Random attractors under discretization

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Deterministic case

Consider the ordinary differential equation in $\mathbb{R}^d$

$$\frac{dx}{dt} = f(x)$$

- f is regular and satisfies the dissipativity condition such as

$$\langle x, f(x) \rangle \leq K - L|x_1 - x_2|^2,$$

on $\mathbb{R}^d$ for some $K \geq 0$ and $L > 0$,

$\Rightarrow$ there exists a global attractor A_0.

Kloeden & Lorenz (SINUM 1986):

• then a one-step numerical scheme with constant step size $h > 0$ has an attractor A_h and the Hausdorff seperation satisfies

$$d(A_h, A_0) \to 0 \quad \text{as} \quad h \to 0+$$

i.e., upper semi continuous convergence of the numerical attractors

More structure, more information

If f satisfies a one-sided dissipative Lipschitz condition

$$\langle x - y, f(x) - f(y) \rangle \leq -L|x_1 - x_2|^2,$$

for all x, $y \in \mathbb{R}^d$ for some $L > 0$

$\Longrightarrow$ there is a unique asymptotically stable steady state $\bar{x}$ with $f(\bar{x}) \neq 0$, i.e., the global attractor is $A_0 = \{\bar{x}\}$, and

• for most one-step numerical schemes the numerical attractor is also $A_h = \{\bar{x}\}$.

Stochastic case

What is the effect of background or environmental noise?

Consider the Ito stochastic differential equation with additive noise

$$dX_t = f(X_t)\,dt + \alpha\,dW_t,$$

where W_t, is a two-sided scalar Wiener process and $\alpha \in \mathbb{R}^d$ is a constant vector.

This has no equilibrium solution but if f satisfies a one-sided dissipative Lipschitz condition, then

$\Rightarrow$ $\exists$ unique stochastic stationary solution $\bar{X}_t$, which is pathwise globally asymptotically stable.

Recall:

- The solutions of Ito stochastic differential equations are pathwise continuous, but not differentiable.

- Ito SDEs are really stochastic integral equations with stochastic integrals defined in the mean-square or L_2 sense.

How do we apply the Lipschitz properties to obtain pathwise estimates?

A technical detour : Consider the Ito SDE

$$dX_t = f(X_t)\,dt + \alpha\,dW_t$$

where f satisfies the one-sided Lipschitz condition.

i.e., the <u>stochastic integral equation</u>

$$X_t = X_{t_0} + \int_{t_0}^{t} f(X_s)\,ds + \alpha \int_{t_0}^{t} dW_t$$

The difference of any two solutions satisfies pathwise

$$X_t^1 - X_t^2 = X_{t_0}^1 - X_{t_0}^2 + \int_{t_0}^{t} \underbrace{\left[f(X_s^1) - f(X_s^2)\right]}_{\text{continuous paths}}\,ds$$

Fundamental theorem of calculus $\Rightarrow$ $X_t^1 - X_t^2$ pathwise differentiable.

$$\frac{d}{dt}\left[X_t^1 - X_t^2\right] = f(X_t^1) - f(X_t^2) \qquad \text{pathwise}$$

- Apply the one-sided Lipschitz condition

$$\frac{d}{dt}\left|X_t^1 - X_t^2\right|^2 = 2\left\langle X_t^1 - X_t^2, f(X_t^1) - f(X_t^2)\right\rangle \leq -2L\left|X_t^1 - X_t^2\right|^2$$

$$\Rightarrow \qquad \left|X_t^1 - X_t^2\right|^2 \leq \left|X_{t_0}^1 - X_{t_0}^2\right|^2 e^{-2L(t-t_0)} \to 0 \qquad \text{as} \quad t \to \infty$$

i.e. all solutions converge pathwise together — but to what?

Special case: Ito SDE with linear drift $f(x) = -x$

$$dX_t = -X_t\,dt + \alpha\,dW_t$$

explicit solution

$$X_t = X_{t_0}e^{-(t-t_0)} + \alpha e^{-t}\int_{t_0}^{t} e^s\,dW_s$$

The forward limit as $t \to \infty$ does not exist — moving target!

