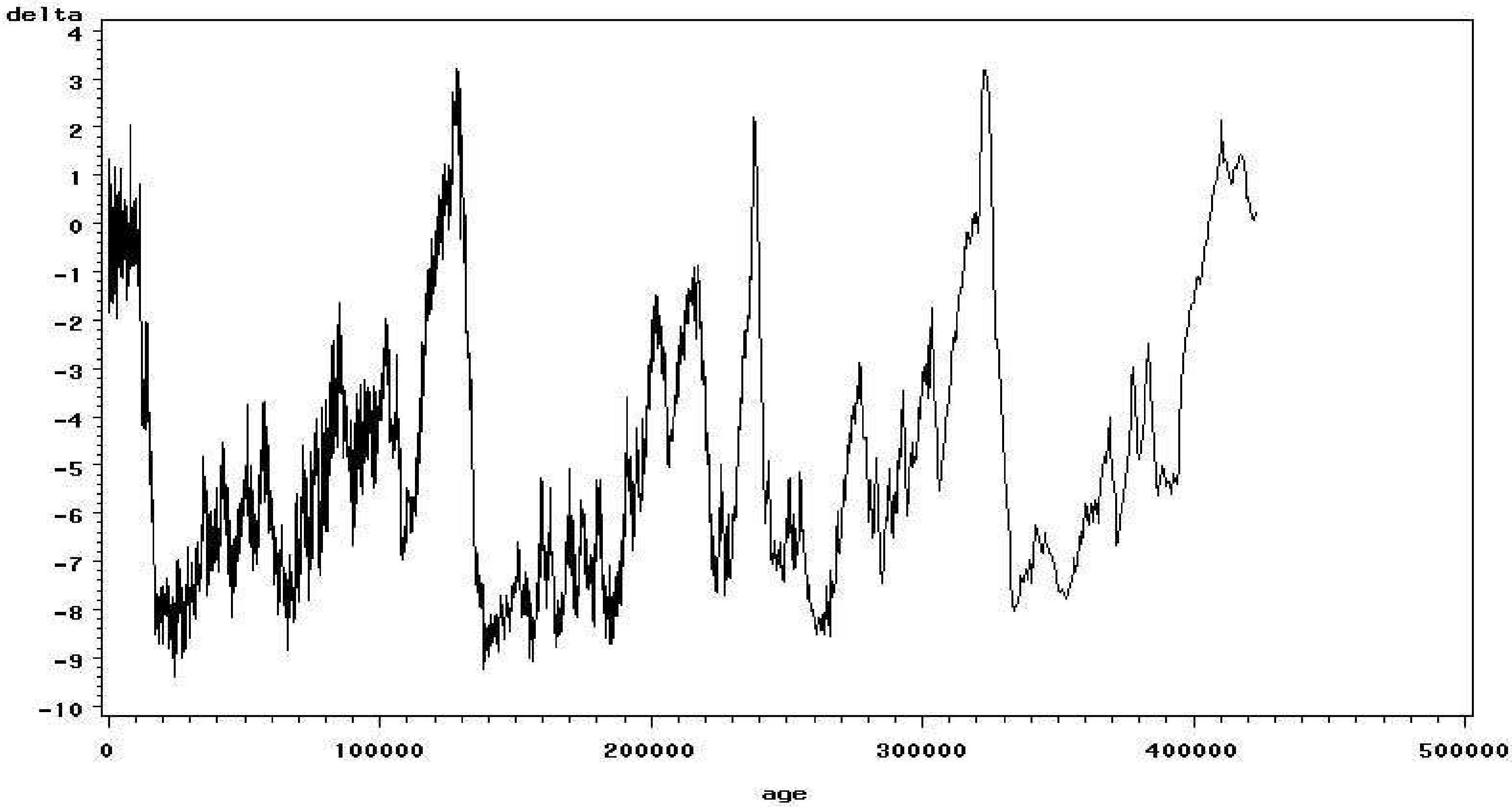
Dynamical systems with Gaussian and Levy noise: analytical and stochastic approaches

Noise is often considered as some disturbing component of the system. In particular physical situations, noise becomes essential: it creates new events (example in climatology: qualitative explanation of the almost periodic recurrence of cold and warm ages in paleoclimatic data).



