delta_\tau^k L_\epsilon$;
- Approximating the small jumps by a Wiener process (optional);

Generating a random variable

$$\hat{\Delta}_\tau^k L_\epsilon$$

Remember

- (Independently scattered property)

The family

$$\{\eta\,(D_j \times [t_{k-1}, t_k)) : j = 1, \ldots, n, k = 1, \ldots, K\}$$

of random variables over $(\Omega, \mathcal{F}, \mathbb{P})$ is mutually independent;

- for any $D \times I \in \{D_j \times [t_{k-1}, t_k) : j = 1, \ldots, n, k = 1, \ldots, K\}$ the random

variable

$$\eta(D \times I)$$

is Poisson distributed with Parameter $\nu(D) \cdot \lambda(I)$.

Generating a random variable

$$\hat{\Delta}^k_\tau L_\epsilon$$

At each time step k generate a family $\{\mathrm{q}^1_k, \ldots, \mathrm{q}^J_k\}$ of independent random variables, where

$$\mathbb{P}\left(\mathrm{q}^j_k = l\right) = \exp(-\tau\nu(B_j))\,\frac{\tau^l \nu(B_j)^l}{l!}, \quad l \geq 0.$$

The approximation $\hat{\Delta}^k_\tau L$ will be given as follows.

$$\hat{\Delta}^k_\tau L_\epsilon := \sum_{j=1}^{J} \left(\mathrm{q}^j_k - \tau\nu(B_j)\right) c_j, \quad k \geq 0.$$

The Lévy random walk is approximated by

$$\left(\hat{\Delta}^0_\tau L_\epsilon, \hat{\Delta}^1_\tau L_\epsilon, \hat{\Delta}^2_\tau L_\epsilon, \ldots, \hat{\Delta}^k_\tau L_\epsilon, \ldots\right),$$

Open Question

At each time step $k \geq 1$ is it sufficient to simulate a family $\{\mathrm{q}_k^1, \ldots, \mathrm{q}_k^J\}$ of independent random variables, where

$$\mathbb{P}\left(\mathrm{q}_k^j = 0\right) = \exp(-\tau\nu(B_j)) \quad j = 1, \ldots, k$$

and

$$\mathbb{P}\left(\mathrm{q}_k^j = 1\right) = 1 - \exp(-\tau\nu(B_j)) \quad j = 1, \ldots, k.$$

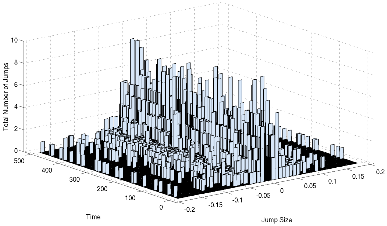
The approximation is now given as follows.

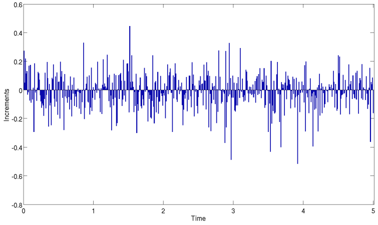
$$\hat{\Delta}_\tau^k L_\epsilon := \sum_{j=1}^{J} \left(\mathrm{q}_k^j - \tau\nu(B_j)\right) c_j, \quad k \geq 0.$$

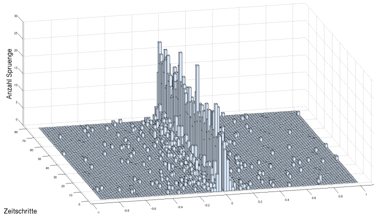
The Lévy random walk is approximated by

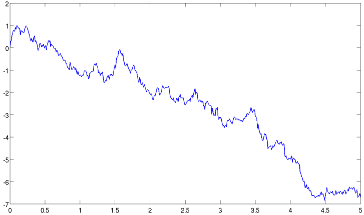
$$\left(\hat{\Delta}_\tau^0 L_\epsilon, \hat{\Delta}_\tau^1 L_\epsilon, \hat{\Delta}_\tau^2 L_\epsilon, \ldots, \hat{\Delta}_\tau^k L_\epsilon, \ldots\right),$$

