Pathwise Approximation of Stochastic Differential Equations

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Oberwolfach, 22.08 – 26.08.2011

Stochastic Differential Equations

Itô-SDE on $\mathbb{R}^d$

$$dX(t) = a(X(t))dt + \sum_{j=1}^{m} b^j(X(t))dW^j(t), \qquad t \in [0, T]$$
$$X(0) = x_0 \in \mathbb{R}^d$$

where

- $a, b^j : \mathbb{R}^d \to \mathbb{R}^d$, $j = 1, \ldots, m$: drift- and diffusion coefficients
- $W = (W^1, \ldots, W^m)$: m-dim. Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$

Assumption: (SDE) has a unique strong solution

$$X = \Phi_{a,b,x_0}(W)$$

with Itô map

$$\Phi_{a,b,x_0} : C([0, T]; \mathbb{R}^m) \to C([0, T]; \mathbb{R}^d)$$

Computational SDEs

$$X = \Phi_{a,b,x_0}(W)$$

Problems

(i) Approximate Itô map Φ_{a,b,x_0}

strong / pathwise approximation

(ii) Approximate law $\mathbf{P}^X$

weak approximation

(iii) Compute expectation $\mathbf{E}f(X)$ for $f : C([0,T];\mathbb{R}^d) \to \mathbb{R}$

quadrature

(iv) control problems, approximation of invariant measure, ...

Maruyama (1955) ... Milstein (1974) ... Kloeden, Platen (1992) ...

Classically: a, b globally Lipschitz, i.e. there exists $L > 0$ s.th.

$$\text{(Lip)} \quad |a(x) - a(y)| + |b(x) - b(y)| \leq L \cdot |x - y|, \quad x, y \in \mathbb{R}^d$$

Pathwise Error Criteria

Approximation

$$\overline{X} = \widehat{\Phi}_{a,b,x_0}(W)$$

typically based on discretisation

$$0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$$

Pathwise error in the discretization points

$$\sup_{i=0,\ldots,n} |X(t_i,\omega) - \overline{X}(t_i,\omega)|, \qquad \omega \in \Omega$$

Global pathwise error

$$\sup_{t\in[0,T]} |X(t,\omega) - \overline{X}(t,\omega)|, \qquad \omega \in \Omega$$

Why?

- Numerical calculation of $\overline{X}$ is carried out path by path
- Theory of random dynamical systems is of pathwise nature
- Solution may be non-integrable
- Natural for other driving noises, ...

Known Results (– 2007)

Milstein scheme for $m = d = 1$ (Talay, 1983):

$$\sup_{i=0,\dots,n} |X(iT/n) - \overline{X}^M(iT/n)| \le C_\varepsilon^M \cdot n^{-1/2+\varepsilon} \quad \text{a.s.}$$

for all $\varepsilon > 0$ where C_ε^M a.s. finite and non-negative random variable

(Proof uses Doss-Sussmann representation)

Milstein scheme $(m = d = 1)$

$$x_{k+1} = x_k + a(x_k)\Delta + b(x_k)\Delta_k W + \frac{1}{2} b' b(x_k)((\Delta_k W)^2 - \Delta)$$

where

$$\Delta = \tfrac{1}{n}, \quad \Delta_k W = W_{(k+1)/n} - W_{k/n}, \quad x_k = \overline{X}^M(kT/n)$$

Known Results (– 2007)

Euler scheme for general SDE under weak assumptions (Gyöngy, 1998; Fleury, 2005):

$$\sup_{i=0,\dots,n} |X(iT/n) - \overline{X}^E(iT/n)| \leq C_\varepsilon^E \cdot n^{-1/2+\varepsilon} \quad \text{a.s.}$$

for all $\varepsilon > 0$

Euler scheme

$$x_{k+1} = x_k + a(x_k)\Delta + b(x_k)\Delta_k W$$

where

$$\Delta = \tfrac{1}{n}, \quad \Delta_k W = W_{(k+1)/n} - W_{k/n}, \quad x_k = \overline{X}^E(kT/n)$$

Convergence order $1/2 - \varepsilon$ "natural" for pathwise approximation?

The Main Lemma

Convergence order $1/2 - \varepsilon$ "natural" for pathwise approximation?

Answer: No!

