Reliable and Unreliable Behavior in Driven Oscillator Networks

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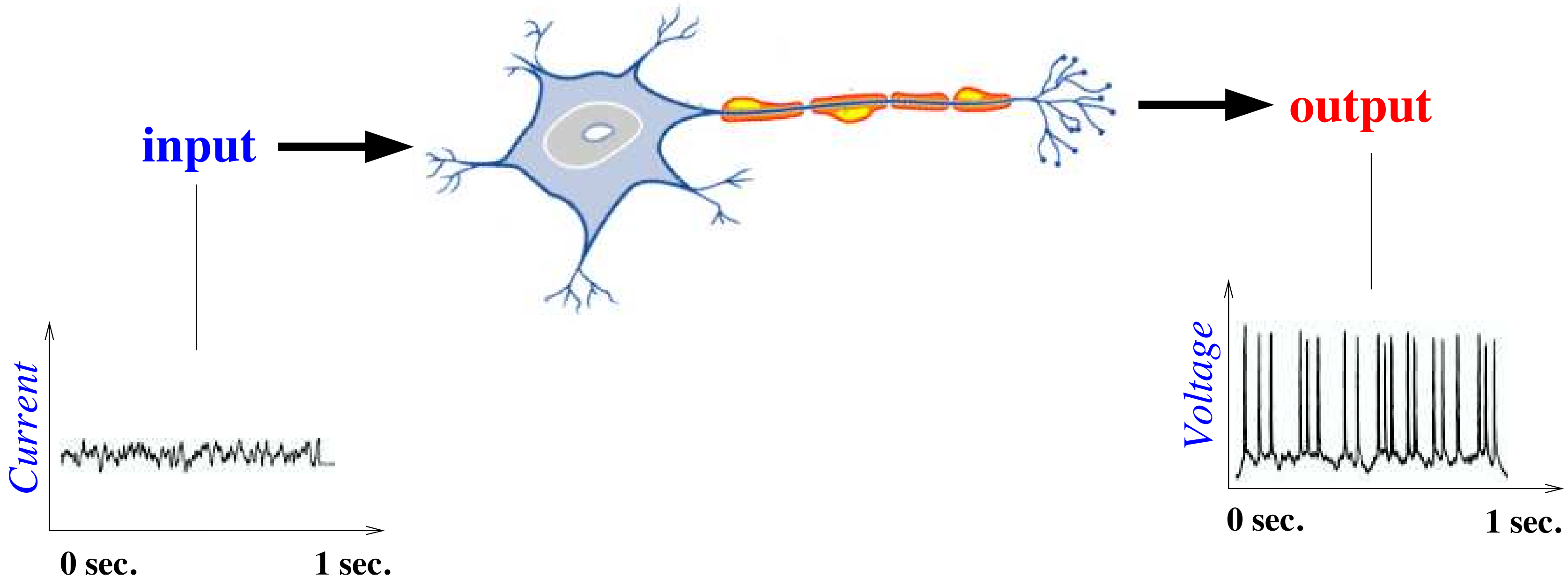
Oberwolfach 2011.08.22

Example: Neural response



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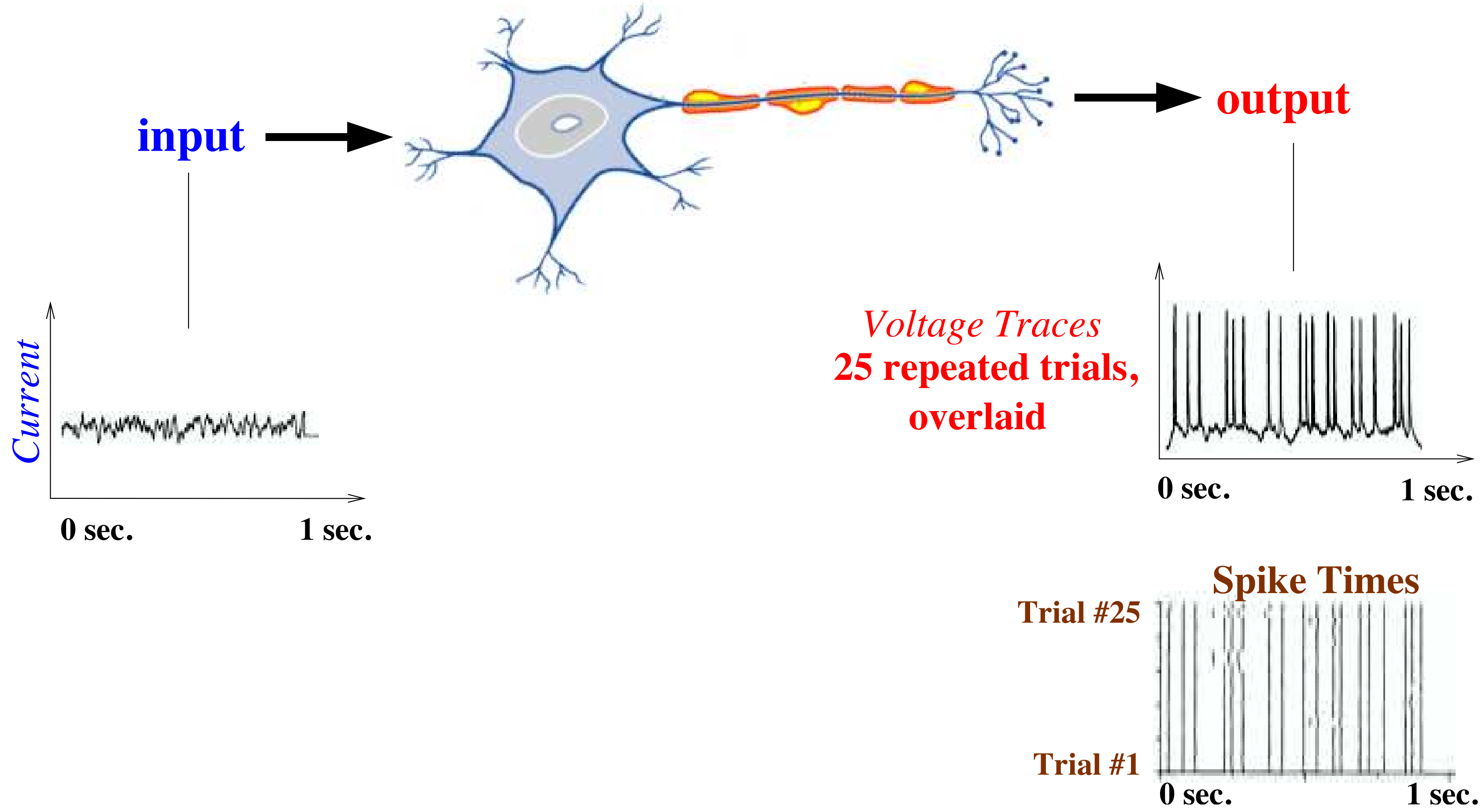

