Random Dynamical Systems, Bielefeld

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Metastability in the random field Curie-Weiss model

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Plan

- ▷ The RFCW model
- ▷ Equilibrium properties
- ▷ Glauber dynamics
- ▷ Main theorem
- ▷ Previous work
- ▷ Elements of proof
- ▷ Conclusions and outlook

Random Hamiltonian:

$$H_N(\sigma) \equiv -\frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right)^2 - \sum_{i=1}^{N} h_i\sigma_i.$$

h_i, $i \in \mathbb{N}$ are (bounded) i.i.d. random variables, $\sigma \in \{-1,1\}^N$.

Equilibrium properties: [see Amaro de Matos, Patrick, Zagrebnov (92), Külske (97)]

Gibbs measure: $\mu_{\beta,N}(\sigma) = \frac{2^{-N}e^{-\beta H_N(\sigma)}}{Z_{\beta,N}}$

Magnetization: $m_N(\sigma) \equiv \frac{1}{N}\sum_{i=1}^{N}\sigma_i$.

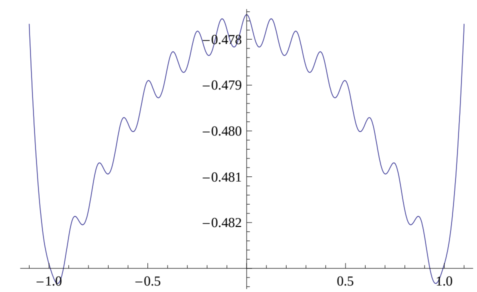
Induced measure: $\mathcal{Q}_{\beta,N} \equiv \mu_{\beta,N} \circ m_N^{-1}$. on the set $\Gamma_N \equiv \{-1, -1+2/N, \dots, +1\}$.

Using sharp large deviation estimates, one gets

$$Z_{\beta,N}\mathcal{Q}_{\beta,N}(m) = \sqrt{\frac{2I_N''(m)}{N\pi}} \exp\left\{-N\beta F_N(x)\right\}(1+o(1)),$$

where $F_N(x) \equiv \frac{1}{2}m^2 - \frac{1}{\beta}I_N(m)$ and $I_N(y)$ is the Legendre-Fenchel transform of

$$U_N(t) \equiv \frac{1}{N}\sum_{i\in\Lambda} \ln\cosh\left(t+\beta h_i\right)$$



Critical points: Solutions of $m^* = \frac{1}{N}\sum_{i\in\Lambda}\tanh(\beta(m^*+h_i))$.
Maxima if $\beta\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right) > 1$.
Moreover, at critical points,

$$Z_{\beta,N}\mathcal{Q}_{\beta,N}(z^*) \quad = \quad \frac{\exp\left\{\beta N\left(-\frac{1}{2}(z^*)^2 + \frac{1}{\beta N}\sum_{i\in\Lambda}\ln\cosh\left(\beta(z^*+h_i)\right)\right)\right\}}{\sqrt{\frac{N\pi}{2}\left(\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right)\right)}}(1+o(1))$$

We consider for definiteness discrete time Glauber dynamics with Metropolis transition probabilities

$$p_N(\sigma,\sigma') \equiv \frac{1}{N}\exp\left\{-\beta[H_N(\sigma') - H_N(\sigma)]_+\right\}$$

if σ and σ' differ on a single coordinate, and zero else.

We will be interested in transition times from a local minimum, m^*, to the set of “deeper” local minima,

$$M \equiv \{m : F_{\beta,N}(m) \le F_{\beta,N}(m^*)\}.$$

Set $S[M] = \{\sigma \in S_N : m_N(\sigma) \in M\}$.

