Random attractors under discretization

Peter Kloeden

Institut für Mathematik
Goethe Universität
Frankfurt am Main

⋆ joint work with with S. Schmalfuß (Paderborn), T. Caraballo, M.-J. Garrido-Atienza (Sevilla), A. Neuenkirch (Kaiserslautern) etc
Deterministic case

Consider the ordinary differential equation in $\mathbb{R}^d$

$$\frac{dx}{dt} = f(x)$$

- $f$ is regular and satisfies the dissipativity condition such as

$$\langle x, f(x) \rangle \leq K - L|x_1 - x_2|^2,$$

on $\mathbb{R}^d$ for some $K \geq 0$ and $L > 0$,

$\implies$ there exists a global attractor $A_0$. 
Kloeden & Lorenz (SINUM 1986):

- then a one-step numerical scheme with constant step size $h > 0$ has an attractor $A_h$ and the Hausdorff separation satisfies

$$d(A_h, A_0) \to 0 \quad \text{as} \quad h \to 0^+$$

i.e., upper semi continuous convergence of the numerical attractors
More structure, more information

If \( f \) satisfies a one-sided dissipative Lipschitz condition

\[
\langle x - y, f(x) - f(y) \rangle \leq -L|x_1 - x_2|^2,
\]

for all \( x, y \in \mathbb{R}^d \) for some \( L > 0 \)

\[ \implies \] there is a unique asymptotically stable steady state \( \bar{x} \) with \( f(\bar{x}) \neq 0 \), i.e., the global attractor is \( A_0 = \{\bar{x}\} \), and

- for most one-step numerical schemes the numerical attractor is also \( A_h = \{\bar{x}\} \).
**Stochastic case**

*What is the effect of background or environmental noise?*

Consider the Ito stochastic differential equation with additive noise

\[ dX_t = f(X_t) \, dt + \alpha \, dW_t, \]

where \( W_t \) is a two-sided scalar Wiener process and \( \alpha \in \mathbb{R}^d \) is a constant vector.

This has no equilibrium solution but if \( f \) satisfies a one-sided dissipative Lipschitz condition, then

\[ \Rightarrow \exists \text{ unique stochastic stationary solution } \bar{X}_t, \text{ which is pathwise globally asymptotically stable.} \]
Recall:

- The solutions of Ito stochastic differential equations are pathwise continuous, but not differentiable.

- Ito SDEs are really stochastic integral equations with stochastic integrals defined in the mean-square or $L_2$ sense.

How do we apply the Lipschitz properties to obtain pathwise estimates?
A technical detour: Consider the Ito SDE

\[ dX_t = f(X_t) \, dt + \alpha \, dW_t \]

where \( f \) satisfies the one-sided Lipschitz condition.

i.e., the stochastic integral equation

\[ X_t = X_{t_0} + \int_{t_0}^t f(X_s) \, ds + \alpha \int_{t_0}^t dW_t \]

The difference of any two solutions satisfies pathwise

\[ X^1_t - X^2_t = X^1_{t_0} - X^2_{t_0} + \int_{t_0}^t \left[ f(X^1_s) - f(X^2_s) \right] \, ds \]

continuous paths
Fundamental theorem of calculus \( \Rightarrow \) \( X_t^1 - X_t^2 \) pathwise differentiable.

\[
\frac{d}{dt} [X_t^1 - X_t^2] = f(X_t^1) - f(X_t^2) \quad \text{pathwise}
\]

- Apply the one-sided Lipschitz condition

\[
\frac{d}{dt} \left| X_t^1 - X_t^2 \right|^2 = 2 \langle X_t^1 - X_t^2, f(X_t^1) - f(X_t^2) \rangle \leq -2L \left| X_t^1 - X_t^2 \right|^2
\]

\[
\Rightarrow \quad \left| X_t^1 - X_t^2 \right|^2 \leq \left| X_{t_0}^1 - X_{t_0}^2 \right|^2 e^{-2L(t-t_0)} \to 0 \quad \text{as} \quad t \to \infty
\]

i.e. all solutions converge pathwise together — but to what?
**Special case:** Ito SDE with linear drift $f(x) = -x$

$$dX_t = -X_t \, dt + \alpha \, dW_t$$

explicit solution

$$X_t = X_{t_0} e^{-(t-t_0)} + \alpha e^{-t} \int_{t_0}^{t} e^s \, dW_s$$

The forward limit as $t \to \infty$ does not exist — moving target!