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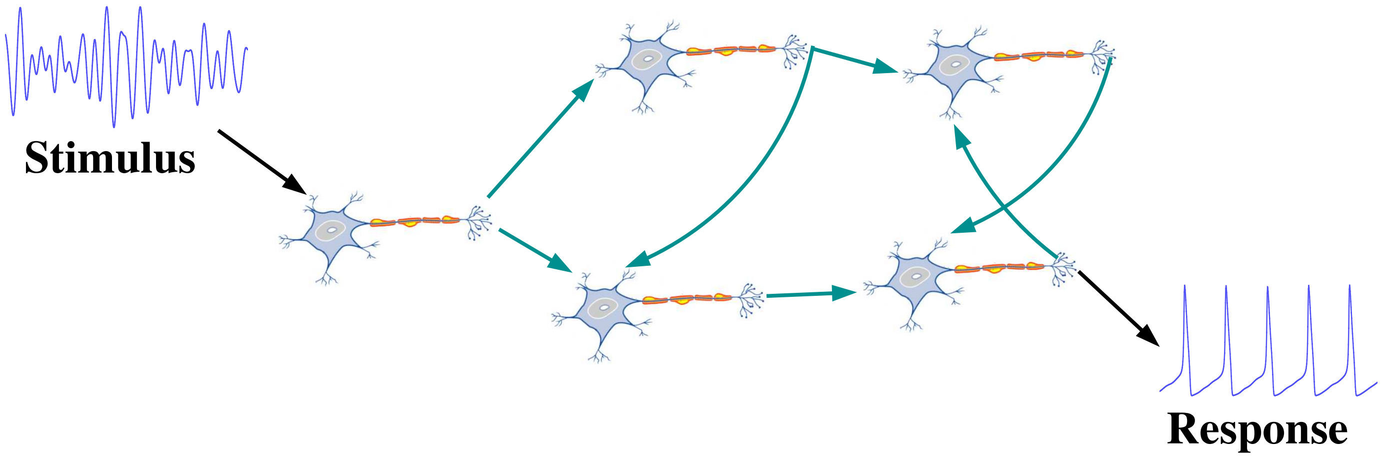
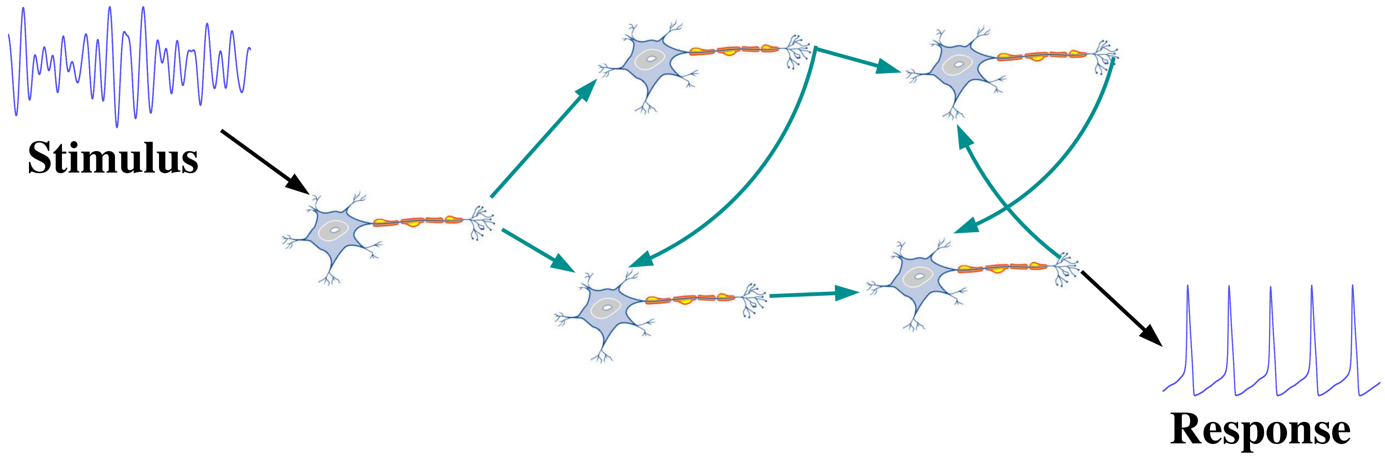


Reliability: same stimulus $\Rightarrow$ (essentially) same response



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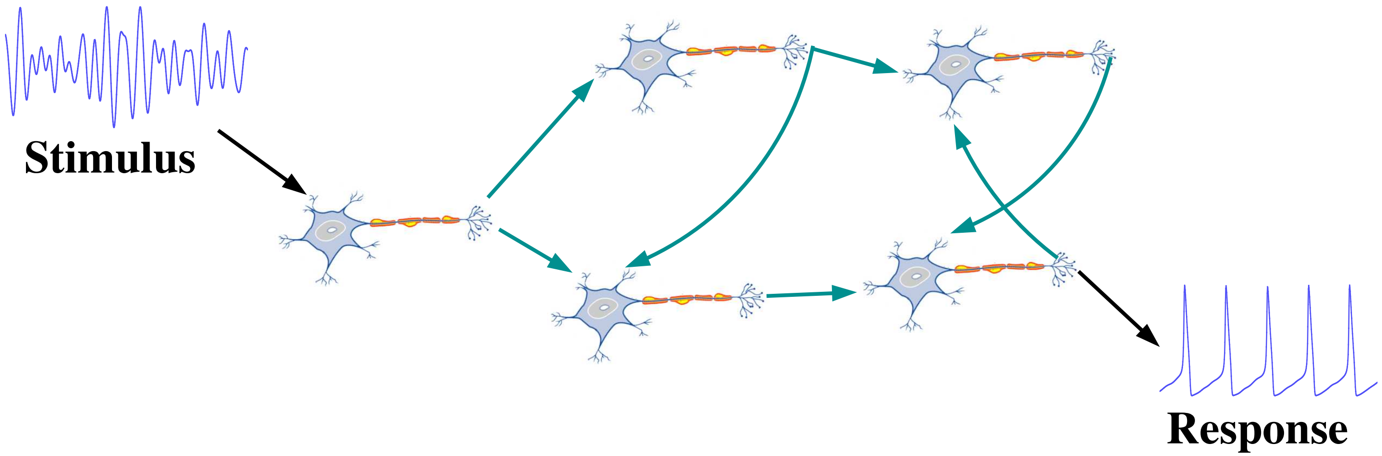
1. *Neural processing via temporal spike patterns*



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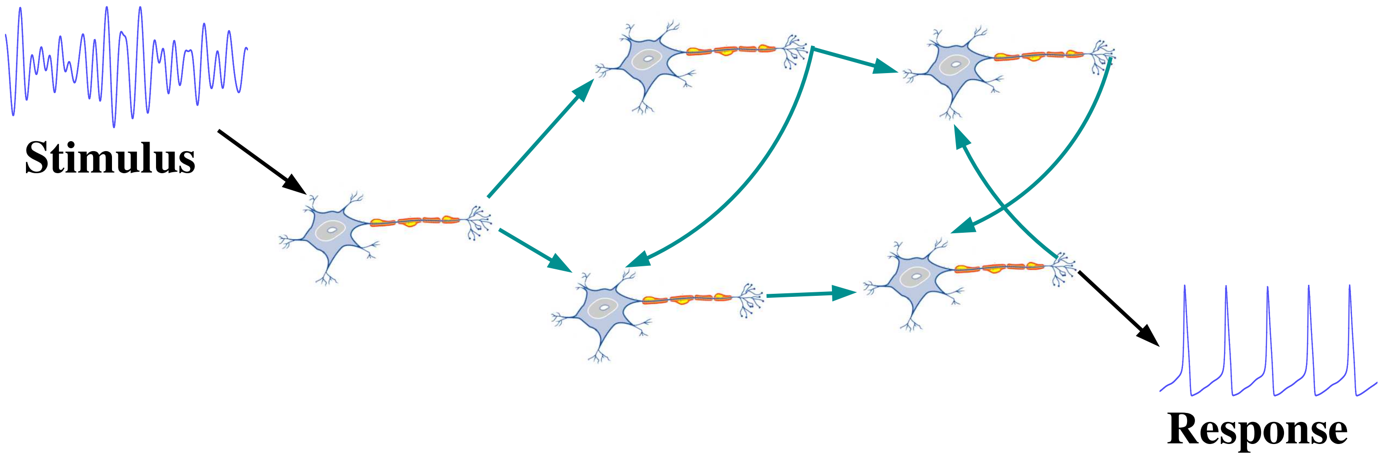

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1. *Neural processing via temporal spike patterns*
2. *Generalized synchrony*
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3. *Uncertainty propagation in dynamical systems* [Mézic]

Goal: Network conditions for reliable and unreliable behavior *via qualitative theory* + *numerics*

Note: Single neurons typically reliable
[Galán-Ermentrout, Goldobin-Pikovsky, Kurths, Teramae, Nakao, Kuske, ...]

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Outline

1. Model and Background
2. A Class of Reliable Networks
3. A Mechanism for Unreliability

1. Model + Background

Model: Phase Oscillator Networks

N phase oscillators $\qquad \theta_i \in \mathbb{S}^1 \sim [0, 2\pi)\ , \quad i = 1, \cdots, N$

$$\dot{\theta}_i(t) = \nu_i$$

1. Model + Background

Model: Phase Oscillator Networks

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$$\dot{\theta}_i(t) = \nu_i + z(\theta_i) \left[I_i(t) \qquad\qquad\qquad \right]$$

$$I_i(t) = \text{External stimulus}$$

$$z(\theta) = \textit{Phase Response}$$

$$= 1 - \cos(\theta)$$ *("Theta neuron" [Ermentrout, Kopell])*

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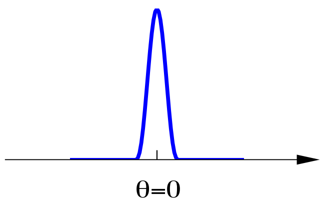

Reliability:

Given $I_i(t)$, $(\theta_1(t), ..., \theta_N(t))$ *essentially independent* of $(\theta_1(0), ..., \theta_N(0))$ for $t > t_0$

i.e. network state reproducible across trials

"Trial": new initial condition (fix stimulus, network, etc.)