Large deviations and a Kramers' type law for self-stabilizing diffusions

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The exit problem is the basic tool for the analysis of almost periodic stochastic paths.

- Introduction: large deviations and exit problem for classical diffusions.
- Existence of self-stabilizing diffusions
- Large deviations for self-stabilizing diffusions
- Exit problem in some convex landscape for self-stabilizing diffusions

Introduction: large deviations for classical diffusions

Let us consider the solution of the stochastic differential equation

$$dY_t^\varepsilon = V(Y_t^\varepsilon)dt + \sqrt{\varepsilon}dW_t, \;\; Y_t^\varepsilon \in \mathbb{R}^d, \;\; Y_0^\varepsilon = y \in D$$

in some bounded domain D. The drift term V is locally Lipschitz and W is some d-dimensional Brownian motion.

➥ Exit problem : it consists in describing both the time needed by the solution Y^ε in order to exit from the bounded domain D and the distribution of the exit point on the boundary. In this dynamical system perturbed by some small noise, we shall focus our attention to the asymptotic behaviour of the exit time as the noise intensity becomes small. Since the diffusion stays close to the solution of the deterministic system

$$dY_t^0 = V(Y_t^0)dt, \;\; Y_0^0 = y$$

on fix time intervals, we shall assume that Y_t^0 doesn't exit from the domain D and converges, as time elapses, towards some stable point $y_{stable} \in D$.

Let us define the rate function

$$I_T(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}_t - V(\varphi_t)\|^2\,dt & \text{if } \varphi \in H_y^1([0,T]) \\ +\infty & \text{otherwise.} \end{cases}$$

Here $H_y^1([0,T])$ is the Cameron-Martin space of absolutely continuous functions on the time interval $[0,T]$ starting in y that possess square integrable derivatives.

Theorem 1 (Freidlin and Wentzell): *The diffusion Y^ε satisfies some large deviations principle with good rate function I_T.* In other words,

$$\mathbb{P}(Y^\varepsilon \in \Gamma) \asymp \exp\left(-\frac{1}{\varepsilon}\inf_{\varphi\in\Gamma} I_T(\varphi)\right), \quad \text{where } \Gamma \subset \mathcal{C}([0,T]).$$

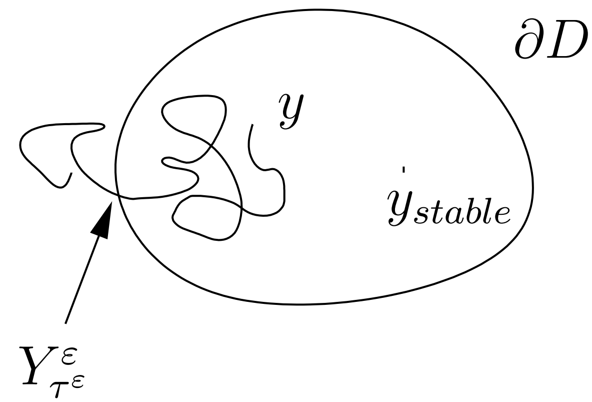
Let us describe the minimal *energy* the diffusion Y^ε needs to exit from the domain D. We define the *quasi-potential*

$$Q(x,z) = \inf\{I_T(\varphi) :\ \varphi \in \mathcal{C}([0,T]),\ \varphi(0) = x,\ \varphi(T) = z,\ T > 0\}.$$

The minimal cost is $\overline{Q} = \inf_{z\in\partial D} Q(y_{stable}, z)$.