But the pullback limit as $t_0 \to -\infty$ with t fixed does exist:

$$\lim_{t_0\to-\infty} X_t = \bar{O}_t := \alpha e^{-t}\int_{-\infty}^{t} e^s\,dW_s \qquad \text{(pathwise)}$$

The Ornstein-Uhlenbeck stochastic stationary process $\bar{O}_t$ is a solution of the linear SDE and all other solutions converge pathwise to it in the forward sense

$$\left|X_t - \bar{O}_t\right| \to 0 \qquad \text{as} \quad t \to \infty \qquad \text{(pathwise)}$$

Random dynamical systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (X, d_X) a metric space.

A random dynamical system (θ, ϕ) on $\Omega \times X$ consists of

• a metric dynamical system θ on Ω, which models the noise,

• a cocycle mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$, which represents the dynamics on the state space X and satisfies

1). $\quad \phi(0, \omega, x_0) = x_0$ (initial condition)

2). $\quad \phi(s + t, \omega, x_0) = \phi(s, \theta_t \omega, \phi(t, \omega, x_0))$ (cocycle property)

3). $\quad (t, x_0) \mapsto \phi(t, \omega, x_0)$ is continuous (continuity)

4). $\quad \omega \mapsto \phi(t, \omega, x_0)$ is $\mathcal{F}$-measurable (measurability)

for all s, $t \geq 0$, $x_0 \in X$ and $\omega \in \Omega$.

Random attractors

A random attractor is a family of nonempty measurable compact subsets of X

$$\widehat{A} = \{A(\omega) : \omega \in \Omega\}$$

which is

- ϕ-invariant $\quad \phi(t, \omega, A(\omega)) = A(\theta_t \omega) \qquad$ for all $t \geq 0$,

- pathwise pullback attracting in the sense that

$$\text{dist}_X \left(\phi\left(t, \theta_{-t}\omega, D(\theta_{-t}\omega)\right), A(\omega)\right) \to 0 \qquad \text{for} \quad t \to +\infty$$

for all suitable families $\widehat{D} = \{D(\omega) : \omega \in \Omega\}$ of nonempty measurable bounded subsets of X.

Theorem (Crauel, Flandoli, Schmalfuß etc)

Let (θ, ϕ) be an RDS on $\Omega \times X$ such that $\phi(t, \omega, \cdot) : X \to X$ is a compact operator for each fixed $t > 0$ and $\omega \in \Omega$.

If there exists a pullback absorbing family $\widehat{B} = \{B(\omega) : \omega \in \Omega\}$ of nonempty closed and bounded measurable subsets of X, i.e. there exists a $T_{\widehat{D},\omega} \geq 0$ such that

$$\phi\left(t, \theta_{-t}\omega, D(\theta_{-t}\omega)\right) \subset B(\omega) \qquad \text{for all} \quad t \geq T_{\widehat{D},\omega}$$

for all $\widehat{D} = \{D(\omega) : \omega \in \Omega\}$ in a given attracting universe.

Then the RDS $\Omega \times X$ has a random attractor $\widehat{A}$ with component subsets given by

$$A(\omega) = \bigcap_{s>0} \overline{\bigcup_{t \geq s} \phi\left(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\right)} \qquad \text{for each} \quad \omega \in \Omega.$$

General case again

Substract the integral version of the linear SDE for $\bar{O}_t$ from the integral version of the nonlinear SDE

$$dX_t = f(X_t)\,dt + \alpha\,dW_t$$

to obtain

$$X_t - \bar{O}_t = X_{t_0} - \bar{O}_{t_0} + \int_{t_0}^{t} \left[f(X_s) + \bar{O}_s\right]\,ds$$