But the pullback limit as $t_0 \to -\infty$ with $t$ fixed does exist:

$$\lim_{t_0 \to -\infty} X_t = \bar{O}_t := \alpha e^{-t} \int_{-\infty}^{t} e^s \, dW_s \quad \text{(pathwise)}$$

The Ornstein-Uhlenbeck stochastic stationary process $\bar{O}_t$ is a solution of the linear SDE and all other solutions converge pathwise to it in the forward sense

$$|X_t - \bar{O}_t| \to 0 \quad \text{as} \quad t \to \infty \quad \text{(pathwise)}$$
Random dynamical systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X, d_X)$ a metric space.

A random dynamical system $(\theta, \phi)$ on $\Omega \times X$ consists of

1. a metric dynamical system $\theta$ on $\Omega$, which models the noise,

2. a cocycle mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$, which represents the dynamics on the state space $X$ and satisfies

   1. $\phi(0, \omega, x_0) = x_0$ (initial condition)
   2. $\phi(s + t, \omega, x_0) = \phi(s, \theta_t \omega, \phi(t, \omega, x_0))$ (cocycle property)
   3. $(t, x_0) \mapsto \phi(t, \omega, x_0)$ is continuous (continuity)
   4. $\omega \mapsto \phi(t, \omega, x_0)$ is $\mathcal{F}$-measurable (measurability)

for all $s, t \geq 0$, $x_0 \in X$ and $\omega \in \Omega$. 
Random attractors

A random attractor is a family of nonempty measurable compact subsets of $X$

$$\hat{A} = \{A(\omega) : \omega \in \Omega\}$$

which is

- $\phi$-invariant $\phi(t, \omega, A(\omega)) = A(\theta_t \omega)$ for all $t \geq 0$,
- pathwise pullback attracting in the sense that

$$\text{dist}_X (\phi (t, \theta_{-t} \omega, D(\theta_{-t} \omega)) , A(\omega)) \to 0 \quad \text{for} \quad t \to +\infty$$

for all suitable families $\hat{D} = \{D(\omega) : \omega \in \Omega\}$ of nonempty measurable bounded subsets of $X$. 
**Theorem** (Crauel, Flandoli, Schmalfuß etc)

Let \((\theta, \phi)\) be an RDS on \(\Omega \times X\) such that \(\phi(t, \omega, \cdot) : X \to X\) is a compact operator for each fixed \(t > 0\) and \(\omega \in \Omega\).

If there exists a pullback absorbing family \(\hat{B} = \{B(\omega) : \omega \in \Omega\}\) of nonempty closed and bounded measurable subsets of \(X\), i.e. there exists a \(T_{\hat{D}, \omega} \geq 0\) such that

\[
\phi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega) \quad \text{for all} \quad t \geq T_{\hat{D}, \omega}
\]

for all \(\hat{D} = \{D(\omega) : \omega \in \Omega\}\) in a given attracting universe.