The most important result is the precise evaluation of the exponential growth rate for the exit time as the intensity of the stochastic perturbation becomes small $\varepsilon \to 0$ (Kramers' time).

$$\tau^\varepsilon = \inf\{t > 0,\ Y_t^\varepsilon \notin D\}.$$



Theorem 2 (Freidlin-Wentzell): $\forall y \in D$ *and* $\delta > 0$

$$\lim_{\varepsilon \to 0} \mathbb{P}_y(e^{(\overline{Q}+\delta)/\varepsilon} > \tau^\varepsilon > e^{(\overline{Q}-\delta)/\varepsilon}) = 1$$

Moreover $\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{E}_y[\tau^\varepsilon] = \overline{Q}$*.*

Furthermore we obtain some informations about the distribution of the exit location: if $N \subset \partial D$ is some closed subset of the boundary and $\inf_{z \in N} Q(y_{stable}, z) > \overline{Q}$, then for all $y \in D$, $\lim_{\varepsilon \to 0} \mathbb{P}_y(Y^\varepsilon_{\tau^\varepsilon} \in N) = 0$.

In particular if there exists some point $z^* \in \partial D$ which satisfies $Q(y_{stable}, z^*) < Q(y_{stable}, z)$ for all $z \neq z^*$, $z \in \partial D$ then

$$\forall \delta > 0,\ \forall y \in D,\ \lim_{\varepsilon \to 0} \mathbb{P}_y(\|Y^\varepsilon_{\tau^\varepsilon} - z^*\| < \delta) = 1.$$

Remark: In the particular gradient case (the drift term is the gradient of some potential $V = -\nabla U$) the quasipotential can be explicitly computed
(under some hypotheses: $U(z) > U(y_{stable})$ and $\nabla U(z) \neq 0$ for all $z \neq y_{stable}$).

$$Q(y_{stable}, z) = 2(U(z) - U(y_{stable})).$$

Hence

$$\overline{Q} = \inf_{z \in \partial D} 2(U(z) - U(y_{stable})).$$

sketch of proof: For any function $\varphi \in \mathcal{C}^1$, we get

$$U(\varphi_T) - U(\varphi_0) = \int_0^T \langle \nabla U(\varphi_s), \dot{\varphi}_s \rangle ds$$

Hence
$$\begin{aligned} I_T(\varphi) &= \frac{1}{2}\int_0^T \|\dot{\varphi}_s - \nabla U(\varphi_s)\|^2 ds + 2\int_0^T \langle \dot{\varphi}_s, \nabla U(\varphi_s) \rangle ds \\ &\geq 2(U(\varphi_T) - U(\varphi_0)). \end{aligned}$$

In order to get the upper-bound, we choose some particular path $\hat{\varphi}$: the unique solution of $\dot{\varphi}_s = \nabla U(\varphi_s)$ with $\varphi_T = z$. Then it suffices to use both the convergence $\hat{\varphi}_0 \longrightarrow y_{stable}$ as $T \longrightarrow \infty$ and the definition of the cost

$$Q(y_{stable}, z) = \inf\{I_T(\varphi) : \ \varphi \in \mathcal{C}([0,T]),\ \varphi_0 = y_{stable},\ \varphi_T = z,\ T > 0\}. \qquad \square$$

In the particular gradient case, there exist precise results introduced by physicists (Kramers was the first one) and developped by several mathematicians (among those Bovier, Eckhoff, Gayrard, Klein 2005) who presented equivalents for the exit time.

Furthermore Day (1983) proved that $\tau^\varepsilon/\mathbb{E}[\tau^\varepsilon] \sim \mathcal{E}(1)$ where $\mathcal{E}(1)$ is some exponential law of parameter 1. More precisely, for any $y \in K \subset D$ where K is some compact set, there exists some constant $\delta > 0$ such that

$$\mathbb{P}_y(\tau^\varepsilon > t) = e^{-\lambda^\varepsilon t}(1 + \mathcal{O}_K(e^{-\delta/\varepsilon}))$$

for all $t \geq 0$ and ε small enough. λ^ε represents the principal eigenvalue of the following operator:

$$\mathcal{L}^\varepsilon u = \frac{\varepsilon}{2}\Delta u + \langle V, \nabla u \rangle.$$

Moreover $\lambda^\varepsilon \mathbb{E}[\tau^\varepsilon] \to 1$ uniformly on each compact subset $K \subset D$. In fact $u(t,y) = \mathbb{P}_y(\tau^\varepsilon > t)$ is solution of the PDE

$$\frac{\partial u}{\partial t} = \mathcal{L}^\varepsilon u \text{ in } D \times (0,\infty), \quad u(t,y) = 0 \text{ for } y \in \partial D,\ t > 0, \quad \text{and } u(0,y) = 1 \text{ for } y \in D.$$