Lemma

Let $\alpha > 0$, $c_p \geq 0$ for $p \geq 1$ and $(Z_n)_{n\in\mathbb{N}}$ a sequence of RVs with

$$(\mathbf{E}|Z_n|^p)^{1/p} \leq c_p \cdot n^{-\alpha}$$

for all $p \geq 1$, $n \in \mathbb{N}$

Then for all $\varepsilon > 0$ there exists a finite and non-negative random variable η_ε such that

$$|Z_n| \leq \eta_\varepsilon \cdot n^{-\alpha+\varepsilon} \quad \text{almost surely}$$

for all $n \in \mathbb{N}$

The Main Lemma

Proof Fix $\varepsilon > 0$ and $p > 1/\varepsilon$. Then

$$\mathbf{P}(n^{\alpha-\varepsilon}|Z_n| > \delta) \leq \frac{\mathbf{E}|Z_n|^p}{\delta^p} n^{p(\alpha-\varepsilon)} \leq \frac{c_p{}^p}{\delta^p} n^{-p\varepsilon}$$

for all $\delta > 0$ using the Chebyshev-Markov inequality

Now, since $p > 1/\varepsilon$,

$$\sum_{n=1}^{\infty} \mathbf{P}(n^{\alpha-\varepsilon}|Z_n| > \delta) < \infty$$

for all $\delta > 0$

Borel-Cantelli Lemma:

$$\lim_{n\to\infty} n^{\alpha-\varepsilon} Z_n = 0 \qquad \text{almost surely}$$

Now set

$$\eta_\varepsilon = \sup_{n\in\mathbb{N}} n^{\alpha-\varepsilon}|Z_n|$$

Itô-Taylor Schemes

- Set of all multi-indices

$$\mathcal{M} = \left\{ \alpha = (j_1, \ldots, j_l) \in \{0, 1, 2, \ldots, m\}^l : l \in \mathbb{N} \right\} \cup \{v\}$$

where

$l(\alpha)$: length of α
ν: multi-index of length 0
$n(\alpha)$: number of zero entries of α

- Differential operators:

$$L^0 = \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{m} b^{k,j} b^{l,j} \frac{\partial^2}{\partial x^k \partial x^l}, \quad L^j = \sum_{k=1}^{d} b^{k,j} \frac{\partial}{\partial x^k}$$

for $j = 1, \ldots, m$

Notation: a^k, $b^{k,j}$ k-th components of a and b^j

Itô-Taylor Schemes

- Iterated integrals and coefficient functions:

$$I_\alpha(s,t) = \int_s^t \cdots \int_s^{\tau_2} dW^{j_1}(\tau_1)\, \ldots dW^{j_l}(\tau_l)$$
$$f_\alpha(x) = L^{j_1} \cdots L^{j_{l-1}} b^{j_l}(x)$$

with $\alpha = (j_1, \ldots, j_l)$
Notation: $dW^0(\tau) = d\tau,\ b^0 = a$

Itô-Taylor scheme of (mean-square) order $\gamma = 0.5, 1.0, 1.5, \ldots$

$$\overline{X}_n^\gamma(t_0) = X_0,$$
$$\overline{X}_n^\gamma(t_{i+1}) = \overline{X}_n^\gamma(t_i) + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{\nu\}} f_\alpha(\overline{X}_n^\gamma(t_i)) \cdot I_\alpha(t_i, t_{i+1})$$

for $i = 0, \ldots, n-1$, where

$$\mathcal{A}_\gamma = \{\alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2\gamma \text{ or } l(\alpha) = n(\alpha) = \gamma + 1/2\}$$

Itô-Taylor Schemes

Theorem 1 (Jentzen, Kloeden and N, 2009)
If $a, b^j \in C^{2\gamma+1}(\mathbb{R}^d; \mathbb{R}^d)$, $j = 1, \ldots, m$, then for all $\varepsilon > 0$

$$\sup_{i=0,\ldots,n} |X(iT/n, \omega) - \overline{X}_n^{\gamma}(iT/n, \omega)| \leq \eta_\varepsilon^\gamma(\omega) \cdot n^{-\gamma+\varepsilon}$$

for almost all $\omega \in \Omega$ and all $n \in \mathbb{N}$

Remarks

- Pathwise convergence rates robust with respect to weak assumptions
  - (i) No global Lipschitz assumptions required
  - (ii) Solution may be non-integrable, $\mathbf{E}|X(t)| = \infty$ for some $t \geq 0$
- Pathwise order of convergence of Milstein scheme: $1 - \varepsilon$
- Random constant η_ε^γ unknown in general:

  Estimator $\eta_\varepsilon^\gamma(\omega) = F_\gamma(\varepsilon, W(\omega), \ldots) + \ldots$?