Conceptual Framework:
Random Dynamical Systems (RDS)

- Idealized inputs: $I_i(t)$ white noise $\Rightarrow$ SDE of the form

$$dX_t = V_0(X_t)\,dt + \sum_{i=1}^{m} V_i(X_t) \circ dW_t^i$$

$X_t \in M$ (for us, $M = \mathbb{T}^N$)
W_t^i IID standard BM
V_i vector fields on M

Standing assumptions: (i) $\exists!$ stationary measure μ ; (ii) $\mu \ll$ Leb

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Dynamics of *ensemble* in response to *one sample input signal*

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- **Stochastic flow of diffeomorphisms**
  [Bismut, Elworthy, Ikeda-Watanabe, Kunita]

RDS Review

Stochastic flow maps

$\exists\ F_\omega^{s,t} \in \mathrm{Diff}(M)$

- $X_t = F_\omega^{s,t}(X_s)$
- $F_{\sigma_s(\omega)}^{s,t} \circ F_\omega^{0,s} = F_\omega^{0,t}$

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Sample measures μ_ω:

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Properties of μ_ω:

(i) $\mu_\omega = \mu$ conditioned on past

(ii) $\mathbb{E}(\mu_\omega) = \mu$

(iii) $(F^{0,t}_\omega)_* \mu_\omega = \mu_{\sigma_t(\omega)}$

$\sigma_t(\omega)_s = \omega_{s+t}$

Lyapunov exponents

- Measure average rate of (local) expansion / contraction
- N dimensions $\quad \Rightarrow \quad \exists N$ exponents
- Largest exponent $\lambda_{\max}$

  $\lambda_{\max} > 0$: *asymptotically unstable*

  $\lambda_{\max} < 0$: *asymptotically stable*

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Theorem (Oseledecs). $\exists\ \lambda_1 > \lambda_2 > ... > \lambda_r$, integers m_i, and subspaces $E_i(x,\omega) \subset T_x M$:

(i) $\dim(E_i(x,\omega)) = m_i\ ;\quad \oplus_{i=1}^r E_i = T_x M$

(ii) $v \in E_i(x,\omega) \quad \Rightarrow \quad \frac{1}{t} \log |DF_\omega^{0,t}(x) \cdot v| \to \lambda_i$

$\lambda_{\max}$ determines structure of μ_ω

Case 1: $\lambda_{\max} < 0$

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Theorem (Le Jan). If $\lambda_{\max} < 0$, then μ_ω is a *random sink*, i.e., $\mu_\omega = \frac{1}{n}\sum_{i=1}^n \delta_{Z^i_\omega(t)}$ $(n \geq 1)$.

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Theorem (Baxendale). Suppose $\lambda_{\max} \leq 0$, and

(i) The "2-point motion" has no compact invariant sets in $M \times M \backslash$diagonal ,
(ii) Lie$(TV_1, \cdots, TV_m)(x, v) = T_{(x,v)}TM$, $\quad (x, v) \in TM, v \neq 0$.

Then $\mu_\omega = \delta_{Z_\omega(t)}$.

Note: "Typically" one expects $n = 1$; confirmed numerically.

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$\lambda_{\max} < 0$ ***indicates reliability***

Note. Single oscillators typically reliable

Proposition (see e.g. [Kifer])
(i) $\sum_i m_i \lambda_i \leq 0$
(ii) $\sum_i m_i \lambda_i = 0$ iff μ is invariant (under $F_\omega^{0,t}$ for a.e. ω)

Can also show directly using Jensen's inequality

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$\lambda_{\max} > 0$ ***indicates unreliability***

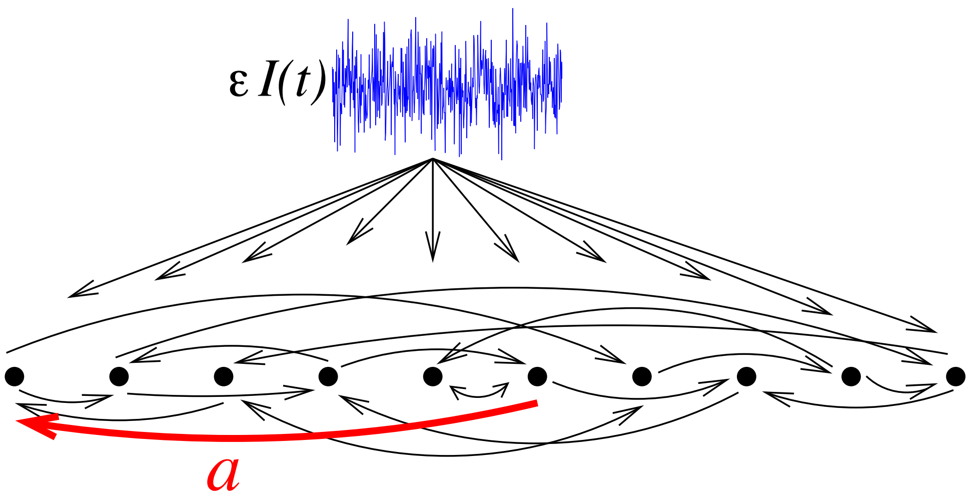
Random Sink ($\lambda_{\max} < 0$)

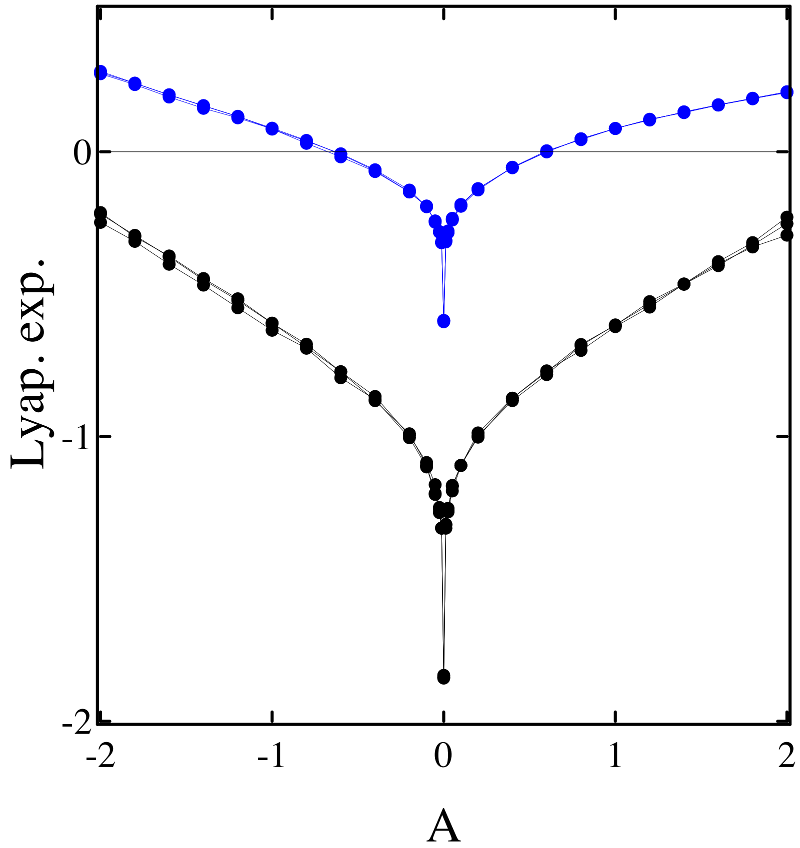
[CLICK TO PLAY MOVIE]

Random SRB Measure ($\lambda_{\max} > 0$)

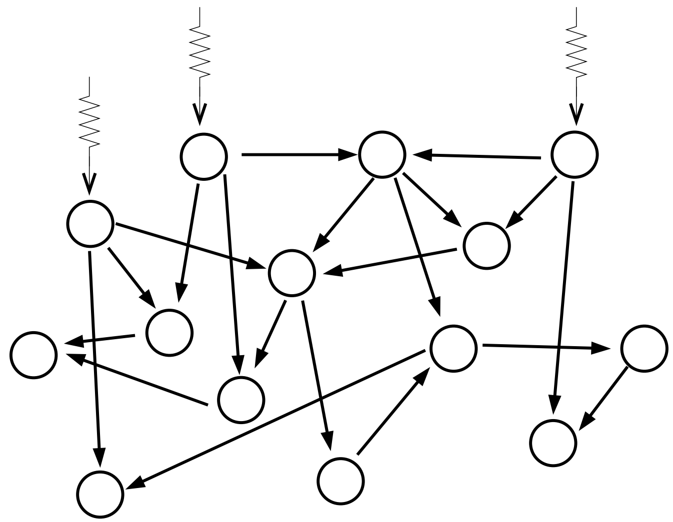
[CLICK TO PLAY MOVIE]

Remark. $\lambda_{\max}$ is useful for *numerically probing* reliability





2. A Condition for Reliable Networks



Theorem (KL–Shea-Brown–Young). If the connection graph of a phase oscillator network is acyclic, then $\lambda_{\max} \leq 0$.