Self-stabilizing diffusions

Let us consider the SDE

$$dX_t^\varepsilon = V(X_t^\varepsilon)dt - \int_{\mathbb{R}^d} \Phi(X_t^\varepsilon - x)u_t^\varepsilon(x)dx\, dt + \sqrt{\varepsilon}dW_t, \quad X_0^\varepsilon = x, \qquad (1)$$

where $u_t^\varepsilon(x)dx$ is the law of X_t^ε. The so-called *self-stabilization* perturbation gives to the diffusion more inertia and so stabilizes it in some particular state space domains. These SDE are obtained as limit of interacting particle systems, as the number of particles tends to infinity.

$$dX_t^{i,N} = V(X_t^{i,N})dt - \frac{1}{N}\sum_{j=1}^{N} \Phi(X_t^{i,N} - X_t^{j,N})dt + \sqrt{\varepsilon}dW_t^i, \quad i = 1, \ldots, N.$$

Assumptions: V and Φ are locally lipschitz, there exists some non decreasing function ϕ with $\phi(0) = 0$ s. t. $\Phi(x) = \frac{x}{\|x\|}\phi(\|x\|)$. Φ increases at most polynomially and $\langle h, DV(x)h\rangle \leq -K_V$ for $\|h\| = 1$ and $\|x\| \geq R_0$.

➥ **Theorem:** There exists some unique strong solution to the equation (1).

More precisely: if the initial condition satisfies $\mathbb{E}[\|X_0\|^{(r+1)^2}] < \infty$ (r describes the growth of the interaction Φ) then there exists some unique strong solution X^ε which satisfies

$$\sup_{t\geq 0} \mathbb{E}(\|X_t^\varepsilon\|^p) < \infty \quad \text{for } p \in [1,(r+1)^2].$$

Sketch of proof : (Benachour, Roynette, Talay, Vallois 1998)

$$\Lambda_T^\nu := \{b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d |\ \|b\|_T < \nu, x \mapsto b(t,x) \quad \text{is locally Lipschitz, uniformly with respect to } t \text{ and satisfies the dissipativity property} \quad \langle x-y, b(t,x)-b(t,y)\rangle \geq 0, \quad x,y \in \mathbb{R}^d.\}.$$

The norm used is $\|b\|_T := \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{\|b(t,x)\|}{1+\|x\|^{2q}}, \quad 2q > r.$

On this function space, we define some map Γ: which associates the function $b \in \Lambda_T^\nu$ with $\Gamma b(t,x) = \mathbb{E}[\Phi(x - X_t^b)]$ where X^b is solution to

$$dX_t = V(X_t) - b(t,X_t)dt + \sqrt{\varepsilon} dW_t.$$

For T small enough, Γ defines some contracting map with Lipschitz constant $1/2$. This implies the existence and uniqueness of the solution which can be extended on $t \geq 0$. $\square$

Large deviations for self-stabilizing diffusions

Let $\psi_t(x)$ the solution of the dynamical system $\dot{\psi}_t = V(\psi_t), \quad \psi_0 = x$.

The rate function is defined by the following expression:

$$I_T(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}_t - V(\varphi_t) + \Phi(\varphi_t - \psi_t(x))\|^2 \, dt, & \text{if } \varphi \in H_x^1, \\ \infty, & \text{otherwise} \end{cases}$$

Theorem: *The unique solution of the SDE*

$$dX_t^\varepsilon = V(X_t^\varepsilon)dt - \int_{\mathbb{R}^d} \Phi(X_t^\varepsilon - x)u_t^\varepsilon(x)dx\, dt + \sqrt{\varepsilon}dW_t,\ X_0^\varepsilon = x,$$

satisfies a large deviations principle with good rate function I_T.

Remark: The drift term can be written as some expectation: $b^\varepsilon(t,y) = V(y) - r^{\varepsilon,x}(t,y)$ with $r^{\varepsilon,x}(t,y) = \mathbb{E}[\Phi(y - X_t^\varepsilon)]$.