Simulation des Lévy walks

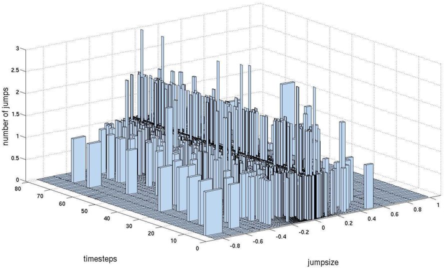


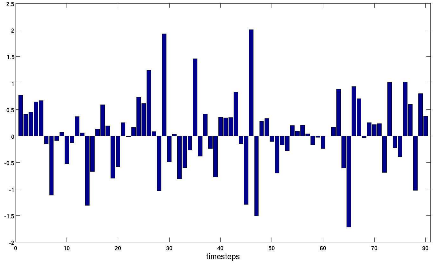






Simulation des Lévy walks





Approximating the small jumps (optional);

At each time step k generate a Gaussian random variables $\Delta_\tau^k W$, where

$$\mathcal{L}(\Delta_\tau^k W_\epsilon) = \sigma(\epsilon)\, \mathcal{N}\,(0, \tau)$$

and

$$\sigma(\epsilon) := \sqrt{\int_{-\epsilon}^{\epsilon} \zeta^2 \nu(d\zeta)}.$$

The Lévy random walk is approximated by

$$\left(\hat{\Delta}_\tau^0 L_\epsilon + \Delta_\tau^0 W_\epsilon, \hat{\Delta}_\tau^1 L_\epsilon + \Delta_\tau^1 W_\epsilon, \hat{\Delta}_\tau^2 L_\epsilon + \Delta_\tau^2 W_\epsilon, \ldots, \hat{\Delta}_\tau^k L_\epsilon + \Delta_\tau^k W_\epsilon, \ldots\right).$$

The Numerical Approximation

Proposition Assume, that the intensity is of type α, $\alpha \in (0, 2]$. Then,

$$\mathbb{E}\left|\Delta_\tau^k L_\epsilon^c - \Delta_\tau^k W_\epsilon\right|^p \leq C\, \tau^{\frac{p}{3}}\, \epsilon^{\frac{p(3-\alpha)}{3}}, \quad \epsilon > 0 \;\; \tau > 0,$$

where $\Delta_\tau^k L_\epsilon^c := \Delta_\tau^k L - \Delta_\tau^k L_\epsilon$. If, in addition, $\tau = \epsilon^\rho$, then

$$\mathbb{E}\left|\Delta_\tau^k L_\epsilon^c - \Delta_\tau^k W_\epsilon\right|^p \leq C\, \tau \epsilon^{\frac{p}{3}(3-\alpha+\rho(1-\frac{3}{p})}, \quad \epsilon > 0 \;\; \tau > 0.$$

The proof is via the Fourier transform, tail estimates and Taylor approximations.

The Approximation of a space time Lévy noise

Let $\mathcal{O}$ be a bounded domain and let $\mathcal{T}$ be a subdivision with

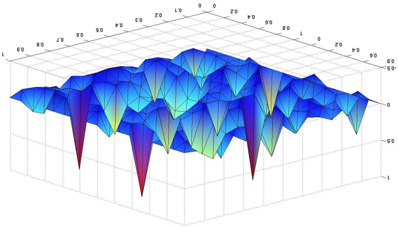
- a set of element domains $\mathcal{K} = \{K_i : i = 1\ldots, l\}$,
- a set of nodals variable $\mathcal{N} = \{N_i, i = 1\ldots, l\}$ and
- a set of space functions $\mathcal{P} = \{\phi_i : i = 1, \ldots, l\}$
- a Voronoi decomposition $\mathcal{J} = \{J_i; i = 1, \ldots, l\}$ induced by $\mathcal{N}$

Let $\mathbf{L} = \{L^i : i = 1, \ldots l\}$ be a family of independent Lévy processes where L^i has characteristic $\rho_i\,\nu$, $\rho_i = \lambda(J_i)$, $i = 1, \ldots, l$. The measure valued process $\tilde{\mathbf{L}} = \{\tilde{\mathbf{L}}(t) : t \geq 0\}$ given by

$$\tilde{\mathbf{L}}(t) : \mathcal{B}(\mathcal{O}) \ni A \mapsto \sum_{i=1}^{N} c_i \int_A \phi_i(x)\, dx\, L^i(t), \quad t \geq 0,$$

where $c_i = (|\phi_i|_{L^1}^{-1})$, is an approximation in space of the space time Lévy noise.