We need to define probability measures on $S[m^*]$ by

$$\nu_{m^*,M}(\sigma) = \frac{\mu_{\beta,N}(\sigma)\mathbb{P}_\sigma\left[\tau_{S[M]} < \tau_{S[m^*]}\right]}{\sum_{\sigma\in S[m^*]}\mu_{\beta,N}(\sigma)\mathbb{P}_\sigma\left[\tau_{S[M]} < \tau_{S[m^*]}\right]}.$$

Main theorem

***Theorem* 1.** *Let m^* be a local minimum of $F_{\beta,N}$; let z^* be the critical point separating m^* from M.*

$$
\begin{aligned}
&\mathbb{E}_{\nu_{m^*}} \tau_{S[M]} \\
&= \exp\left\{\beta N\left(\tfrac{(z^*)^2-(m^*)^2}{2} - \tfrac{1}{\beta N}\textstyle\sum_{i\in\Lambda}\left[\ln\cosh\left(\beta(z^*+h_i)\right) - \ln\cosh\left(\beta(m^*+h_i)\right)\right]\right)\right\} \\
&\times \frac{2\pi N}{\beta|\hat{\gamma}_1|}\sqrt{\frac{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)-1}{1-\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(m^*+h)\right)\right)}}\left(1+o(1)\right),
\end{aligned}
$$

where $\hat{\gamma}_1$ is the unique negative solution of the equation

$$
\mathbb{E}_h\left[\frac{1-\tanh(\beta(z^*+h))}{\left[\beta\left(1+\tanh(\beta(z^*+h))\right)\right]^{-1}-\gamma}\right] = 1.
$$

Note that a naive approximation by a one-dimensional chain would give the same result except the wrong constant

$$
\gamma = \frac{1}{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)} - 1
$$

Previous work

The model was studied in

- ▷ F. den Hollander and P. dai Pra (JSP 1996) [large deviations, logarithmic asymptotics]
- ▷ P. Mathieu and P. Picco (JSP, 1998) [binary distribution; up to polynomial errors in N]
- ▷ A.B, M. Eckhoff, V. Gayrard, M. Klein (PTRF, 2001) [discrete distribution, up to multiplicative constants]

Both MP and BEGK made heavy use of exact mapping to finite-dimensional Markov chain!

The main goal of the present work was to show that potential theoretic methods allow to get sharp estimates (i.e. precise pre-factors of exponential rates) in spin systems at finite temperature when no symmetries are present. The RFCW model is the simplest model of this kind.

Elements of the proof: 1. Potential theory

Equilibrium potential for $A \cap B = \emptyset$, $-L = P - 1$ generator, solution of

$$(Lh_{B,A})(\sigma) = 0, \quad \sigma \notin A \cup B,$$

with boundary conditions

$$h_{B,A}(\sigma) = \begin{cases} 1, & \text{if } \sigma \in B \\ 0, & \text{if } \sigma \in A \end{cases}.$$

Equilibrium measure $e_{B,A}(\sigma) \equiv -(Lh_{B,A})(\sigma)$.

Capacity: $\sum_{\sigma \in B} \mu(\sigma) e_{B,A}(\sigma) \equiv \text{cap}(B, A)$.

Dirichlet form $\Phi_N(f) \equiv \frac{1}{2} \sum_{\sigma,\sigma' \in S_N} \mu(\sigma) p_N(\sigma, \sigma')[f(\sigma) - f(\sigma')]^2$.

Dirichlet principle: $\text{cap}(B, A) = \Phi(h_{B,A}) = \inf_{h \in \mathcal{H}_{B,A}} \Phi_N(h)$.

Probabilistic interpretation:

$$\mathbb{P}_\sigma[\tau_B < \tau_A] = \begin{cases} h_{B,A}(\sigma), & \text{if } \sigma \notin A \cup B \\ e_{B,A}(\sigma), & \text{if } \sigma \in A. \end{cases}$$

Elements of the proof: 1. Potential theory

Equilibrium potentials and equilibrium measures also determine the Green's function:

$$h_{B,A}(\sigma) = \sum_{\sigma' \in B} G_{S_N \setminus A}(\sigma, \sigma') e_{A,B}(\sigma')$$

Mean hitting times:

$$\sum_{\sigma \in B} \mu(\sigma) e_{A,B}(\sigma) \mathbb{E}_\sigma \tau_A = \sum_{\sigma' \in S_N} \mu(\sigma') h_{A,B}(\sigma'),$$

or

$$\sum_{\sigma \in B} \nu_{B,A}(\sigma) \mathbb{E}_\sigma \tau_A = \frac{1}{\text{cap}(B,A)} \sum_{\sigma' \in S_N} \mu(\sigma') h_{B,A}(\sigma').$$

Thus we need

- ▷ precise control of capacities and some
- ▷ rough control of equilibrium potential.