Then the RDS \(\Omega \times X\) has a random attractor \(\hat{A}\) with component subsets given by

\[
A(\omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \phi(t, \theta_{-t} \omega, B(\theta_{-t} \omega)) \quad \text{for each} \quad \omega \in \Omega.
\]
**General case again**

Substract the integral version of the linear SDE for $\bar{O}_t$ from the integral version of the nonlinear SDE

$$dX_t = f(X_t) \, dt + \alpha \, dW_t$$

to obtain

$$X_t - \bar{O}_t = X_{t_0} - \bar{O}_{t_0} + \int_{t_0}^{t} [f(X_s) + \bar{O}_s] \, ds$$

$\Rightarrow \quad V_t := X_t - \bar{O}_t$ is pathwise differentiable and satisfies the pathwise ODE

$$\frac{d}{dt} V_t = f(V_t + \bar{O}_t) + \bar{O}_t \quad \text{(pathwise)}$$
• Apply the one-sided Lipschitz condition pathwise to

$$\frac{d}{dt} [X_t - \bar{O}_t] = [f(X_t) - f(\bar{O}_t)] + [f(\bar{O}_t) + \bar{O}_t] \quad \text{(pathwise)}$$

to obtain the pathwise estimate

$$|V_t|^2 \leq |V_{t_0}|^2 e^{-L(t-t_0)} + \frac{2}{L} \int_{t_0}^{t} e^{Ls} \left( |f(\bar{O}_s)|^2 + |\bar{O}_s|^2 \right) ds$$

• Take pathwise pullback convergence as $t_0 \to -\infty$ to obtain

$$|X_t - \bar{O}_t| \leq \bar{R}_t := 1 + \frac{2}{L} \int_{-\infty}^{t} e^{Ls} \left( |f(\bar{O}_s)|^2 + |\bar{O}_s|^2 \right) ds$$

for $t \geq T$ depending on suitable bounded sets of initial values.
• i.e., there exists a family of compact pullback absorbing balls centered on \( \tilde{O}_t \) with random radius \( \tilde{R}_t \).

• Dynamical systems limit set ideas
  \[ \implies \text{there exists a compact setvalued stochastic process } A_t \text{ inside} \]
  \[ \text{these absorbing balls which pathwise pullback attracts the solutions}. \]

• **BUT** the solutions converge together pathwise in forwards sense, so
  the sets \( A_t \) are in fact all singleton sets

\[ \implies \exists \text{stochastic stationary solution } \bar{X}_t. \]
General Principles

- All regular Ito SDE in $\mathbb{R}^d$ can be transformed into pathwise ODE
  
  \cite{Imkeller & Schmalfuß (2001), Imkeller & Lederer (2001,2002)}

- and generate random dynamical systems

  ⇒ pathwise theory and numerics for Ito SDE

- Pullback convergence enables us to construct moving targets.
• Stochastic stationary solutions are a simple singleton set version of more general random attractors

⇒ theory of random dynamical systems

e.g., Ludwig Arnold (Bremen)

• parallel theory of deterministic skew product flows

  e.g., almost periodic ODE : George Sell (Minneapolis)

⇒ A theory of nonautonomous dynamical systems

  e.g., pullback attractors
Effects of discretization

Numerical Ornstein-Uhlenbeck process

For the linear SDE with additive noise,

\[ dX_t = -X_t \, dt + \alpha \, dW_t, \]

the drift-implicit Euler-Maruyama scheme with constant step size \( h > 0 \) is

\[ X_{n+1} = X_n - hX_{n+1} + \alpha \Delta W_n, \quad n = n_0, n_0 + 1, \ldots, \]

which simplifies algebraically to

\[ X_{n+1} = \frac{1}{1 + h} X_n + \frac{\alpha}{1 + h} \Delta W_n, \]

Here the \( \Delta W_n = W_{h(n+1)} - W_{hn} \) are mutually independent and \( N(0, h) \) distributed
It follows that

\[ X_n = \frac{1}{(1 + h)^{n-n_0}} X_{n_0} + \frac{\alpha}{1 + h} \sum_{j=n_0}^{n-1} \frac{1}{(1 + h)^{n-1-j}} \Delta W_j \]

and the pathwise pullback limit, i.e. with \( n \) fixed and \( n_0 \to -\infty \), gives the discrete time numerical Ornstein-Uhlenbeck process

\[ \hat{O}_n^{(h)} := \frac{\alpha}{1 + h} \sum_{j=-\infty}^{n-1} \frac{1}{(1 + h)^{n-1-j}} \Delta W_j, \quad n \in \mathbb{Z}. \quad (1) \]

which is an entire solution of the numerical scheme and a discrete time stochastic stationary process.