$\Rightarrow$ $V_t := X_t - \bar{O}_t$ is pathwise differentiable and satisfies the <u>pathwise ODE</u>

$$\frac{d}{dt}V_t = f(V_t + \bar{O}_t) + \bar{O}_t \qquad \text{(pathwise)}$$

- Apply the one-sided Lipschitz condition pathwise to

$$\frac{d}{dt}\left[X_t - \bar{O}_t\right] = \left[f(X_t) - f(\bar{O}_t)\right] + \left[f(\bar{O}_t) + \bar{O}_t\right] \qquad \text{(pathwise)}$$

  to obtain the pathwise estimate

$$|V_t|^2 \leq |V_{t_0}|^2 \, e^{-L(t-t_0)} + \frac{2}{L}e^{-Lt}\int_{t_0}^{t} e^{Ls}\left(|f(\bar{O}_s)|^2 + |\bar{O}_s|^2\right)\, ds$$

- Take pathwise pullback convergence as $t_0 \to -\infty$ to obtain

$$|X_t - \bar{O}_t| \;\leq\; \bar{R}_t := 1 + \frac{2}{L}e^{-Lt}\int_{-\infty}^{t} e^{Ls}\left(|f(\bar{O}_s)|^2 + |\bar{O}_s|^2\right)\, ds$$

for $t \geq T$ depending on suitable bounded sets of initial values.

- i.e., there exists a family of compact pullback absorbing balls centered

on $\bar{O}_t$ with random radius $\bar{R}_t$.

- Dynamical systems limit set ideas

  $\Rightarrow$ there exists a compact setvalued stochastic process A_t inside these absorbing balls which pathwise pullback attracts the solutions.

- BUT the solutions converge together pathwise in forwards sense, so the sets A_t are in fact all singleton sets

$\Rightarrow \quad \exists$ stochastic stationary solution $\bar{X}_t$.

General Principles

- All regular Ito SDE in $\mathbb{R}^d$ can be transformed into pathwise ODE

  [*Imkeller* & *Schmalfuß* (2001), *Imkeller* & *Lederer* (2001,2002)]

- and generate random dynamical systems

  $\Rightarrow$ **pathwise theory and numerics for Ito SDE**

- Pullback convergence enables us to construct moving targets.

- Stochastic stationary solutions are a simple singleton set version of more general random attractors

$\Rightarrow$ **theory of random dynamical systems**

e.g., Ludwig Arnold (Bremen)

- parallel theory of **deterministic skew product flows**

e.g., almost periodic ODE : George Sell (Minneapolis)

$\Rightarrow$ **A theory of nonautonomous dynamical systems**

e.g., pullback attractors

Effects of discretization

Numerical Ornstein-Uhlenbeck process

For the linear SDE with additive noise,

$$dX_t = -X_t\,dt + \alpha\,dW_t,$$

the drift-implicit Euler-Maruyama scheme with constant step size $h > 0$ is

$$X_{n+1} = X_n - hX_{n+1} + \alpha\Delta W_n, \qquad n = n_0, n_0+1, \ldots,$$

which simplifies algebraically to

$$X_{n+1} = \frac{1}{1+h}X_n + \frac{\alpha}{1+h}\,\Delta W_n,$$

Here the $\Delta W_n = W_{h(n+1)} - W_{hn}$ are mutually independent and $N(0,h)$ distributed

It follows that

$$X_n = \frac{1}{(1+h)^{n-n_0}} X_{n_0} + \frac{\alpha}{1+h} \sum_{j=n_0}^{n-1} \frac{1}{(1+h)^{n-1-j}} \, \Delta W_j$$

and the pathwise pullback limit, i.e. with n fixed and $n_0 \to -\infty$, gives the discrete time numerical Ornstein-Uhlenbeck process

$$\widehat{O}_n^{(h)} := \frac{\alpha}{1+h} \sum_{j=-\infty}^{n-1} \frac{1}{(1+h)^{n-1-j}} \, \Delta W_j, \qquad n \in \mathbb{Z}. \tag{1}$$

which is an entire solution of the numerical scheme and a discrete time stochastic stationary process.