As ε tends to 0, the drift term tends to $b^0(t,y) = V(y) - \Phi(y - \psi_t(x))$.

Exit problem for self-stabilizing diffusions

Let D be some compact domain which contains some unique stable equilibrium point x_{stable}. We assume that all paths, solutions of the deterministic system, $\dot{\psi}_t = V(\psi_t), \quad \psi_0 = x$ are attracted towards x_{stable}. Let I_T^∞ the rate function defined by

$$I_T^\infty = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}_t - V(\varphi_t) + \Phi(\varphi_t - x_{stable})\|^2 \, dt & \text{if } \varphi \in H_y^1([0,T]) \\ +\infty & \text{otherwise.} \end{cases}$$

$Q^\infty(x,z)$ is the associated quasi-potential and $\overline{Q}^\infty = \inf_{z\in\partial D} Q^\infty(x_{stable}, z)$.

Theorem : For any $x \in D$ and $\delta > 0$,

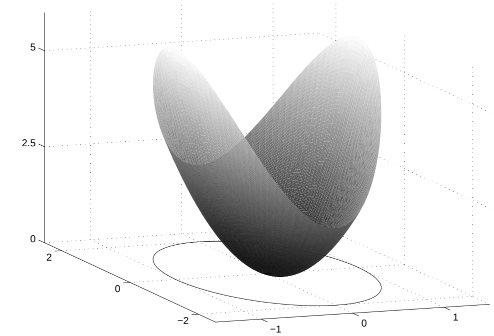
$$\lim_{\varepsilon\to 0} \mathbb{P}_x(e^{(\overline{Q}^\infty+\delta)/\varepsilon} > \tau^\varepsilon > e^{(\overline{Q}^\infty-\delta)/\varepsilon}) = 1$$

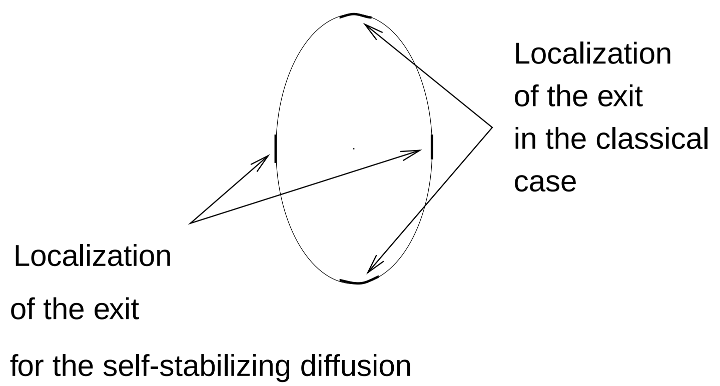
and $\lim_{\varepsilon\to 0} \varepsilon \ln \mathbb{E}_x[\tau^\varepsilon] = \overline{Q}^\infty$.

If $N \subset \partial D$ is some closed subset such that $\inf_{z\in N} Q^\infty(x_{stable}, z) > \overline{Q}^\infty$, then the probability that N contains the exit point is asymptotically negligible: $\lim_{\varepsilon\to 0} \mathbb{P}_x(X^\varepsilon_{\tau^\varepsilon} \in N) = 0$.

Example in $\mathbb{R}^2$

Let us consider V the gradient of the following potential: $V = -\nabla U$, with $U(x,y) = 6x^2 + y^2/2$. The exit problem is studied in some elliptic domain defined by $D = \{(x,y) \in \mathbb{R}^2 : \; x^2 + y^2/4 = 1\}$. The origin $x_{stable} = 0$ is the unique stable equilibrium point.





Without self-stabilization, Freidlin and Wentzell's theory implies

$$\mathbb{E}_0[\tau_D^\varepsilon] \asymp e^{4/\varepsilon},$$

with self-stabilization: $\Phi = \nabla \mathcal{A}$ where $\mathcal{A}(x,y) = 2x^2 + 2y^2$, we get

$$\mathbb{E}_0[\tau_D^\varepsilon] \asymp e^{16/\varepsilon}$$

Moreover the locations of the exits are totally different in these two particular cases.