Sketch of Proof

Step 1: For bounded coefficients with bounded derivatives use **Burkholder-Davis-Gundy inequality**

$$\mathbf{E} \sup_{s\in[0,t]} \left| \int_0^s X(\tau)\, dW(\tau) \right|^p \leq k_p \cdot \mathbf{E} \left| \int_0^t |X(\tau)|^2\, d\tau \right|^{p/2}$$

and standard error analysis to show

$$\mathbf{E} \sup_{i=0,\dots,n} |X(iT/n) - \overline{X}_n^{\gamma}(iT/n)|^p \leq c_p^p \cdot n^{-p\gamma}$$

Now use Main Lemma

Step 2: General case by localisation procedure

Sketch of Proof

Step 2:

(i) Choose open and bounded sets $D_q \subset D$, $q \in \mathbb{N}$, s.th.

$$\ldots \subset D_q \subset D_{q+1} \subset \ldots$$

and

$$\cup_{q \in \mathbb{N}} D_q = D$$

Define stopping times

$$\tau^{(q)} = \inf\{t \geq 0 : X(t) \notin D_q\}$$
$$\tau_n^{(q)} = \inf\{t \geq 0 : \overline{X}_n^{\gamma}(t) \notin D_q\}$$

(ii) By Step 1

$$\sup_{i=0,\ldots,n} |X(iT/n,\omega) - \overline{X}_n^{\gamma}(iT/n,\omega)| \mathbf{1}_{\{\tau^{(q)} > T, \tau_n^{(q)} > T\}} \leq \eta_{\varepsilon,q}^{\gamma}(\omega) \cdot n^{-\gamma+\varepsilon}$$

(Choose coefficients a_q and b_q with compact support s.th. $a_q = a, b_q = b$ on D_q)

Sketch of Proof

(i)+(ii)

$$\tau^{(q)} = \inf\{t \geq 0 : X(t) \notin D_q\}$$
$$\tau_n^{(q)} = \inf\{t \geq 0 : \overline{X}_n^{\gamma}(t) \notin D_q\}$$

$$\sup_{i=0,\ldots,n} |X(iT/n,\omega) - \overline{X}_n^{\gamma}(iT/n,\omega)| \mathbf{1}_{\{\tau^{(q)} > T, \tau_n^{(q)} > T\}} \leq \eta_{\varepsilon,q}^{\gamma}(\omega) \cdot n^{-\gamma+\varepsilon}$$

(iii) Since

$$\sup_{t\in[0,T]} |X(t,\omega)| < \infty$$

it follows now

$$\mathbf{1}_{\{\tau^{(q)}(\omega) > T, \tau_n^{(q)}(\omega) > T\}} = 1$$

for $n \geq n_0(\omega)$, $q \geq q_0(\omega)$

Numerical Example I

Duffing-van der Pol oscillator with multiplicative noise on the velocity:

$$\ddot{y}(t) - \beta\dot{y}(t) + y^3(t) + y^2(t)\dot{y}(t) + y(t) - \sigma\dot{y}(t) \circ \dot{W}(t) = 0$$

with $\beta, \sigma \in \mathbb{R}$, $\sigma \neq 0$

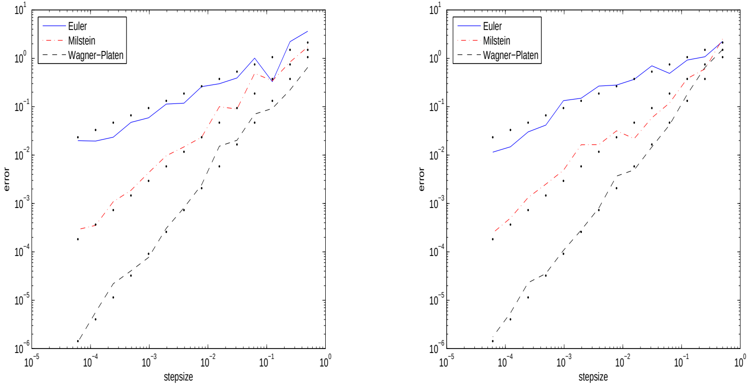
Corresponding Itô-SDE:

$$dX_1(t) = X_2(t)\,dt$$
$$dX_2(t) = \big(-X_1(t) + \bar{\beta}X_2(t) - X_1^3(t) - X_1^2(t)X_2(t)\big)\,dt + \sigma X_2(t)\,dW(t)$$