One can show that it converges pathwise to the continuous time Ornstein-Uhlenbeck process in the sense that

\[ \hat{O}_0^{(h)} \to \hat{O}_0 \quad \text{as} \quad h \to 0. \]
Discretization of an nonlinear stochastic system

Consider the nonlinear SDE in $\mathbb{R}^d$ with additive noise,

$$dX_t = f(X_t) \, dt + \alpha \, dW_t,$$

where the drift coefficient $f$ is continuously differentiable and satisfies the one-sided dissipative Lipschitz condition with constant $L$.

The drift-implicit Euler-Maruyama scheme with constant step size $h > 0$ applied to this SDE is

$$X_{n+1} = X_n + hf(X_{n+1}) + \alpha \Delta W_n,$$

which is, in general, an implicit algebraic equation and must be solved numerically for $X_{n+1}$ for each $n$. 
The difference of any two solutions

\[ X_{n+1} - X'_{n+1} = (X_n - X'_n) + h \left( f(X_{n+1}) - f(X'_{n+1}) \right), \]

does not contain a driving noise term. Then

\[
|X_{n+1} - X'_{n+1}|^2 = \langle X_{n+1} - X'_{n+1}, X_n - X'_n \rangle \\
+ h \langle X_{n+1} - X'_{n+1}, f(X_{n+1}) - f(X'_{n+1}) \rangle \\
\leq |X_{n+1} - X'_{n+1}| |X_n - X'_n| - hL |X_{n+1} - X'_{n+1}|^2,
\]

\[ \Rightarrow |X_{n+1} - X'_{n+1}| \leq \frac{1}{1 + Lh} |X_n - X'_n|, \]

\[ \Rightarrow |X_n - X'_n| \leq \frac{1}{(1 + Lh)^n} |X_0 - X'_0| \to 0 \quad \text{as} \quad n \to \infty. \]

i.e. all numerical solutions converge pathwise to each other forward in time.
Change variables to $U_n := X_n - \hat{O}_n^{(h)}$, where $\hat{O}_n^{(h)}$ is the numerical Ornstein-Uhlenbeck process, to obtain the numerical scheme

$$U_{n+1} = U_n + hf\left(U_{n+1} + \hat{O}_{n+1}^{(h)}\right) + h\hat{O}_{n}^{(h)}.$$

Taking the inner product of both sides with $U_{n+1}$ we obtain

$$|U_{n+1}|^2 = \langle U_{n+1}, U_n \rangle + h \langle U_{n+1}, f\left(U_{n+1} + \hat{O}_{n+1}^{(h)}\right) \rangle + h \langle U_{n+1}, \hat{O}_n^{(h)} \rangle$$

$$\leq |U_{n+1}| \langle U_n \rangle + h \langle U_{n+1}, f\left(U_{n+1} + \hat{O}_{n+1}^{(h)}\right) \rangle + h |U_{n+1}| \left|\hat{O}_n^{(h)}\right|.$$  

Rearranging, using the one-sided Lipschitz condition and simplifying gives

$$|U_{n+1}| \leq |U_n| - Lh |U_{n+1}| + h \left|f\left(\hat{O}_{n+1}^{(h)}\right)\right| + h \left|\hat{O}_n^{(h)}\right|.$$
⇒ \[ |U_{n+1}| \leq \frac{1}{1 + Lh} |U_n| + \frac{h}{1 + Lh} B_n^{(h)}, \]

where

\[
B_n(h) := |f(\hat{\varnothing}^{(h)}_{n+1})| + |\hat{\varnothing}^{(h)}_n|,
\]