Space - Time - Poissonian Noise



The Equation to Approximate

Let $\mathcal{O} = [0,1] \times [0,1]$. The Equation:

$$(\star) \quad \begin{cases} \frac{du(t,\xi)}{dt} &= \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \alpha \nabla u(t,\xi) + g(u(t,\xi))\dot{L}(t) \\ & \quad + f(u(t,\xi)), \quad \xi \in \mathcal{O}, \; t > 0; \\ u(0,\xi) &= u_0(\xi) \quad \xi \in \mathcal{O}; \\ u(t,\xi) &= u(t,\xi) = 0, \quad t \geq 0, \xi \in \partial\mathcal{O}; \end{cases}$$

where $u_0 \in L^p(0,1)$, $p \geq 1$, g a certain mapping and $L(t)$ is a Lévy process taking values in a certain Banach space.

Problem: To find a function

$$u : \mathbb{R}_+ \times \mathcal{O} \longrightarrow \mathbb{R}$$

solving Equation $(\star)$.

Numeric of SPDEs

The implicit Euler scheme. Here

$$\frac{u_n(t+\tau_n)-u_n(t)}{\tau_n} \approx Au_n(t+\tau_n) \\ f(u_n(t))+g(u_n(t))\Delta_n L(t),$$

Again, if v_n^k denotes the approximation of $u_n(k\tau_n)$, then

$$\begin{cases} v_n^{k+1} &= (1-\tau_n A_n)^{-1} v_n^k + \tau_n f(v_n^k) \\ & \quad + \sqrt{\tau_n}\, g(u_n(t))\, \xi_k^n, \\ v_n^0 &= x. \end{cases}$$

Between the points $k\tau_n$ and $(k+1)\tau_n$ the solution is linear interpolated.

Numerical Analysis

Theorem

(E.H. 2008)

- *Let $u = \{u(t); 0 \le t < \infty\}$ be the mild solution of an SPDE wit space time Lévy noise*
- *Let $\hat{u} = \{\hat{u}_k, k \in \mathbb{N}\}$ be the approximation of u*
  - *Discretization in space: affine finite Elements with length h;*
  - *Discretization in time: implicit Euler scheme with time step τ_h.*

Then, the error is given by

$$\left[\mathbb{E}\, \left\| u(k\tau_h) - \hat{u}_h^k \right\|_{L^p(\mathcal{O})}^p \right]^{\frac{1}{p}} \le C_0\, \tau^\alpha + C_1\, h^\delta.$$

where $\alpha < \frac{2}{p} - d + \frac{d}{p}$ and $\delta < \left(2 + \frac{d}{2} - \frac{d}{p}\right)\left(\frac{2}{p} - d + \frac{d}{p}\right)$.

The Numerical Approximation

Assume in addition that the space time Lévy white noise is approximated as described before. Then The following holds.

- For any $M \in \mathbb{N}$ there exist constants C_1 and C_2 such that

$$\sup_{0\le m\le M} \mathbb{E}\, \left\| v_h^m - \hat{v}_{\epsilon_h}^m \right\|_{L^p(\mathcal{O})}^p \le C_1 \epsilon_h^{p-\alpha}$$

$$+ C_2 \int_{\zeta\in\mathbb{R}\setminus(-\epsilon_h,\epsilon_h)} |\zeta|^p d(\hat{\nu}^{\epsilon_h} - \nu^{\epsilon_h})(d\zeta).$$

- If the stability condition $(\star)$ is satisfied, then there exist constants C_1 and C_2 such that for any $\theta > 0$, and ϵ_h with $\epsilon_h \sim \tau_h^\theta$,

$$\sup_{0\le m\le M} \mathbb{E}\, \left\| v_h^m - \hat{v}_{h,\epsilon}^{w,m} \right\|_{L^p(\mathcal{O})}^p \le C_1\, \tau_h^{\frac{p}{3}}\, \epsilon_h^{\frac{p(3-\alpha)}{3}}$$

$$+ C_2 \int_{\zeta\in\mathbb{R}\setminus(-\epsilon_h,\epsilon_h)} |\zeta|^p d(\hat{\nu}^{c,\epsilon_h} - \nu^{\epsilon_h})(d\zeta).$$