Elements of Proof 2: Coarse graining

$I_\ell, \ell \in \{1, \dots, n\}$: partition of the support of the distribution of the random field.

Random partition of the set $\Lambda \equiv \{1, \dots, N\}$

$$\Lambda_k \equiv \{i \in \Lambda : h_i \in I_k\}$$

Order parameters

$$\boldsymbol{m}_k(\sigma) \equiv \frac{1}{N} \sum_{i \in \Lambda_k} \sigma_i$$

$$H_N(\sigma) = -NE(\boldsymbol{m}(\sigma)) + \sum_{\ell=1}^{n} \sum_{i \in I_\ell} \sigma_i \tilde{h}_i$$

where $\tilde{h}_i = h_i - \bar{h}_\ell$, $i \in \Lambda_\ell$. Note $|\tilde{h}_i| \leq c/n$;

$$E(\boldsymbol{x}) \equiv \frac{1}{2} \left(\sum_{\ell=1}^{n} \boldsymbol{x}_\ell \right)^2 + \sum_{\ell=1}^{n} \bar{h}_\ell \boldsymbol{x}_\ell$$

Equilibrium distribution of the variables $\boldsymbol{m}[\sigma]$

$$\mu_{\beta,N}(\boldsymbol{m}(\sigma) = \boldsymbol{x}) \equiv \mathcal{Q}_{\beta,N}(\boldsymbol{x})$$

Elements of Proof 2: Coarse graining

Coarse grained Dirichlet form:

$$\widehat{\Phi}(g) \equiv \sum_{\boldsymbol{x},\boldsymbol{x}'\in\Gamma_N} \mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x}) r_N(\boldsymbol{x},\boldsymbol{x}') \left[g(\boldsymbol{x}) - g(\boldsymbol{x}')\right]^2$$

with

$$r_N(\boldsymbol{x},\boldsymbol{x}') \equiv \frac{1}{\mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x})} \sum_{\sigma:\boldsymbol{m}(\sigma)=\boldsymbol{x}} \mu_{\beta,N}[\omega](\sigma) \sum_{\sigma':\boldsymbol{m}(\sigma)=\boldsymbol{x}'} p(\sigma,\sigma').$$

The key step in the proof of both upper and lower bounds is to find a function that is almost harmonic in a small neighborhood of the relevant saddle point. This will be given by

$$h(\sigma) = g(\boldsymbol{m}(\sigma)) = f((\boldsymbol{v}, (\boldsymbol{z}^* - \boldsymbol{m}(\sigma))))$$

for suitable vector $\boldsymbol{v} \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}_+$

$$f(a) = \sqrt{\frac{\beta N \hat{\boldsymbol{\gamma}}_1^{(n)}}{2\pi}} \int_{-\infty}^{a} e^{-\beta N |\hat{\gamma}_1| u^2/2} du.$$

This yields a straightforward upper bound for capacities which will turn out to be the correct answer, as $n \uparrow \infty$!

Lower bounds use a variational principle from Berman and Konsowa [1990]:

Let $f : \mathcal{E} \to \mathbb{R}_+$ be a non-negative unit flow from $A \to B$, i.e. a function on edges such that

- ▷ $\sum_{a\in A} \sum_b f(a,b) = 1$
- ▷ for any a, $\sum_b f(b,a) = \sum_b f(a,b)$ (Kirchhoff's law).

Set $q^f(a,b) \equiv \frac{f(a,b)}{\sum_b f(a,b)}$, and let the initial distribution for $a \in A$ be $F(a) \equiv \sum_b f(a,b)$.

This defines a Markov chain on paths $\mathcal{X} : A \to B$, with law $\mathbb{P}^f$.

Theorem 2. *For any non-negative unit flow, f, one has that, for $\mathcal{X} = (a_0, a_1, \ldots, a_{|\mathcal{X}|})$,*

$$\textit{cap}(A,B) \geq \mathbb{E}^f_{\mathcal{X}} \left[\sum_{\ell=0}^{|\mathcal{X}|-1} \frac{f(a_\ell, a_{\ell+1})}{\mu(a_\ell) p(a_\ell, a_{\ell+1})} \right]^{-1}$$

Note: the variational principle is sharp, as equality is reached for the harmonic flow

$$f(a,b) = \frac{1}{\text{cap}(A,B)} \mu(a) p(a,b) \left[h^*(b) - h^*(a) \right]_+$$

Elements of proof: Construction of flow

Again, care has to be taken in the construction of the flow only near the saddle point.