⇒ \[ |U_n| \leq \frac{1}{(1 + Lh)^{n-n_0}} |U_{n_0}| + \frac{h}{1 + Lh} \sum_{j=n_0}^{n-1} \frac{1}{(1 + h)^{n-1-j}} B_j^{(h)}. \]

Taking the pullback limit as \( n_0 \to -\infty \) with \( n \) fixed, it follows that \( U_n \) is pathwise pullback absorbed into the ball \( B_d[0, \bar{R}_n] \) in \( \mathbb{R}^d \) centered on the origin with squared radius

\[
\bar{R}_n^2 := 1 + \frac{h}{1 + Lh} \sum_{j=-\infty}^{n-1} \frac{1}{(1 + h)^{n-1-j}} B_j^{(h)}.
\]

Note that \( \bar{R}_n \) is random, but finite.
From the theory of random dynamical systems we conclude that the discrete time random dynamical system generated by drift-implicit Euler-Maruyama scheme has a random attractor with component sets in the corresponding balls $B_d[0, R_n]$.

Since all of the trajectories converge together pathwise forward in time, the random attractor consists of a single stochastic stationary process which we shall denote by $\hat{U}_n^{(h)}$.

Transforming back to the original variable, we have shown that the drift-implicit Euler-Maruyama scheme applied to the nonlinear SDE has a stochastic stationary solution

$$\hat{X}_n^{(h)} := \hat{U}_n^{(h)} + \hat{O}_n^{(h)}, \quad n \in \mathbb{Z},$$

taking values in the random balls $B_d[\hat{O}_n^{(h)}, \check{R}_n]$, which attracts all other solutions pathwise in both the forward and pullback senses.
References


Stochastic differential equations with nonlocal sample dependence

Based on joint work with Thomas Lorenz

P.E. Kloeden and T. Lorenz,
Stochastic differential equations with nonlocal sample dependence,
We consider the existence and uniqueness of strong solutions of Itô stochastic differential equations of the form

\[ dX_t = a(t, X_t, \mathbb{E}(X_t), \mathbb{E}(|X_t|^2)) \, dt \]
\[ + b(t, X_t, \mathbb{E}(X_t), \mathbb{E}(|X_t|^2)) \, dW_t \]
A strong solution of a scalar Itô stochastic differential equation
\[ dX_t = a(t, X_t) \, dt + b(t, X_t) \, dW_t, \quad (2) \]
on a given time interval \([0, T]\) is a function \(X : [0, T] \times \Omega \to \mathbb{R}\) with \((t, \omega) \mapsto X(t, \omega) =: X_t(\omega)\) such that
(1.) \(X\) is jointly \(\text{Leb}^1 \times \mathcal{A}\)-measurable with
\[ \int_0^T \mathbb{E}(|X_t|^2) \, dt < \infty, \]
(2.) \(X_t : \Omega \to \mathbb{R}\) is \(\mathcal{A}_t\)-measurable with \(\mathbb{E}(|X_t|^2) < \infty\) for every \(t \in [0, T]\),
(3.) \(X\) satisfies for \(t \in [0, T]\) the Itô stochastic integral equation
\[ X_t = X_0 + \int_0^t a(s, X_s) \, ds + \int_0^t b(s, X_s) \, dW_s, \]
(4.) the solution \(X\) is unique in the sense that
\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} |X_t - Y_t| > 0 \right) = 0 \]
for every solution \(Y_t\) of the above integral equation with \(Y_0 = X_0\).
Theorem 1  Suppose that

(i)  \( a, b : [0,T] \times \mathbb{R} \to \mathbb{R} \) are jointly Leb\(^2\)-measurable,

(ii) there exists a constant \( \Lambda > 0 \) such that for all \( t \in [0,T] \) and \( x, y \in \mathbb{R} \),
\[
|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq \Lambda |x - y|,
\]