Note: [Mézic et al] have analogous result for uncertainty propagation

Proof (sketch):

(i) Acyclic graph $\Rightarrow$ $\exists$ ordering so $i < j$ iff $i \rightsquigarrow j$

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(ii) Skew product structure: $\forall k < N$, can write flow as

$$F^{0,t}_{\omega,N}(\Theta^{(N)}_t) = \left(F^{0,t}_{\omega,k}(\Theta^{(k)}_t) \ , \ \ G^{0,t}_{\omega,\Theta^{(k)}_t}(\theta_{k+1}, \cdots, \theta_N) \right)$$

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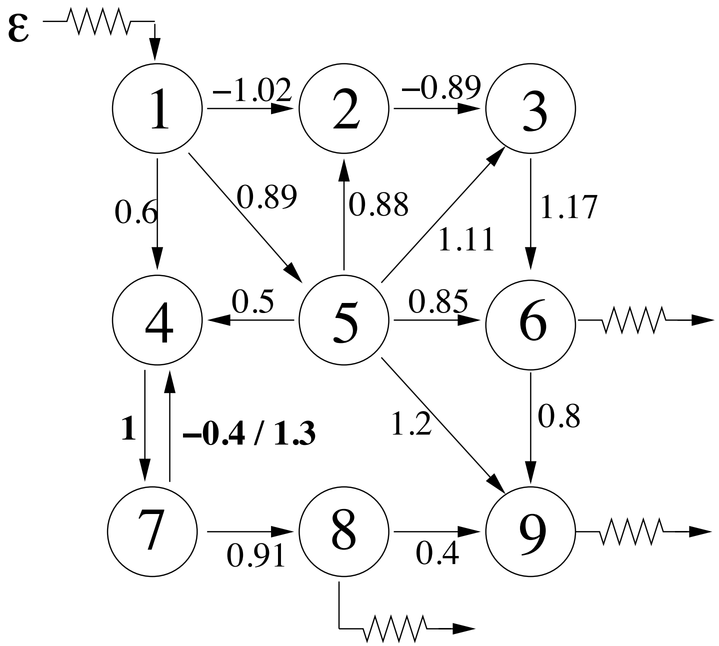
(iii) *Induction on k, using*

$$|\pi_{k+1}DF^{0,t}_\omega(x)v| = \frac{|\pi_k DF^{0,t}_\omega(x)v|}{|\sin\angle(v_{k+1},\pi_{k+1}DF^{0,t}_\omega(x)v)|} \text{ and}$$

$$\frac{1}{t}\log|\sin\angle(DF^{0,t}_\omega(x)u, DF^{0,t}_\omega(x)v)| \to 0 \text{ as } t\to\infty$$

Extension: Acyclic Networks of Modules



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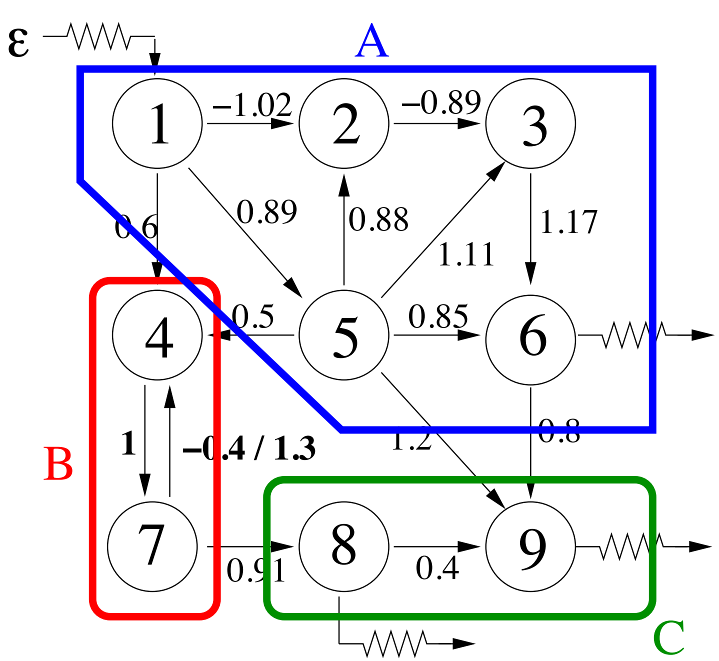

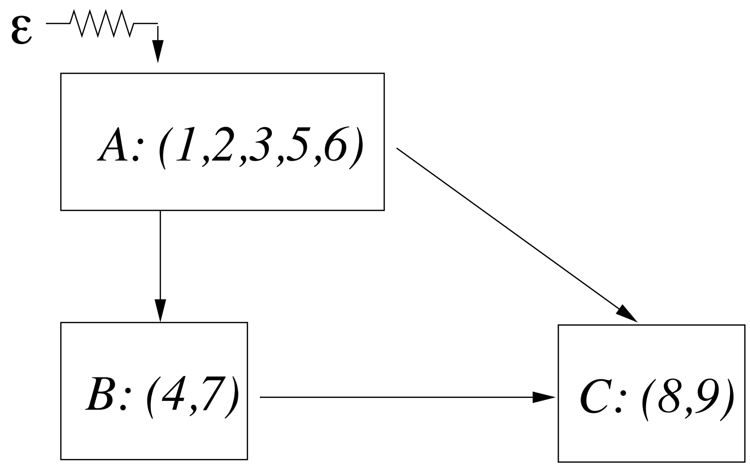

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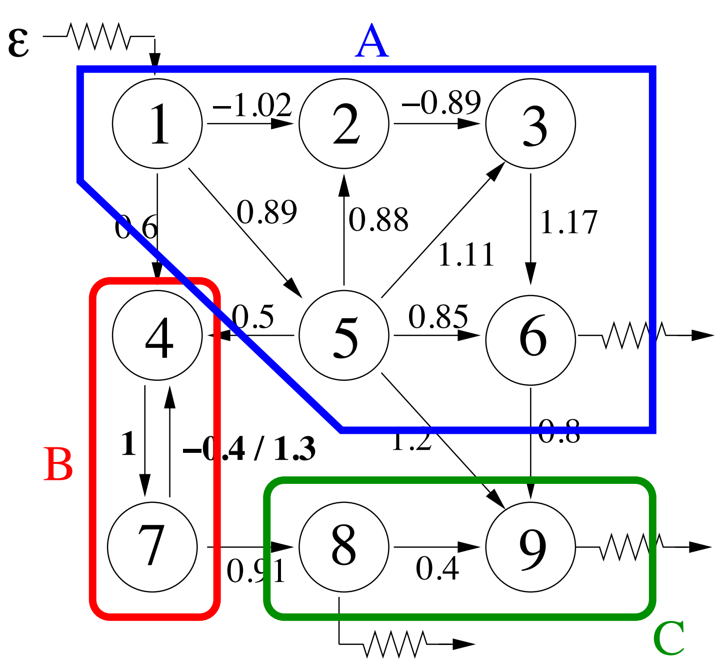
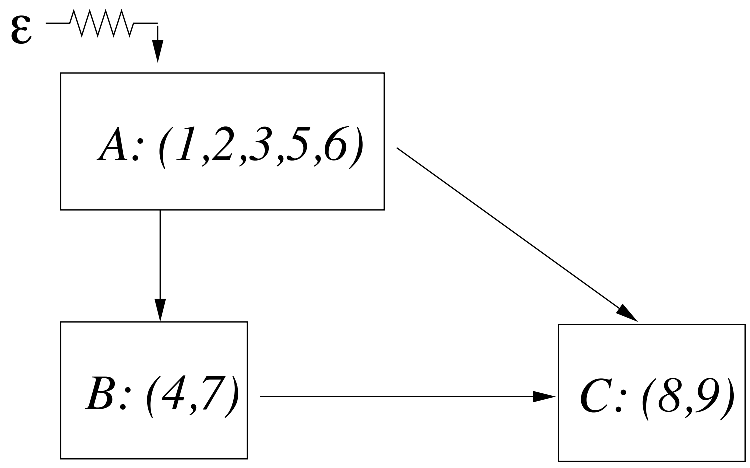
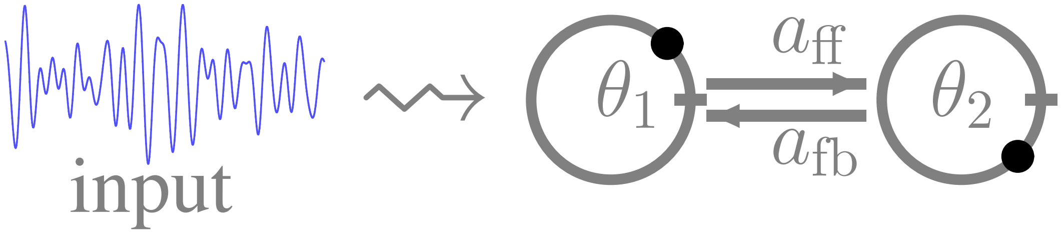