Two scale construction:

- ▷ Construct mesoscopic flow on variables $\boldsymbol{m}$ from approximate harmonic function used in upper bound. This gives good lower bound in the mesoscopic Dirichlet form.
- ▷ Construct microscopic flow for each mesoscopic path.
- ▷ Use the magnetic field is almost constant and averaging that conductance of most mesoscopic paths give the same values as in mesoscopic Dirichlet function.

This yields upper lower bound that differs from upper bound only by factor $1 + O(1/n))$.

If $A = \{\sigma : \boldsymbol{m}_N(\sigma) = \boldsymbol{m}_1\}$, $B = \{\sigma : \boldsymbol{m}_N(\sigma) = \boldsymbol{m}_2\}$, and $\boldsymbol{z}^*$ is the essential saddle point connecting them, then

$$\text{cap}(A,B) = \mathcal{Q}_{\beta,N}(\boldsymbol{z}^*)\frac{\beta|\hat{\gamma}_1|}{2\pi N}\left(\prod_{\ell=1}^{n}\sqrt{r_\ell}\right)\left(\frac{\pi N}{2\beta}\right)^{n/2}\frac{1}{\sqrt{\prod_{j=1}^{n}|\hat{\gamma}_j|}}(1+O(\epsilon))$$

This can be re-written as:

***Theorem* 3.**

$$Z_{\beta,N}\textit{cap}(A,B) = \frac{\beta|\hat{\gamma}_1^{(n)}|}{2\pi N}\frac{\exp\left\{\beta N\left(-\frac{1}{2}\left(z^*\right)^2+\frac{1}{\beta N}\sum_{i\in\Lambda}\ln\cosh\left(\beta(z^*+h_i)\right)\right)\right\}(1+o(\epsilon))}{\sqrt{\beta\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right)-1}}.$$

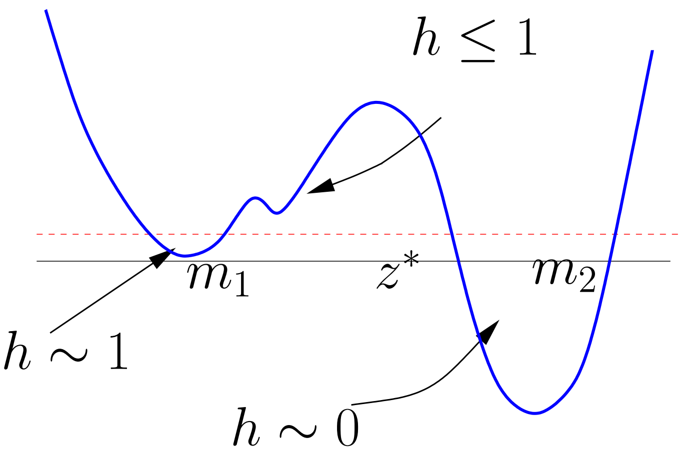
Final step in control of mean hitting times:
Compute

$$\sum_{\sigma} \mu_{\beta,N}(\sigma) h_{A,B}(\sigma) \sim \mathcal{Q}_{\beta,N}([\rho + m_1, m_1 - \rho])$$

This requires to show that: $h_{A,B}(\sigma) \sim 1$, if σ near A, and

$$h_{A,B}(\sigma) \leq \exp\{-\mathbb{N}(F_{\beta,N}(z^*) - F_{\beta,N}(m_N(\sigma)) - \delta)\}$$

if $F_{\beta,N}(m_N(\sigma) \leq F_{b,N}(m_1)$.



Can be done using super-harmonic barrier function.

Conclusions

Nice features:

- ▷ We have obtained sharp estimates on exit times in a model without symmetry when entropy is relevant.
- ▷ Avoided use of renewal estimates for harmonic functions.

Future challenges:

- ▷ Control of small eigenvalues!
- ▷ Beyond mean field models: Kac model should be next candidate.
- ▷ Full scale Glauber or Kawasaki dynamics for lattice Ising!

Work on all this is in progress with Alessandra Bianchi, Frank den Hollander, Dima Ioffe, and Cristian Spitoni