with $\bar{\beta} = \beta + \frac{1}{2}\sigma^2$

Numerical Example I

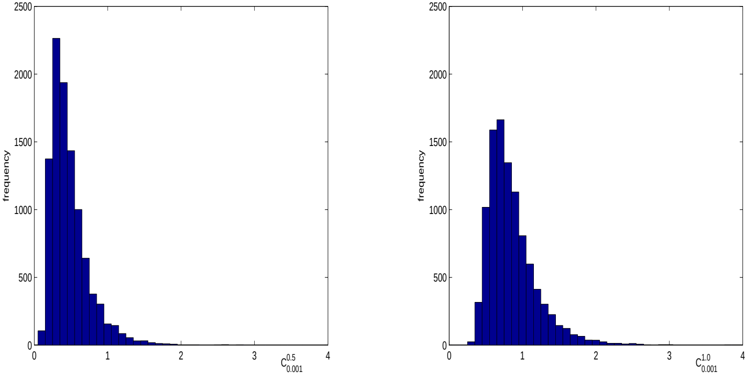
$$\beta = -1, \sigma = 2, X_1(0) = X_2(0) = 1, T = 1$$



pathwise maximum error vs. stepsize for two sample paths

Numerical Example II

$$dX(t) = -(1 + X(t))(1 - X^2(t))dt + (1 - X^2(t))dW(t), \;\; t \in [0,1],$$
$$X(0) = 0$$



empirical distribution of $\eta_{0.001}^{0.5}$ and $\eta_{0.001}^{1.0}$ (sample size: $N = 10^4$)

Extension: SDE on Domains

Assumption

$$\mathbf{P}(X(t) \in D \ \text{for all} \ \ t \geq 0) = 1$$

where D open subset of $\mathbb{R}^d$

Standard Itô-Taylor schemes may leave D, thus choose auxiliary functions $g, h^j \in C^s(\overline{D}^c; \mathbb{R}^d)$ and set

$$\widetilde{a}(x) = a(x) \cdot 1_D(x) + g(x) \cdot 1_{\overline{D}^c}(x), \qquad x \in \mathbb{R}^d$$
$$\widetilde{b}^j(x) = b^j(x) \cdot 1_D(x) + h^j(x) \cdot 1_{\overline{D}^c}(x), \qquad x \in \mathbb{R}^d$$

Modified Itô-Taylor scheme: Standard Itô-Taylor scheme of order γ applied to SDE

$$dX(t) = \widetilde{a}(X(t))dt + \sum_{j=1}^{m} \widetilde{b}^j(X(t))dW^j(t), \qquad t \geq 0$$

Extension: SDE on Domains

Modified Itô-Taylor scheme: Standard Itô-Taylor scheme of order γ applied to SDE

$$dX(t) = \widetilde{a}(X(t))dt + \sum_{j=1}^{m} \widetilde{b}^j(X(t))dW^j(t), \qquad t \geq 0$$

Theorem 2 (Jentzen, Kloeden and N, 2009)
If $a, b^j \in C^{2\gamma+1}(D; \mathbb{R}^d)$, $g, h^j \in C^{2\gamma-1}(\overline{D}^c; \mathbb{R}^d)$, $j = 1, \ldots, m$, then

$$\sup_{i=0,\ldots,n} |X(iT/n) - \widetilde{X}_n^\gamma(iT/n)| \leq \eta_{\varepsilon,g,h}^\gamma \cdot n^{-\gamma+\varepsilon} \quad \text{a.s.}$$

Proof Use Theorem 1 and localisation procedure

Example

Wright-Fisher-type diffusion:

$$dX(t) = (\kappa_1(1 - X(t)) - \kappa_2 X(t))\,dt + \sqrt{X(t)(1 - X(t))}\,dW(t)$$

If $\min\{\kappa_1, \kappa_2\} \geq 1/2$ and $X_0 \in (0,1)$, then

$$\mathbf{P}(X(t) \in (0,1) \;\text{ for all }\; t \geq 0) = 1$$

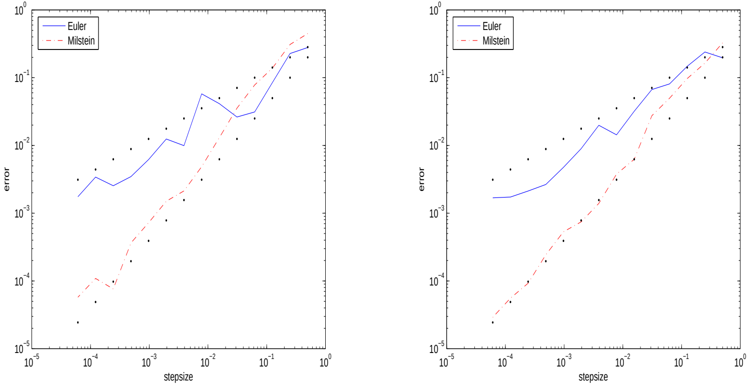
Choose new coefficients outside $[0,1]$, e.g.