- If every "module" is reliable, then $\lambda_{\max} \leq 0$

- Can measure reliability of modules via fiber Lyapunov exponents

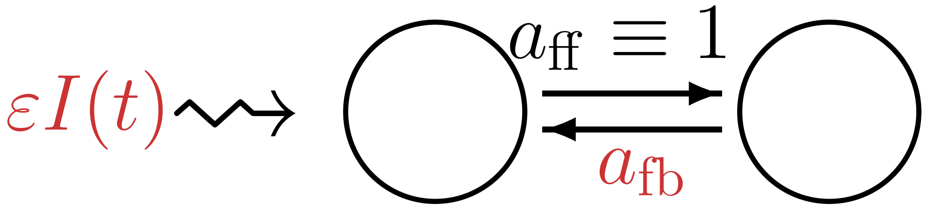
3. A Mechanism for Unreliability

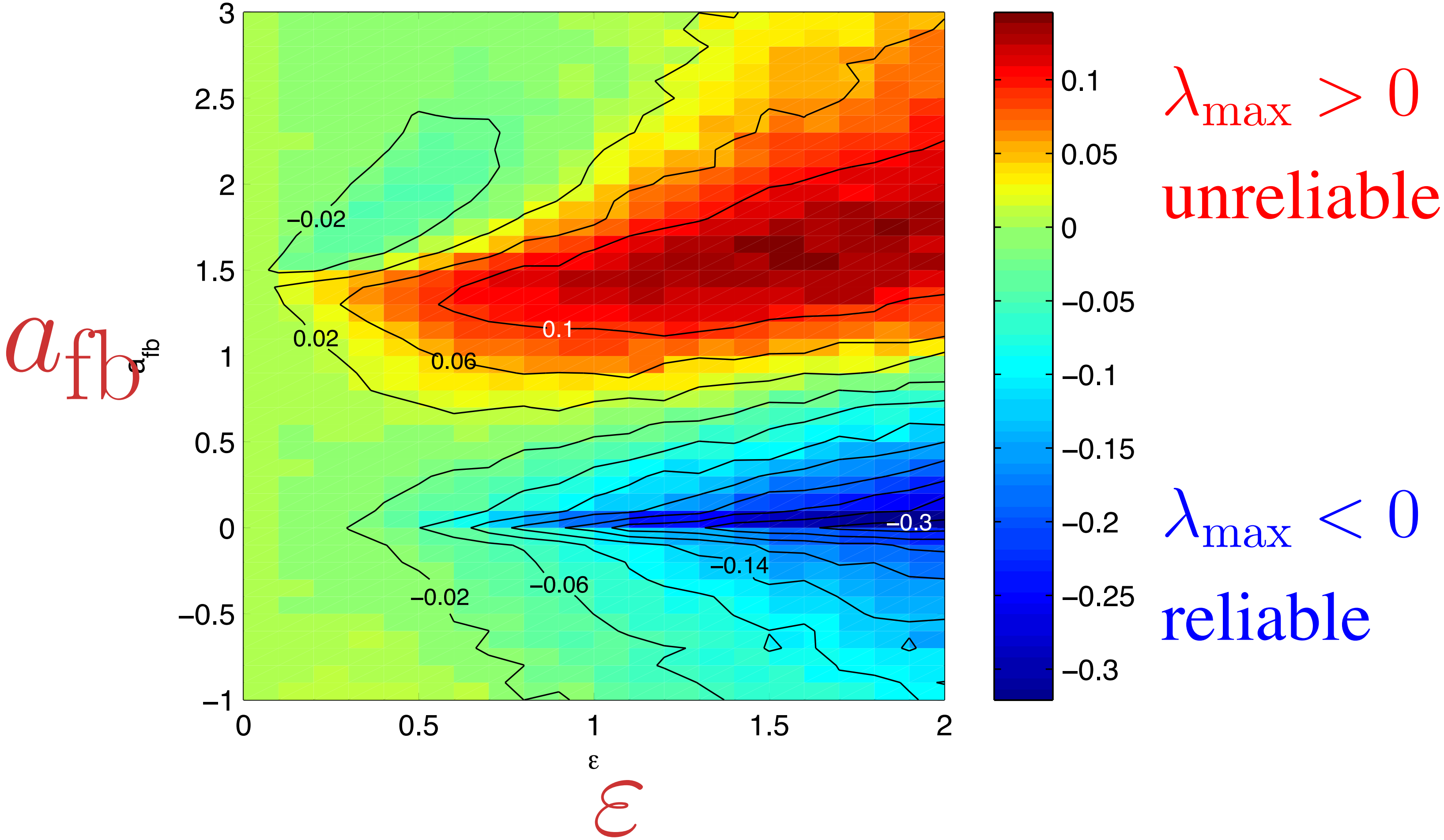


$$\dot{\theta}_1 = \nu_1 + z(\theta_1) \cdot [a_{\mathrm{fb}} \cdot g(\theta_2) + \varepsilon I(t)]$$

$$\dot{\theta}_2 = \nu_2 + z(\theta_2) \cdot [a_{\mathrm{ff}} \cdot g(\theta_1)]$$

How do $a_{\rm ff}$, $a_{\rm fb}$, and ε affect reliability?





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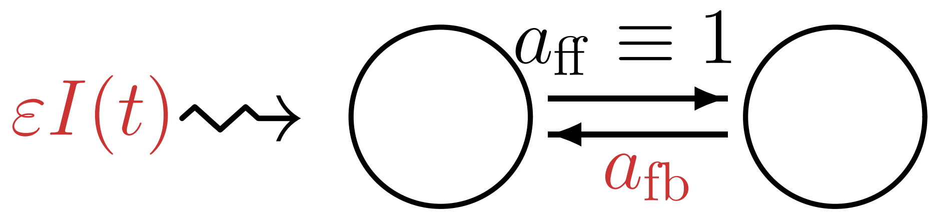