One can show that it converges pathwise to the continuous time Ornstein-Uhlenbeck process in the sense that

$$\widehat{O}_0^{(h)} \to \widehat{O}_0 \qquad \text{as} \quad h \to 0.$$

Discretization of an nonlinear stochastic system

Consider the nonlinear SDE in $\mathbb{R}^d$with additive noise,

$$dX_t = f(X_t)\,dt + \alpha\,dW_t,$$

where the drift coefficient f is continuously differentiable and satisfies the one-sided dissipative Lipschitz condition with constant L.

The drift-implicit Euler-Maruyama scheme with constant step size $h > 0$ applied to this SDE is

$$X_{n+1} = X_n + hf(X_{n+1}) + \alpha\Delta W_n,$$

which is, in general, an implicit algebraic equation and must be solved numerically for X_{n+1} for each n.

The difference of any two solutions

$$X_{n+1} - X'_{n+1} = (X_n - X'_n) + h\left(f(X_{n+1}) - f(X'_{n+1})\right),$$

does not contain a driving noise term. Then

$$\begin{aligned}
\left|X_{n+1} - X'_{n+1}\right|^2 &= \left\langle X_{n+1} - X'_{n+1}, X_n - X'_n \right\rangle \\
&\quad + h \left\langle X_{n+1} - X'_{n+1}, f(X_{n+1}) - f(X'_{n+1}) \right\rangle \\
&\leq \left|X_{n+1} - X'_{n+1}\right| \, |X_n - X'_n| - hL \left|X_{n+1} - X'_{n+1}\right|^2,
\end{aligned}$$

$$\Rightarrow \qquad \left|X_{n+1} - X'_{n+1}\right| \leq \frac{1}{1+Lh} \, |X_n - X'_n|,$$

$$\Rightarrow \qquad |X_n - X'_n| \leq \frac{1}{(1+Lh)^n} \, |X_0 - X'_0| \to 0 \quad \text{as } n \to \infty.$$

i.e. all numerical solutions converge pathwise to each other forward in time.

Change variables to $U_n := X_n - \widehat{O}_n^{(h)}$, where $\widehat{O}_n^{(h)}$ is the numerical Ornstein-Uhlenbeck process, to obtain the numerical scheme

$$U_{n+1} = U_n + hf\left(U_{n+1} + \widehat{O}_{n+1}^{(h)}\right) + h\widehat{O}_n^{(h)}.$$

Taking the inner product of both sides with U_{n+1} we obtain

$$\begin{aligned} |U_{n+1}|^2 &= \langle U_{n+1}, U_n\rangle + h\left\langle U_{n+1}, f\left(U_{n+1} + \widehat{O}_{n+1}^{(h)}\right)\right\rangle + h\left\langle U_{n+1}, \widehat{O}_n^{(h)}\right\rangle \\ &\le |U_{n+1}|\;|U_n| + h\left\langle U_{n+1}, f\left(U_{n+1} + \widehat{O}_{n+1}^{(h)}\right)\right\rangle + h\,|U_{n+1}|\;\left|\widehat{O}_n^{(h)}\right|. \end{aligned}$$

Rearranging, using the one-sided Lipschitz condition and simplifying gives

$$|U_{n+1}| \le |U_n| - Lh\,|U_{n+1}| + h\left|f\left(\widehat{O}_{n+1}^{(h)}\right)\right| + h\left|\widehat{O}_n^{(h)}\right|.$$

$$\Rightarrow \qquad |U_{n+1}| \leq \frac{1}{1+Lh}\,|U_n| + \frac{h}{1+Lh} B_n^{(h)},$$

where

$$B_n(h) := \left| f\left(\widehat{O}_{n+1}^{(h)}\right) \right| + \left| \widehat{O}_n^{(h)} \right|,$$

$$\Rightarrow \qquad |U_n| \leq \frac{1}{(1+Lh)^{n-n_0}}\,|U_{n_0}| + \frac{h}{1+Lh} \sum_{j=n_0}^{n-1} \frac{1}{(1+h)^{n-1-j}}\, B_j^{(h)}.$$

Taking the pullback limit as $n_0 \to -\infty$ with n fixed, it follows that U_n is pathwise pullback absorbed into the ball $B_d[0, \bar{R}_n]$ in $\mathbb{R}^d$ centered on the origin with squared radius

$$\bar{R}_n^2 := 1 + \frac{h}{1+Lh} \sum_{j=-\infty}^{n-1} \frac{1}{(1+h)^{n-1-j}}\, B_j^{(h)}.$$

Note that $\bar{R}_n$ is random, but finite.