(iii) there exists a constant \( \gamma < \infty \) such that for all \( t \in [0,T] \) and \( x \in \mathbb{R} \),
\[
|a(t,x)| + |b(t,x)| \leq \gamma (1 + |x|),
\]

(iv) \( X_0 : \Omega \to \mathbb{R} \) is \( \mathcal{A}_0 \)-measurable with \( \mathbb{E}(|X_0|^2) < \infty \).
Then the stochastic differential equation (2) has a pathwise unique strong solution \((X_t)_{0 \leq t \leq T}\) with initial value \(X_0\), which satisfies

\[
\sup_{0 \leq t \leq T} \mathbb{E}(|X_t|^2) < \infty.
\]

If, in addition, \(\mathbb{E}(|X_0|^{2n}) < \infty\) for some integer \(n \geq 1\), then there exist constants \(C_1, C_2\) and \(C_3\), depending only on \(\gamma, \Lambda, n\) and \(T\), such that

\[
\mathbb{E}(|X_t|^{2n}) \leq (\mathbb{E}(|X_0|^{2n}) + C_2 t) e^{C_1 t},
\]

\[
\mathbb{E}(|X_t - X_0|^{2n}) \leq C_3 (\mathbb{E}(|X_0|^{2n}) + 1) \times e^{C_1 |t-s|} \cdot |t - s|^n
\]

for every \(s, t \in [0, T]\).
Definition 2 \textit{Lip}(\mathbb{R}) denotes the set of Lipschitz continuous functions \(\mathbb{R} \to \mathbb{R}\) and for each \(f \in \text{Lip}(\mathbb{R})\), set
\[
\|f(\cdot)\|_{lg} := \sup_{x \in \mathbb{R}} \frac{|f(x)|}{1 + |x|}.
\]

Remark 3 \(\| \cdot \|_{lg} : \text{Lip}(\mathbb{R}) \to \mathbb{R}^+\) is a norm on \(\text{Lip}(\mathbb{R})\) with
\[
\|f\|_{lg} \leq |f(0)| + \text{Lip} f,
\]
\[
\sup_{|x| \leq r} |f(x)| \leq \|f\|_{lg} (1 + r)
\]
for every \(f \in \text{Lip}(\mathbb{R})\) and radius \(r \geq 0\).
Lemma 4 Suppose that $a_1, b_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $a_2, b_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy the assumptions of Theorem 1 with common parameters $\Lambda, \gamma > 0$. Let $(X^1_t)_{0 \leq t \leq T}$ and $(X^2_t)_{0 \leq t \leq T}$ denote the strong solutions of

$$dX^k_t = a_k(t, X^k_t)dt + b_k(t, X^k_t)dW_t,$$

for $k = 1$ and 2, respectively, with uniformly bounded second moments. Then,

$$\mathbb{E}(|X^1_T - X^2_T|^2) \leq$$

$$3 \cdot \mathbb{E}(|X^1_0 - X^2_0|^2) + 81 \left(1 + \mathbb{E}(|X^1_0|^2)\right) \times$$

$$\int_0^T \left(\|a_1(s, \cdot) - a_2(s, \cdot)\|_{L^2}^2 + \|b_1(s, \cdot) - b_2(s, \cdot)\|_{L^2}^2\right) ds$$

$$\times e^{(2\Lambda^2 e^T + 1 + C_1 + C_2) T}.$$
Define

\[ E_A := \{ (t, X) \mid t \in \mathbb{R}, X : \Omega \to \mathbb{R} \text{ is } \mathcal{A}_{t^+} \text{-measurable with } t^+ := \max\{t, 0\} \text{ and } \mathbb{E}(|X|^2) < \infty \}, \]

and

\[ |(t, X)|_{E_A} := |t| + \mathbb{E}(|X|^2) \quad \text{for every } (t, X) \in E_A. \]

**Remark 5** Note that \( | \cdot |_{E_A} \) is not a norm on \( E_A \) because the triangle inequality does not hold in general due to the mean square.