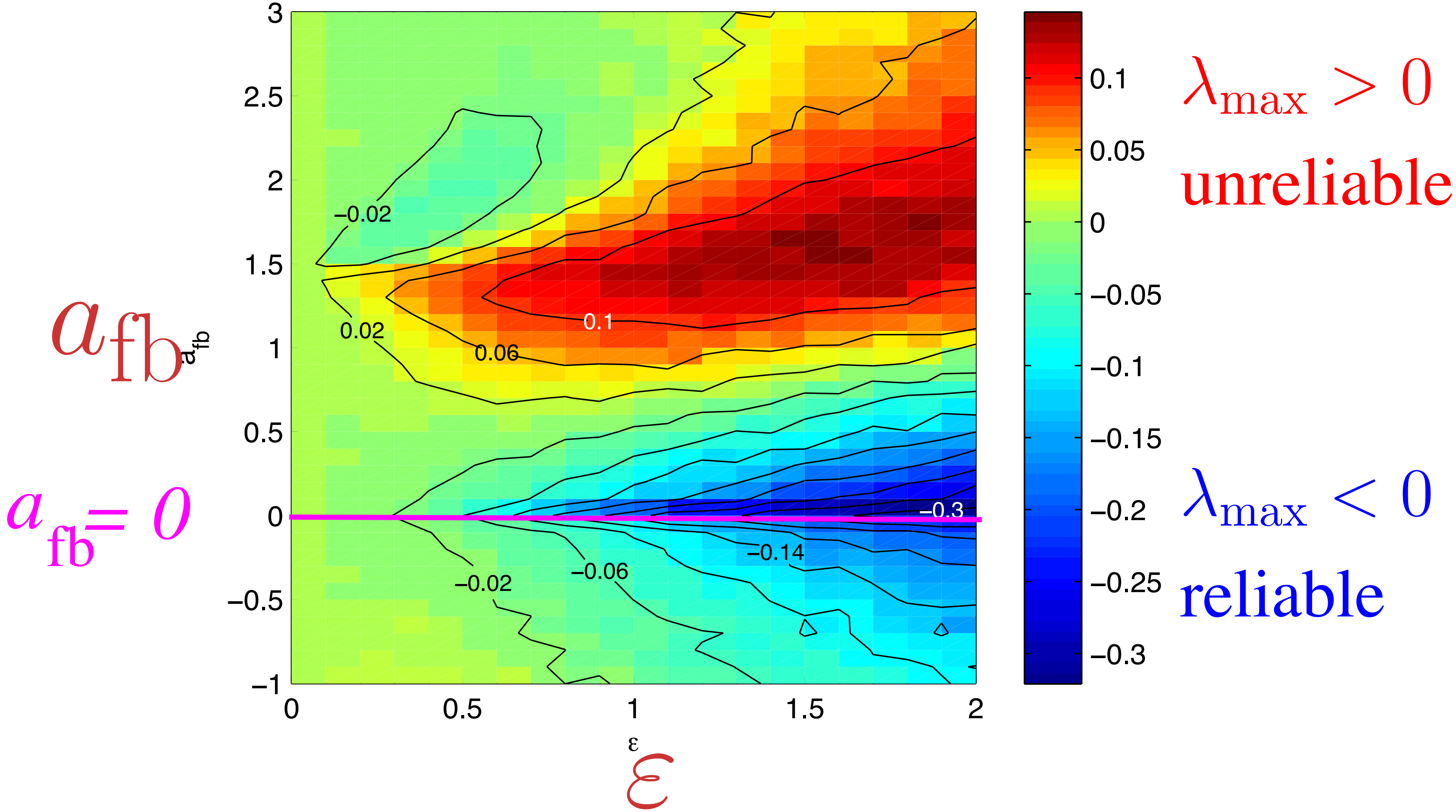

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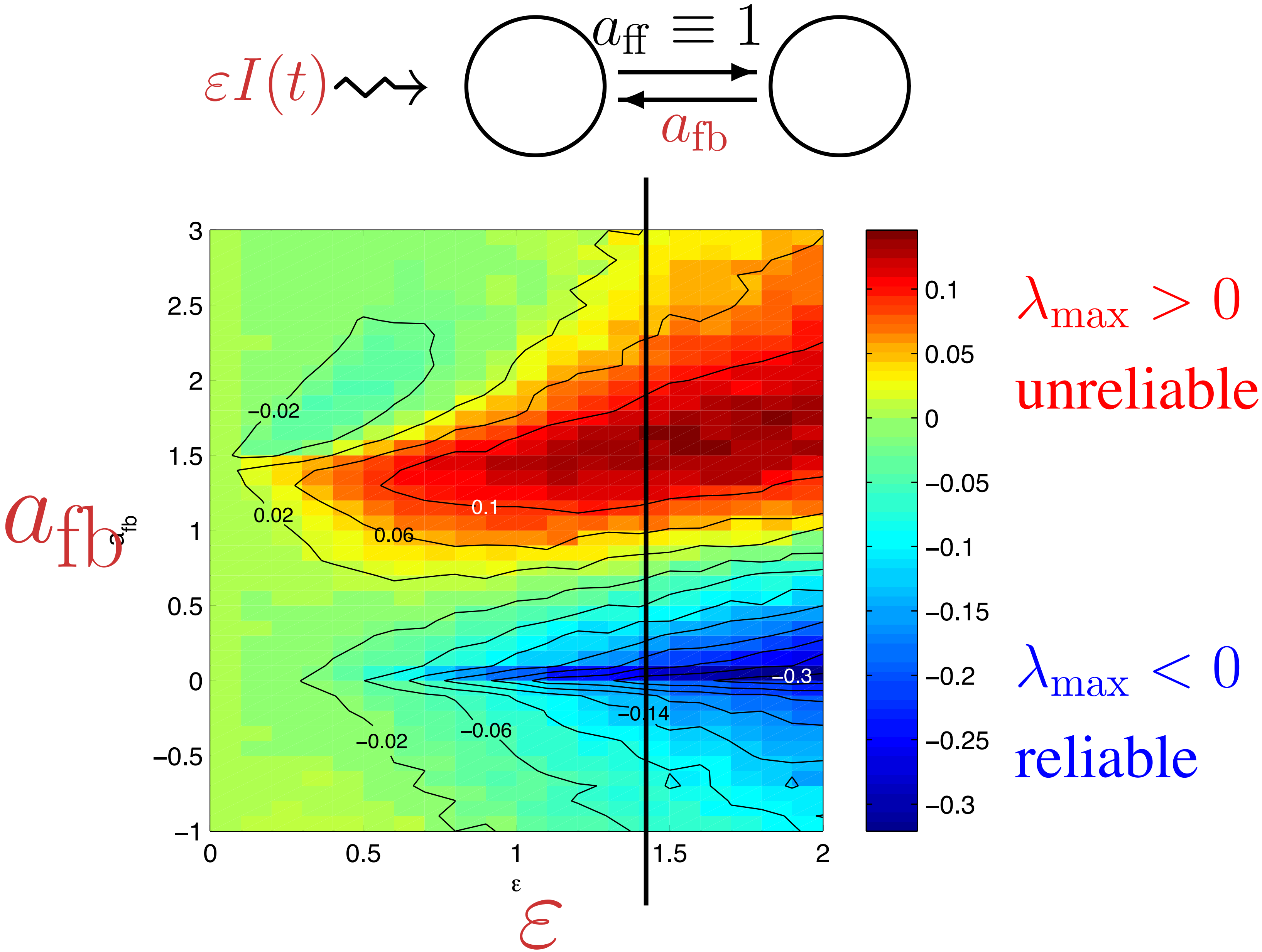

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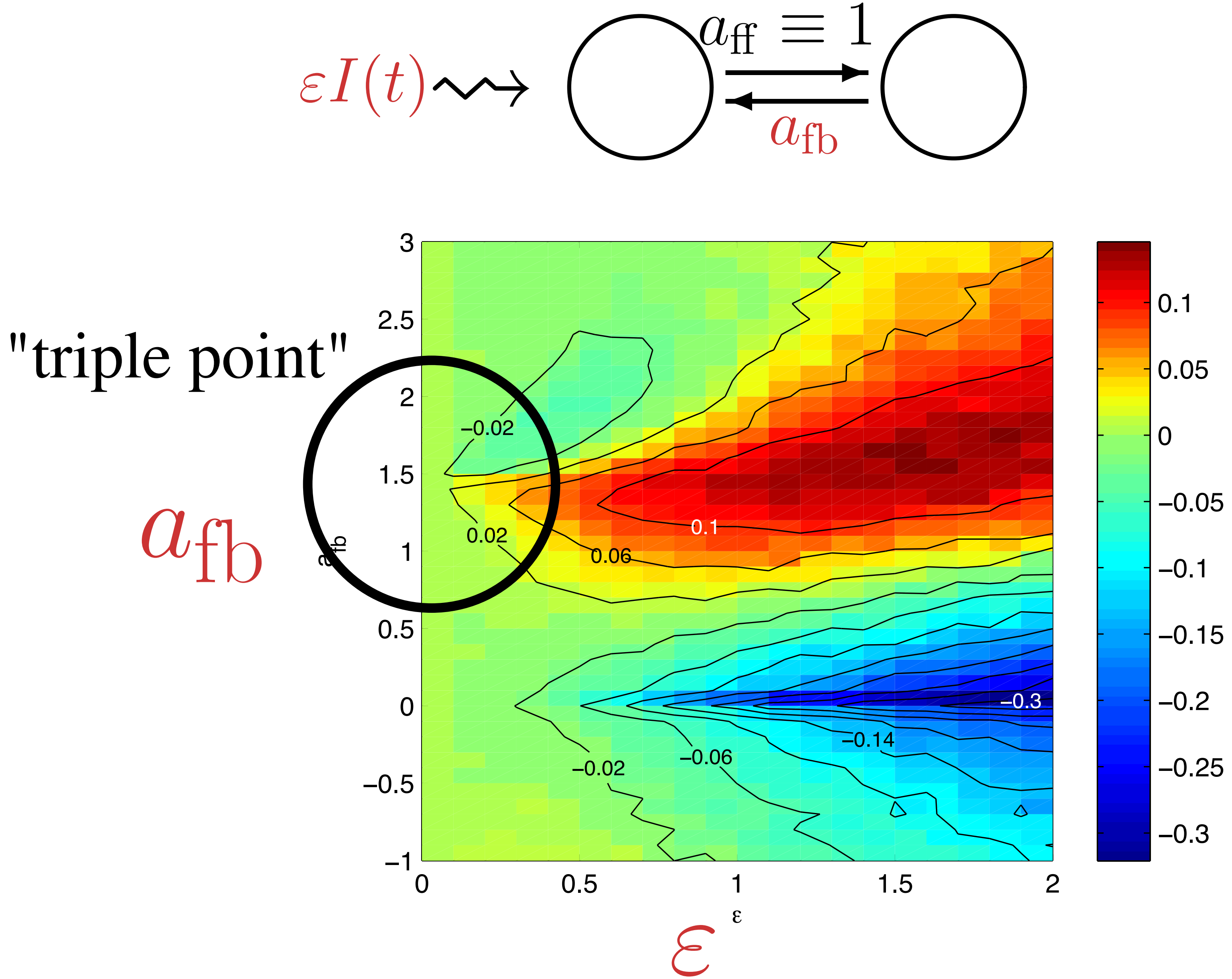
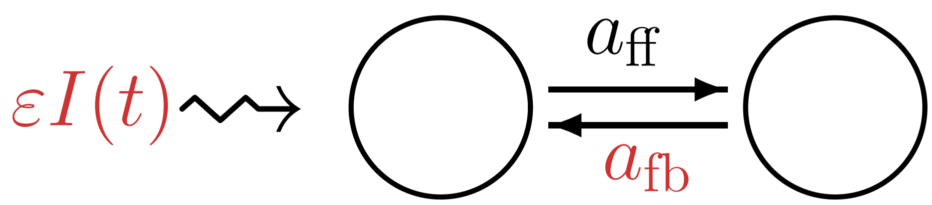