From the theory of random dynamical systems we conclude that the discrete time random dynamical system generated by drift-implicit Euler-Maruyama scheme has a random attractor with component sets in the corresponding balls $B_d[0, \bar{R}_n]$.

Since all of the trajectories converge together pathwise forward in time, the random attractor consists of a single stochastic stationary process which we shall denote by $\widehat{U}_n^{(h)}$.

Transforming back to the original variable, we have shown that the drift-implicit Euler-Maruyama scheme applied to the nonlinear SDE has a stochastic stationary solution

$$\widehat{X}_n^{(h)} := \widehat{U}_n^{(h)} + \widehat{O}_n^{(h)}, \qquad n \in \mathbb{Z},$$

taking values in the random balls $B_d[\widehat{O}_n^{(h)}, \bar{R}_n]$, which attracts all other solutions pathwise in both the forward and pullback senses.

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Stochastic differential equations with nonlocal sample dependence

Based on joint work with Thomas Lorenz

P.E. Kloeden and T. Lorenz,
Stochastic differential equations with nonlocal sample dependence,
J. Stoch. Anal. Applns., **28** (2010), 937–948.

We consider the existence and uniqueness of strong solutions of Itô stochastic differential equations of the form

$$
\begin{aligned}
d\,X_t &= a(t, X_t, \mathbb{E}(X_t), \mathbb{E}(|X_t|^2))\; dt \\
&\quad + b(t, X_t, \mathbb{E}(X_t), \mathbb{E}(|X_t|^2))\; dW_t
\end{aligned}
$$

A strong solution of a scalar Itô stochastic differential equation

$$dX_t = a(t, X_t)\ dt + b(t, X_t)\ dW_t, \tag{2}$$

on a given time interval $[0, T]$ is a function $X : [0, T] \times \Omega \to \mathbb{R}$ with $(t, \omega) \mapsto X(t, \omega) =: X_t(\omega)$ such that
(1.) X is jointly $\mathrm{Leb}^1 \times \mathcal{A}$-measurable with

$$\int_0^T \mathbb{E}(|X_t|^2)\ dt \ < \ \infty,$$

(2.) $X_t : \Omega \to \mathbb{R}$ is $\mathcal{A}_t$-measurable with $\mathbb{E}(|X_t|^2) < \infty$ for every $t \in [0, T]$,
(3.) X satisfies for $t \in [0, T]$the Itô stochastic integral equation

$$X_t = X_0 + \int_0^t a(s, X_s)\ ds + \int_0^t b(s, X_s)\ dW_s,$$

(4.) the solution X is unique in the sense that

$$\mathbb{P}\left(\sup_{0 \le t \le T} |X_t - Y_t| > 0 \right) \ = \ 0$$

for every solution Y_t of the above integral equation with $Y_0 = X_0$.

Theorem 1 *Suppose that*

(i) $a,\, b : [0,T] \times \mathbb{R} \to \mathbb{R}$ *are jointly* Leb^2*–measurable,*

(ii) there exists a constant $\Lambda > 0$ *such that for all* $t \in [0,T]$ *and* $x,\, y \in \mathbb{R}$*,*

$$|a(t,x) \;-\; a(t,y)| + |b(t,x) \;-\; b(t,y)| \leq \Lambda\, |x-y|,$$

(iii) there exists a constant $\gamma < \infty$ *such that for all* $t \in [0,T]$ *and* $x \in \mathbb{R}$*,*

$$|a(t,x)| + |b(t,x)| \;\leq\; \gamma\,(1+|x|),$$

(iv) $X_0 : \Omega \to \mathbb{R}$ *is* $\mathcal{A}_0$*-measurable with* $\mathbb{E}\left(|X_0|^2\right) < \infty$*.*