It could be made into a norm by taking the square root of the expectation term, but that is not necessary and would complicate estimates later.
Consider what we call a \textit{nonlocal} stochastic ordinary differential equation

\[ dX_t(\omega) = F_1[t, X_t(\cdot)] (X_t(\omega)) dt \]

\[ + F_2[t, X_t(\cdot)] (X_t(\omega)) dW_t(\omega). \]

We represent the coefficient functions here in the form

\[ F = (F_1, F_2) : E_A \to \operatorname{Lip}(\mathbb{R}) \times \operatorname{Lip}(\mathbb{R}) \]

rather than

\[ F : \mathbb{R}^+ \times L^2(\Omega, \mathcal{A}, \mathbb{P}) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, \]

since this allows us to state the regularity assumptions in a more transparent way, which is also more convenient to use in estimates.
Theorem 6 Suppose that $F = (F_1, F_2) : E_A \to Lip(\mathbb{R}) \times Lip(\mathbb{R})$ satisfies:

(i) $\sup_{(t,Y) \in E_A} \{ \| F[t,Y](\cdot) \|_{tg} + LipF[t,Y](\cdot) \} < \infty$,

(ii) $F$ is locally Lipschitz w.r.t. the random variables and continuous in the following sense: For each $R > 0$, there exist a constant $L_R > 0$ and a modulus of continuity $\omega_R(\cdot)$ such that

$$\| F[t_1,Y_1](\cdot) - F[t_2,Y_2](\cdot) \|_{tg}^2 \leq L_R \cdot E(|Y_1 - Y_2|^2) + \omega_R(|t_1 - t_2|)$$

for all $(t_k,Y_k) \in E_A$ with $|(t_k,Y_k)|_{E_A} \leq R$ for $k = 1$ and 2.

Then for any $T \in (0, \infty)$ and any $A_0$-measurable random variable $X_0$ with bounded second moment, there exists a unique strong solution $(X_t)_{0 \leq t \leq T}$ of the nonlocal SDE (3) with the initial value $X_0$.

If, in addition, $X_0$ has bounded fourth moment, then $(X_t)_{0 \leq t \leq T}$ is sample-path continuous.
Idea of Proof.

We use interpolated Euler-like approximations on equi-distant partitions of \([0,T]\), which turn out to be a uniform Cauchy sequence in the mean-square norm. The completeness of \(L^2(\Omega, \mathcal{A}, P)\) then provides a m.s. continuous candidate \(X : [0, T] \to L^2(\Omega, \mathcal{A}, P)\) for the sought solution.

For \(X^n(0) = X_0\) and for each \(k = 1, \ldots, 2^n\), define \(X^n : (t^n_k, t^n_{k+1}] \to L^2(\Omega, \mathcal{A}, \mathbb{P})\) inductively as the pathwise unique strong solution of the local SDE

\[
\begin{align*}
    dX^n_t(\omega) &= F_1[t^n_k, X^n_{t^n_k}(\cdot)](X^n_t(\omega))dt \\
                     &+ F_2[t^n_k, X^n_{t^n_k}(\cdot)](X^n_t(\omega))dW_t(\omega)
\end{align*}
\]

as guaranteed by Theorem 1.
Mean-square evolution processes.

- The solutions of nonlocal SDEs can be formulated as mean-square evolution processes.

Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space and let \((\mathcal{A}_t)_{t \in \mathbb{R}}\) be a filtration.