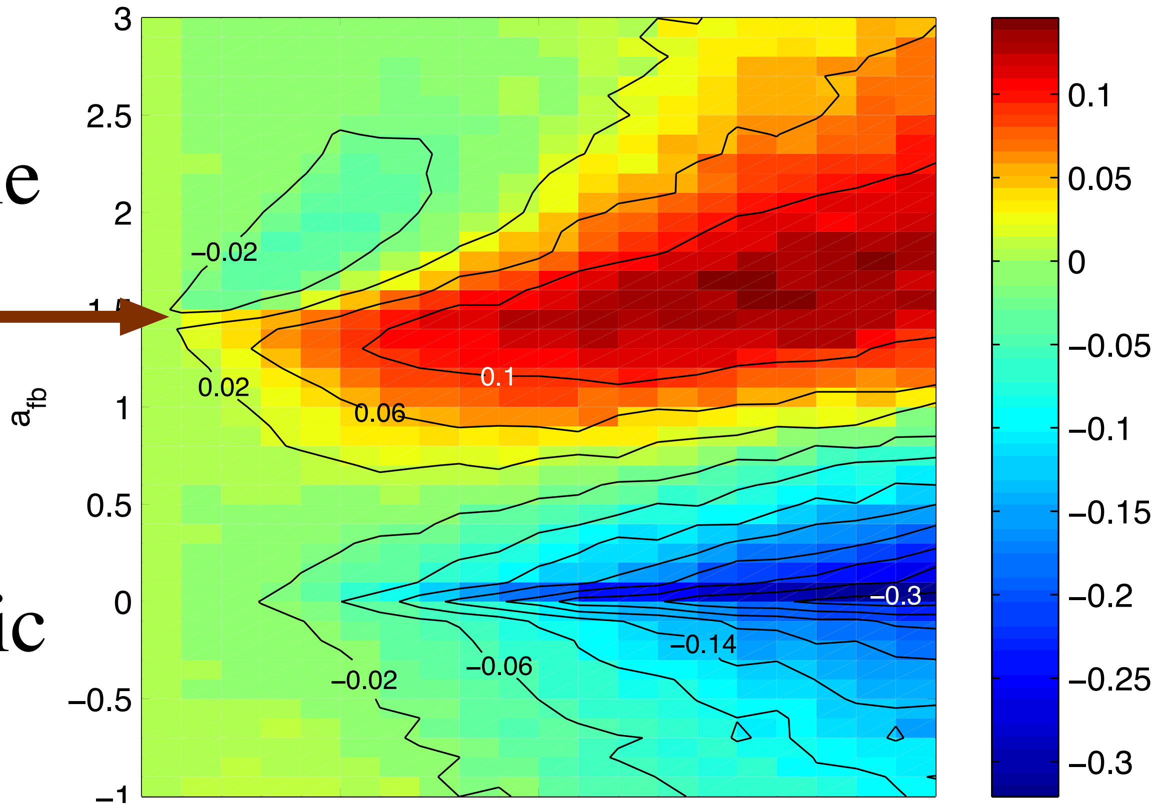
For $\varepsilon = 0$, $\exists a^*_{\mathrm{fb}}(a_{\mathrm{ff}})$ at which a limit cycle emerges

[KL–Shea-Brown–Young]

limit cycle

a^*_{fb}

~quasiperiodic



$\varepsilon = 0$

$\lambda_{\max} > 0$ unreliable

$\lambda_{\max} < 0$ reliable

Geometric Mechanism for Positive Exponent

Dynamical setting: Limit cycle + perturbation

Setting close to rigorous work of Wang and Young on **shear-induced chaos** [Wang, Young, $\cdots$]

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Setting close to rigorous work of Wang and Young on **shear-induced chaos** [Wang, Young, $\cdots$]

Q. Wang and L.-S. Young studied

$$\dot{x} = f(x) + g(x) \sum_k \delta(t - kT) \ .$$

Proved in a variety of settings (Hopf, reaction-diffusion PDEs, mechanical oscillators...): $\lambda_{\max} > 0$, SRB measures , exponential mixing....

Simplest example:

$$\begin{aligned}\dot{\theta} &= 1 + \sigma y \\ \dot{y} &= -\lambda y + A \cdot H(\theta) \cdot \textstyle\sum_{n=0}^{\infty} \delta(t - nT)\end{aligned}$$

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Theorem (Wang-Young). If

$$\frac{\sigma}{\lambda} \cdot A \equiv \frac{\text{shear}}{\text{contraction rate}} \cdot (\text{kick "amplitude"}) \gg 1 ,$$

then $\exists$ a positive measure set $\Delta \subset \mathbb{R}^+$ such that $\forall T \in \Delta$,