Then the stochastic differential equation (2) *has a pathwise unique strong solution* $(X_t)_{0\le t\le T}$ *with initial value* X_0*, which satisfies*

$$\sup_{0\le t\le T} \mathbb{E}\left(|X_t|^2\right) < \infty.$$

If, in addition, $\mathbb{E}(|X_0|^{2n}) < \infty$ *for some integer* $n \ge 1$*, then there exist constants* C_1*,* C_2 *and* C_3*, depending only on* γ*,* Λ, n *and* T*, such that*

$$\begin{aligned}
\mathbb{E}\left(|X_t|^{2n}\right) &\le \left(\mathbb{E}\left(|X_0|^{2n}\right) + C_2 t\right) e^{C_1\, t}, \\
\mathbb{E}\left(|X_t - X_0|^{2n}\right) &\le C_3\left(\mathbb{E}\left(|X_0|^{2n}\right) + 1\right) \times e^{C_1\,|t-s|} \cdot |t-s|^n
\end{aligned}$$

for every $s, t \in [0, T]$.

Definition 2 *$Lip(\mathbb{R})$ denotes the set of Lipschitz continuous functions $\mathbb{R} \to \mathbb{R}$ and for each $f \in Lip(\mathbb{R})$, set*

$$\|f(\cdot)\|_{lg} := \sup_{x \in \mathbb{R}} \frac{|f(x)|}{1 + |x|}.$$

Remark 3 *$\|\cdot\|_{lg} : Lip(\mathbb{R}) \to \mathbb{R}^+$ is a norm on $Lip(\mathbb{R})$ with*

$$\begin{aligned} \|f\|_{lg} &\leq |f(0)| + Lip\, f, \\ \sup_{|x| \leq r} |f(x)| &\leq \|f\|_{lg}\,(1 + r) \end{aligned}$$

for every $f \in Lip(\mathbb{R})$ and radius $r \geq 0$.

Lemma 4 *Suppose that $a_1,\ b_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $a_2,\ b_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy the assumptions of Theorem 1 with common parameters $\Lambda,\ \gamma > 0$. Let $(X_t^1)_{0 \le t \le T}$ and $(X_t^2)_{0 \le t \le T}$ denote the strong solutions of*

$$dX_t^k = a_k(t, X_t^k)dt + b_k(t, X_t^k)dW_t,$$

for $k = 1$ and 2, respectively, with uniformly bounded second moments. Then,

$$\mathbb{E}(|X_T^1 \ - \ X_T^2|^2) \le$$

$$\Big(3 \cdot \mathbb{E}\left(|X_0^1 \ - \ X_0^2|^2\right) + 81\left(1 + \mathbb{E}(|X_0^1|^2)\right) \times$$

$$\int_0^T \left(\left\|a_1(s,\cdot) - a_2(s,\cdot)\right\|_{lg}^2 + \left\|b_1(s,\cdot) - b_2(s,\cdot)\right\|_{lg}^2\right) ds\Big)$$

$$\times e^{(2\,\Lambda^2\ e^T\ + 1 + C_1 + C_2)\ \ T}\ .$$

Nonlocal stochastic differential equations

Define

$$\begin{aligned} E_{\mathcal{A}} \quad := \quad & \Big\{ (t, X) \;\Big|\; t \in \mathbb{R}, X : \Omega \to \mathbb{R} \text{ is} \\ & \mathcal{A}_{t^+} - \text{measurable with } t^+ := \max\{t, 0\} \\ & \text{and } \mathbb{E}(|X|^2) < \infty \Big\}, \end{aligned}$$

and

$$|(t, X)|_{E_{\mathcal{A}}} := |t| + \mathbb{E}(|X|^2) \quad \text{for every } (t, X) \in E_{\mathcal{A}}.$$

Remark 5 *Note that $|\cdot|_{E_{\mathcal{A}}}$ is <u>not</u> a norm on $E_{\mathcal{A}}$ because the triangle inequality does not hold in general due to the mean square.*

It could be made into a norm by taking the square root of the expectation term, but that is not necessary and would complicate estimates later.