auxiliary drift coefficient: $a(x) = \kappa_1(1-x) - \kappa_2 x, \quad x \notin [0,1]$

auxiliary diffusion coefficient: $b(x) = 0, \quad x \notin [0,1]$

Numerical Example III

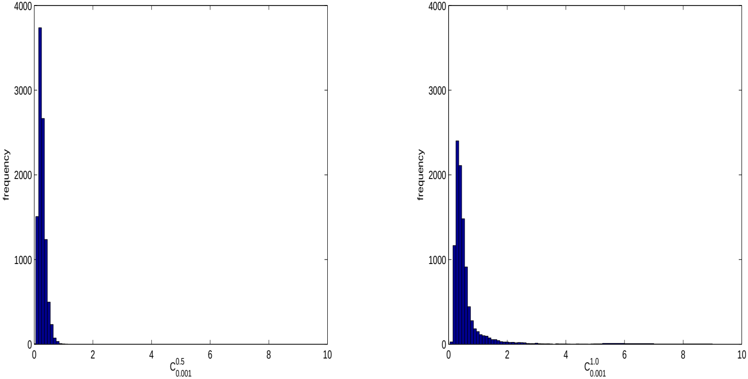
$$\kappa_1 = 0.5, \kappa_2 = 1, X(0) = 0.1, T = 1$$



pathwise maximum error vs. stepsize for two sample paths

Numerical Example III (cont'd)

$\kappa_1 = 0.5, \kappa_2 = 1, X(0) = 0.1, T = 1$



empirical distribution of $\eta^{0.5}_{0.001}$ and $\eta^{1.0}_{0.001}$ (sample size: $N = 10^4$)

Addendum: Rough Path Theory

Pathwise Theory (via rough differential equations) for Stratonovich-SDEs

$$dY(t) = a(Y(t))dt + \sum_{j=1}^{m} b^j(Y(t)) \circ dW^j(t), \qquad t \in [0, T]$$
$$Y(0) = y_0 \in \mathbb{R}^d$$

Lyons (1994); Lyons (1998); Friz, Victoir (2010); ...

Pathwise existence and uniqueness (in rough paths sense) of solution Y for smooth coefficients a, b

Addendum: Rough Path Theory

$$dY(t) = a(Y(t))dt + \sum_{j=1}^{m} b^j(Y(t)) \circ dW^j(t)$$

Pathwise existence and uniqueness for smooth a, b in particular

$$Y = \Psi_{a,b,y_0}(W, \mathbf{W^2})$$

where

- Itô-Lyons map Ψ_{a,b,y_0} locally Lipschitz in appropriate Hölder-type spaces
- $\mathbf{W^2}$ Lévy area associated to W, i.e.

$$\mathbf{W}^2_{st}(i,j) := \int_s^t (W_u^{(i)} - W_s^{(i)}) \circ dW_u^{(j)}, \; i,j = 1, \ldots, m, \; 0 \le s \le t \le T$$

Remarks

- For pathwise approximation of Itô-/Stratonovich-SDEs: no "new" results
- Smoothness of Itô-Lyons map crucial for numerics for SDEs driven by fractional Brownian motion (Deya, N, Tindel, 2011)

Summary

- Itô-Taylor scheme of mean-square order γ:
  pathwise order of convergence $\gamma - \varepsilon$ for all $\varepsilon > 0$
- No global Lipschitz conditions, no integrability assumptions
- Same pathwise convergence order also for modified Itô-Taylor schemes for SDEs on domains

(Some) References

(1) I. Gyöngy. *A note on Euler's approximations.* Potential Anal. 8, p. 205-216, 1998.

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(3) A. Jentzen, P. E. Kloeden and A. Neuenkirch. *Pathwise approximation of stochastic differential equations on domains: Higher order convergence rates without global Lipschitz coefficients.* Numer. Math. 112(1), p. 41-64, 2009.