(i) $\lambda_{\max} > 0$

(ii) the system above has an SRB measure

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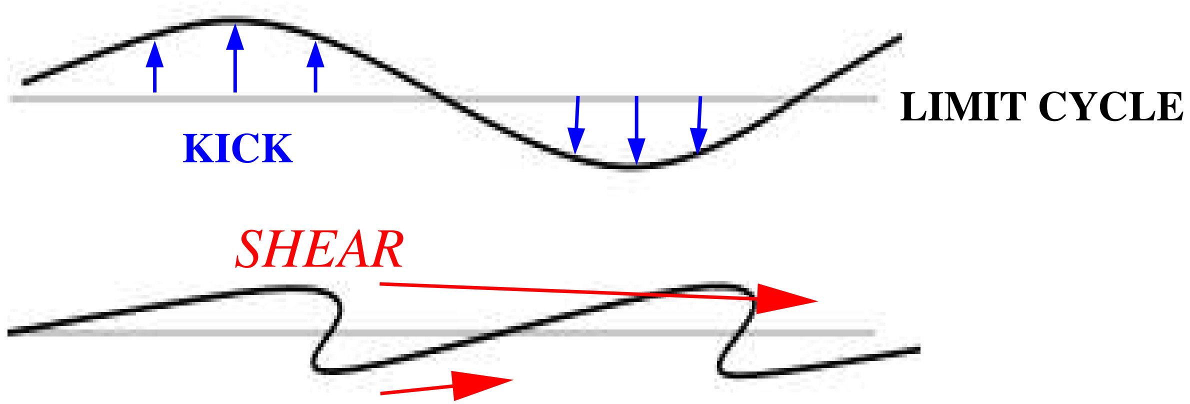
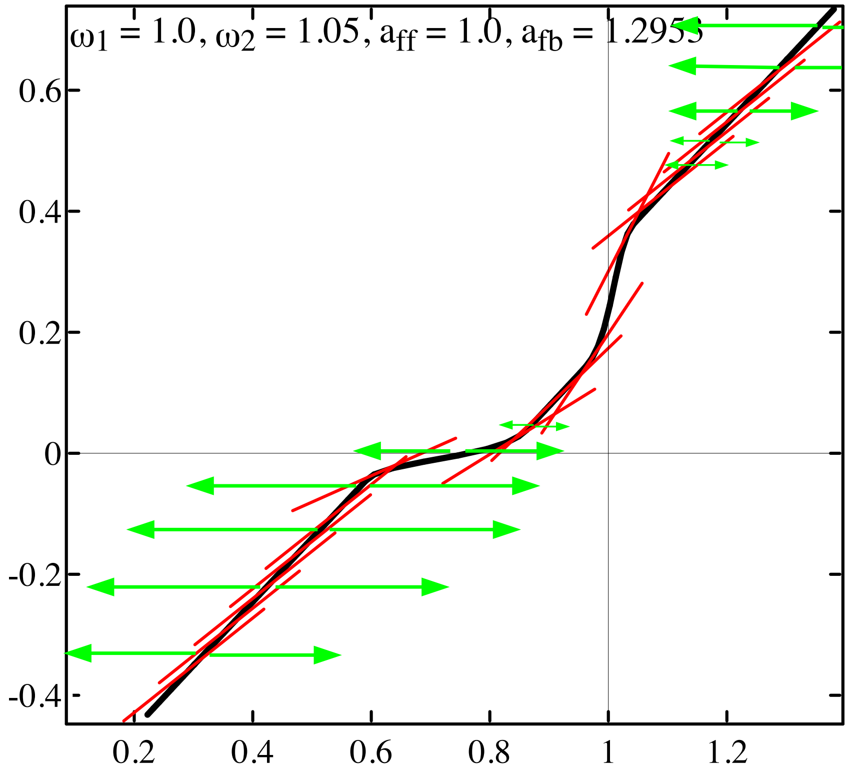


At triple point (where limit cycle emerges):

- Geometry indicates strong shear in relevant parameter region



At triple point (where limit cycle emerges):

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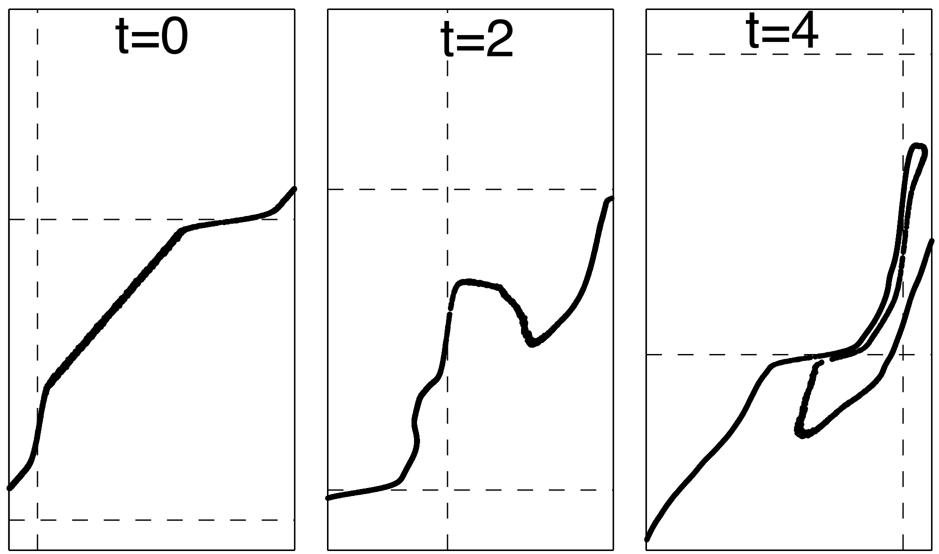
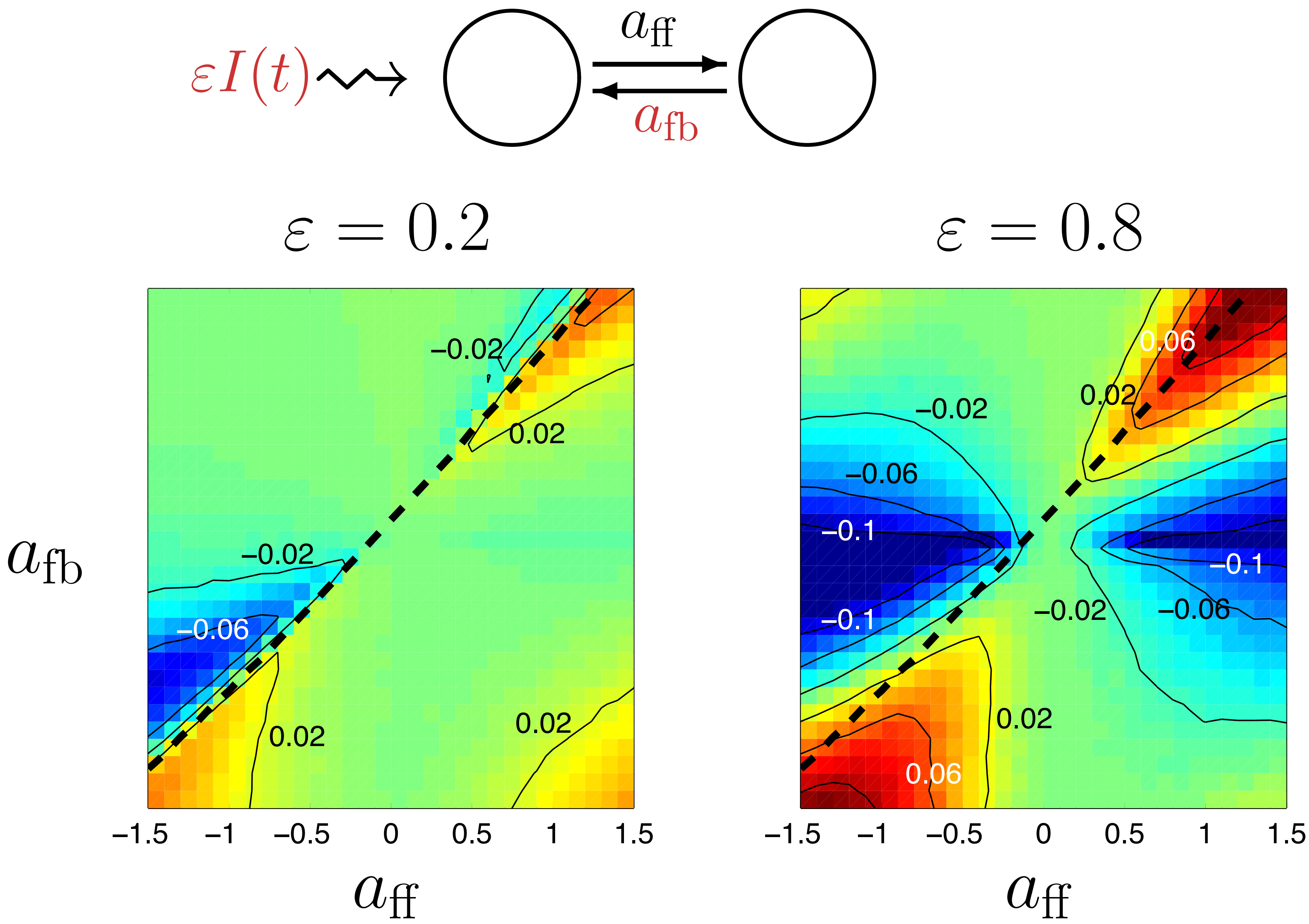
- Suggests: Weakly-attracting cycle + strong shear + perturb.

  $\Rightarrow$ Folding of limit cycle

  $\Rightarrow$ Unreliability




$\varepsilon I(t)$
a_{ff}
a_{fb}
$\varepsilon = 0.2$
$\varepsilon = 0.8$
a_{fb}
−0.02
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−0.06
0.06
−0.1
−1.5 −1 −0.5 0 0.5 1 1.5
a_{ff}

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(ii) In 2-osc systems, shear-induced chaos may play a role in $\lambda_{\max} > 0$

RDS provides a useful framework for studying reliability

Conclusions

Driven oscillator networks can be reliable or unreliable depending on network conditions, and

(i) Acyclic networks cannot be unreliable

(ii) In 2-osc systems, shear-induced chaos may play a role in $\lambda_{\max} > 0$

RDS provides a useful framework for studying reliability

References: http://math.arizona.edu/~klin

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