Consider what we call a *nonlocal* stochastic ordinary differential equation

$$\begin{aligned} dX_t(\omega) &= F_1\left[t, X_t(\cdot)\right]\,(X_t(\omega))dt \\ &\quad + F_2\left[t, X_t(\cdot)\right]\,(X_t(\omega))dW_t(\omega). \end{aligned} \tag{3}$$

We represent the coefficient functions here in the form

$$F = (F_1, F_2) : E_{\mathcal{A}} \to \mathrm{Lip}(\mathbb{R}) \times \mathrm{Lip}(\mathbb{R})$$

rather than

$$F : \mathbb{R}^+ \times L^2(\Omega, \mathcal{A}, \mathbb{P}) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R},$$

since this allows us to state the regularity assumptions in a more transparent way, which is also more convenient to use in estimates.

Theorem 6 *Suppose that $F = (F_1, F_2) : E_{\mathcal{A}} \to Lip(\mathbb{R}) \times Lip(\mathbb{R})$ satisfies:*

(i) $\sup_{(t,Y)\in E_{\mathcal{A}}} \{\|F[t,Y](\cdot)\|_{lg} + LipF[t,Y](\cdot)\} < \infty$,

(ii) F is locally Lipschitz w.r.t. the random variables and continuous in the following sense: For each $R > 0$, there exist a constant $L_R > 0$ and a modulus of continuity $\omega_R(\cdot)$ such that

$$\begin{aligned}
&\left\|F[t_1,Y_1](\cdot) - F[t_2,Y_2](\cdot)\right\|_{lg}^2 \\
&\qquad \leq\; L_R \cdot \mathbb{E}(|Y_1 - Y_2|^2) + \omega_R(|t_1 - t_2|)
\end{aligned}$$

for all $(t_k, Y_k) \in E_{\mathcal{A}}$ with $|(t_k,Y_k)|_{E_{\mathcal{A}}} \leq R$ for $k = 1$ and 2.

Then for any $T \in (0,\infty)$ and any $\mathcal{A}_0$-measurable random variable X_0 with bounded second moment, there exists a unique strong solution $(X_t)_{0\leq t\leq T}$ of the nonlocal SDE (3) *with the initial value X_0.*

If, in addition, X_0 has bounded fourth moment, then $(X_t)_{0\leq t\leq T}$ is sample-path continuous.

Idea of Proof.

We use interpolated Euler-like approximations on equi-distant partitions of $[0,T]$, which turn out to be a uniform Cauchy sequence in the mean-square norm. The completeness of $L^2(\Omega,\mathcal{A},P)$ then provides a m.s. continuous candidate $X:[0,T]\to L^2(\Omega,\mathcal{A},P)$ for the sought solution.
For $X^n(0)=X_0$ and for each $k=1,\,\ldots,\,2^n$, define $X^n:(t_k^n,\;t_{k+1}^n]\to L^2(\Omega,\mathcal{A},\mathbb{P})$ inductively as the pathwise unique strong solution of the local SDE

$$\begin{aligned} dX_t^n(\omega) &= F_1\left[t_k^n, X_{t_k^n}^n(\cdot)\right](X_t^n(\omega))dt \\ &\quad + F_2\left[t_k^n, X_{t_k^n}^n(\cdot)\right](X_t^n(\omega))dW_t(\omega) \end{aligned}$$

as guaranteed by Theorem 1.

Mean-square evolution processes.

• The solutions of nonlocal SDEs can be formulated as mean-square evolution processes.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(\mathcal{A}_t)_{\in\mathbb{R}}$ be a filtration.