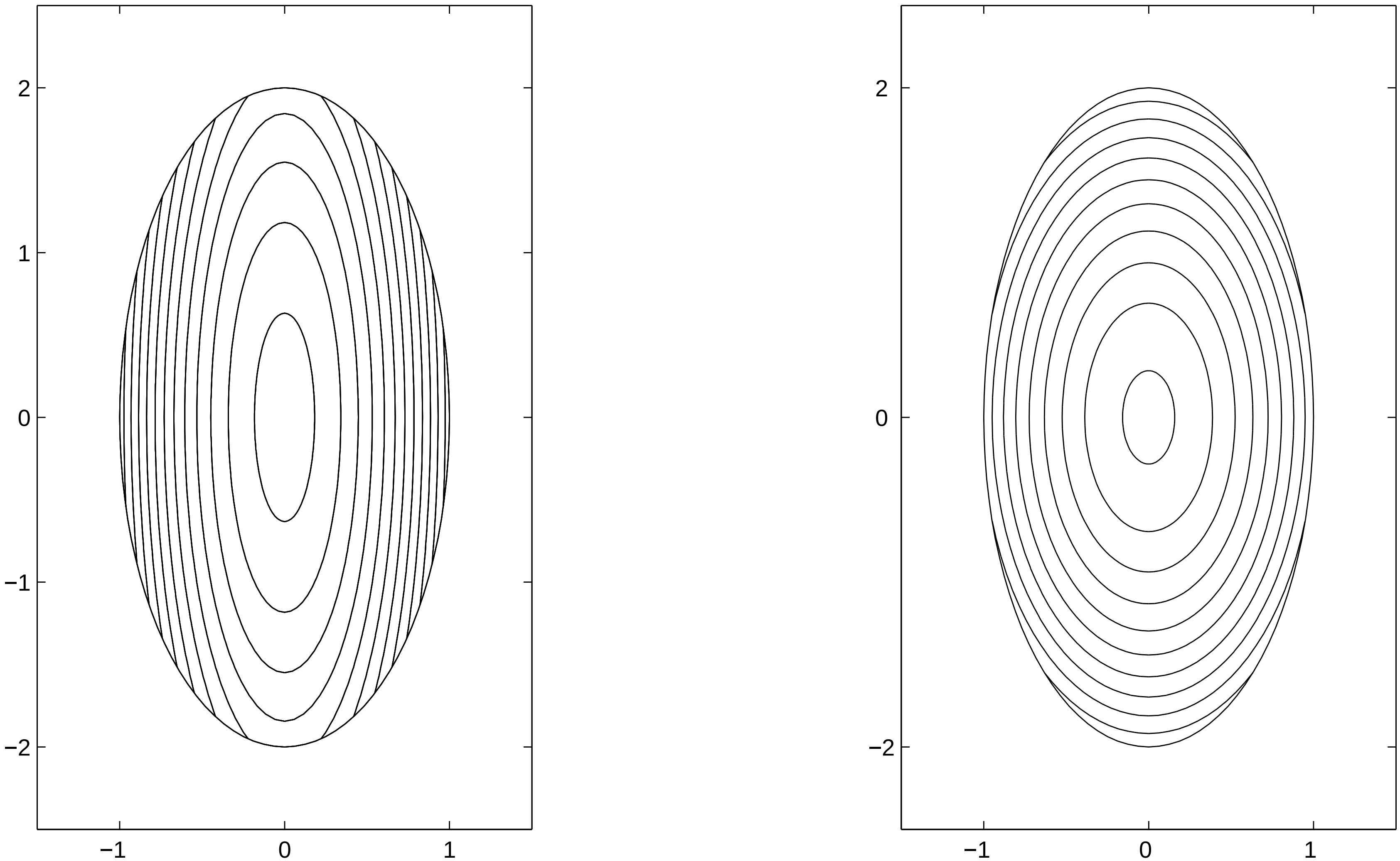


Figure 1: Level sets without (resp. with) interaction