Define \(\mathcal{X} := L^2 (\Omega, \mathcal{A}; \mathbb{R}^d)\) and \(\mathcal{X}_t := L^2 ((\Omega, \mathcal{A}_t, \mathbb{P}), \mathbb{R}^d)\) for each \(t \in \mathbb{R}\) and

\(\mathbb{R}^2_\geq := \{(t, s) \in \mathbb{R}^2 : t \geq s\}\)
Definition 7 A mean-square evolution process $\phi$ on an underlying space $\mathbb{R}^d$ with a probability set-up $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ is a family of mappings

$$\phi(t, t_0, \cdot) : \mathcal{X}_{t_0} \to \mathcal{X}_t, \quad (t, t_0) \in \mathbb{R}^2_+,$$

which satisfies

1) initial value property: $\phi(t_0, t_0, X_0) = X_0$ for every $X_0 \in \mathcal{X}_{t_0}$ and any $t_0 \in \mathbb{R}$;

2) two-parameter semigroup property: for each $X_0 \in \mathcal{X}_{t_0}$ and all $(t_2, t_1)$, $(t_1, t_0) \in \mathbb{R}^2_+$

$$\phi(t_2, t_0, X_0) = \phi(t_2, t_1, \phi(t_1, t_0, X_0));$$

3) continuity property:

$$(t, t_0, X_0) \mapsto \phi(t, t_0, X_0) \text{ is continuous in the space } \mathbb{R}^2_+ \times \mathcal{X}.$$
Definition 8  A family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$ of nonempty subsets of $X$ with $B_t \subset X_t$ for each $T \in \mathbb{R}$ is said to be $\phi$-invariant if

$$\phi(t, t_0, B_{t_0}) = B_t \quad \text{for all } (t, t_0) \in \mathbb{R}^2_\geq \text{ and every } t \in \mathbb{R}$$

and $\phi$-positively invariant if

$$\phi(t, t_0, B_{t_0}) \subset B_t \quad \text{for all } (t, t_0) \in \mathbb{R}^2_\geq \text{ and every } t \in \mathbb{R}$$

For simplicity, we will say that $\mathcal{B}$ is a family of subsets of $\{X_t\}_{t \in \mathbb{R}}$ and that $\mathcal{B}$ is uniformly bounded if there is an $R := R_D < \infty$ such that $\mathbb{E}\|X_t\|^2 \leq R$ for all points $X_t \in B_t$ for every $t \in \mathbb{R}$

Definition 9  A $\phi$-invariant family $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$ of nonempty compact subsets of $\{X_t\}_{t \in \mathbb{R}}$ is called a forward attractor if it forward attracts all families $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$ of uniformly bounded subsets of $\{X_t\}_{t \in \mathbb{R}}$, i.e.,

$$\text{dist} (\phi(t, t_0, B_{t_0}), A_t) \to 0 \quad \text{as } t \to \infty \quad (t_0 \text{ fixed}) \quad (4)$$

and a pullback attractor if it pullback attracts all families $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$ of uniformly bounded subsets of $\{X_t\}_{t \in \mathbb{R}}$, i.e.,

$$\text{dist} (\phi(t, t_0, B_{t_0}), A_t) \to 0 \quad \text{as } t_0 \to -\infty \quad (t \text{ fixed}) \quad (5)$$
Theorem 10 Suppose that a mean-square evolution $\phi$ on $\mathbb{R}^d$ has a $\phi$-positively invariant pullback absorbing family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$ of nonempty uniformly bounded subsets of $\{\mathcal{X}_t\}_{t \in \mathbb{R}}$ and that the mappings $\phi(t, t_0, \cdot) : \mathcal{X}_{t_0} \to \mathcal{X}_t$ are asymptotically compact.

Then $\phi$ has a unique global pullback attractor $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$ with component sets determined by

$$A_t = \bigcap_{t_0 \leq t} \phi(t, t_0, B_{t_0}) \quad \text{for each } t \in \mathbb{R} \quad (6)$$

If $\mathcal{B}$ is not $\phi$-positively invariant, then

$$A_t = \bigcap_{s \geq 0} \bigcup_{t_0 \leq t-s} \phi(t, t_0, B_{t_0}) \quad \text{for each } t \in \mathbb{R}$$
Big technical problem

How do we characterise compact subsets of a space of mean-square random variables

\[ L_2 ((\Omega, \mathcal{A}_t, \mathbb{P}), \mathbb{R}^d) \]
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