Define $\mathfrak{X} := L^2\left(\Omega, \mathcal{A}; \mathbb{R}^d\right)$ and $\mathfrak{X}_t := L_2\left((\Omega, \mathcal{A}_t, \mathbb{P}), \mathbb{R}^d\right)$ for each $t \in \mathbb{R}$ and

$$\mathbb{R}^2_{\geq} := \left\{(t, s) \in \mathbb{R}^2 \,:\, t \geq s\right\}$$

Definition 7 *A* <u>*mean-square evolution process*</u> ϕ *on an underlying space* $\mathbb{R}^d$ *with a probability set-up* $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ *is a family of mappings*

$$\phi(t, t_0, \cdot) : \mathfrak{X}_{t_0} \to \mathfrak{X}_t, \qquad (t, t_0) \in \mathbb{R}^2_{\geq},$$

which satisfies
1) <u>initial value property</u>*:* $\phi(t_0, t_0, X_0) = X_0$ *for every* $X_0 \in \mathfrak{X}_{t_0}$ *and any* $t_0 \in \mathbb{R}$*;*

2) <u>two-parameter semigroup property</u>*: for each* $X_0 \in \mathfrak{X}_{t_0}$ *and all* (t_2, t_1), $(t_1, t_0) \in \mathbb{R}^2_{\geq}$

$$\phi(t_2, t_0, X_0) = \phi\left(t_2, t_1, \phi(t_1, t_0, X_0)\right);$$

3) <u>continuity property</u>*:*

$$(t, t_0, X_0) \mapsto \phi(t, t_0, X_0) \quad \textit{is continuous in the space } \mathbb{R}^2_{\geq} \times \mathfrak{X}.$$

Definition 8 *A family* $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ *of nonempty subsets of* $\mathfrak{X}$ *with* $B_t \subset \mathfrak{X}_t$ *for each* $T \in \mathbb{R}$ *is said to be* ϕ-*invariant if*

$$\phi\,(t, t_0, B_{t_0}) = B_t \qquad \textit{for all } (t, t_0) \in \mathbb{R}^2_{\geq} \textit{ and every } t \in \mathbb{R}$$

and ϕ-*positively invariant if*

$$\phi\,(t, t_0, B_{t_0}) \subset B_t \qquad \textit{for all } (t, t_0) \in \mathbb{R}^2_{\geq} \textit{ and every } t \in \mathbb{R}$$

For simplicity, we will say that $\mathcal{B}$ is a family of subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$ and that $\mathcal{B}$ is uniformly bounded if there is an $R := R_{\mathcal{D}} < \infty$ such that $\mathbb{E}\|X_t\|^2 \leq R$ for all points $X_t \in B_t$ for every $t \in \mathbb{R}$

Definition 9 *A* ϕ-*invariant family* $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ *of nonempty compact subsets of* $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$ *is called a forward attractor if it forward attracts all families* $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ *of uniformly bounded subsets of* $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$, *i.e.,*

$$\operatorname{dist}(\phi(t, t_0, B_{t_0}), A_t) \to 0 \quad \textit{as } t \to \infty \quad (t_0 \;\; \textit{fixed}) \tag{4}$$

and a pullback attractor if it pullback attracts all families $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ *of uniformly bounded subsets of* $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$, *i.e.,*

$$\operatorname{dist}(\phi(t, t_0, B_{t_0}), A_t) \to 0 \quad \textit{as } t_0 \to -\infty \quad (t \;\; \textit{fixed}). \tag{5}$$

Theorem 10 *Suppose that a mean-square evolution ϕ on $\mathbb{R}^d$ has a ϕ-positively invariant pullback absorbing family $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ of nonempty uniformly bounded subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$ and that the mappings $\phi(t, t_0, \cdot) : \mathfrak{X}_{t_0} \to \mathfrak{X}_t$ are asymptotically compact.*

Then ϕ has a unique global pullback attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ with component sets determined by

$$A_t = \bigcap_{t_0 \le t} \phi(t, t_0, B_{t_0}) \quad \text{for each } t \in \mathbb{R} \tag{6}$$

If $\mathcal{B}$ is not ϕ-positively invariant, then

$$A_t = \bigcap_{s\ge 0} \overline{\bigcup_{t_0 \le t-s} \phi(t, t_0, B_{t_0})} \quad \text{for each } t \in \mathbb{R}$$

Big technical problem

How do we characterise compact subsets of a space of mean-square random variables

$$L_2\left((\Omega, \mathcal{A}_t, \mathbb{P}), \mathbb{R}^d\right)?$$

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