Random attractors and the preservation of synchronization in the presence of noise*

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Deterministic case

Consider the ordinary differential equations in $\mathbb{R}^d$

$$\frac{dx}{dt} = f(x), \qquad \frac{dy}{dt} = g(y),$$

- f, g regular and satisfy the one-sided Lipschitz conditions

$$\langle x_1 - x_2, f(x_1) - f(x_2) \rangle \leq -L|x_1 - x_2|^2,$$

$$\langle y_1 - y_2, g(y_1) - g(y_2) \rangle \leq -L|y_1 - y_2|^2,$$

on $\mathbb{R}^d$ for some $L > 0$,

$\Rightarrow$ globally asymptotically stable equilibria $\bar{x}$ and $\bar{y}$.

Now consider the dissipatively coupled system

$$\frac{dx}{dt} = f(x) + \nu(y - x), \qquad \frac{dy}{dt} = g(y) + \nu(x - y)$$

with $\nu > 0$.

$\Rightarrow$ globally asymptotically stable equilibrium $(\bar{x}^\nu, \bar{y}^\nu)$.

$(\bar{x}^\nu, \bar{y}^\nu) \longrightarrow (\bar{z}, \bar{z})$ as $\nu \longrightarrow \infty$, where $\bar{z}$ is the unique globally asymptotically stable equilibrium of the <u>averaged</u> system

$$\frac{dz}{dt} = \frac{1}{2}\left(f(z) + g(z)\right).$$

This is known as <u>synchronization</u>

Stochastic case

What is the effect of environmental noise on synchronization?

Coupled Ito stochastic differential equations with additive noise

$$\begin{aligned} dX_t &= (f(X_t) + \nu(Y_t - X_t))\, dt + \alpha\, dW_t^1, \\ dY_t &= (g(Y_t) + \nu(X_t - Y_t))\, dt + \beta\, dW_t^2 \end{aligned}$$

where W_t^1, W_t^2 are independent two-sided scalar Wiener processes and α, $\beta \in \mathbb{R}^d$ are constant vectors.

$\Rightarrow$ $\exists$ unique stochastic stationary solution $(\bar{X}_t^\nu, \bar{Y}_t^\nu)$, which is pathwise globally asymptotically stable.

Moreover on finite time intervals $[T_1, T_2]$ of $\mathbb{R}$

$$(\bar{X}_t^\nu, \bar{Y}_t^\nu) \to (\bar{Z}_t^\infty, \bar{Z}_t^\infty) \quad \text{as} \quad \nu \to \infty \qquad \underline{\text{pathwise}}$$

where $\bar{Z}_t^\infty$ is the unique pathwise globally asymptotically stable stationary solution of the averaged SDE

$$dZ_t = \frac{f(Z_t) + g(Z_t)}{2}\, dt + \frac{\alpha}{2}\, dW_t^1 + \frac{\beta}{2}\, dW_t^2.$$

[*Caraballo & Kloeden*, Proc. Roy. Soc. London (2005)]

Recall:

- The solutions of Ito stochastic differential equations are pathwise continuous, but not differentiable.

- Ito SDEs are really stochastic integral equations with stochastic integrals defined in the mean-square or L_2 sense.

How do we apply the Lipschitz properties to obtain pathwise estimates?

A technical detour : Consider the Ito SDE

$$dX_t = f(X_t)\,dt + \alpha\,dW_t$$

where f satisfies the one-sided Lipschitz condition.

i.e., the stochastic integral equation

$$X_t = X_{t_0} + \int_{t_0}^{t} f(X_s)\,ds + \alpha \int_{t_0}^{t} dW_t$$

The difference of any two solutions satisfies pathwise

$$X_t^1 - X_t^2 = X_{t_0}^1 - X_{t_0}^2 + \int_{t_0}^{t} \underbrace{\left[f(X_s^1) - f(X_s^2)\right]}_{\text{continuous paths}} ds$$

Fundamental theorem of calculus $\Rightarrow$ $X_t^1 - X_t^2$ pathwise differentiable.

$$\frac{d}{dt}\left[X_t^1 - X_t^2\right] = f(X_t^1) - f(X_t^2) \qquad \text{pathwise}$$

- Apply the one-sided Lipschitz condition

$$\frac{d}{dt}\left|X_t^1 - X_t^2\right|^2 = 2\left\langle X_t^1 - X_t^2, f(X_t^1) - f(X_t^2)\right\rangle \leq -2L\left|X_t^1 - X_t^2\right|^2$$

$$\Rightarrow \qquad \left|X_t^1 - X_t^2\right|^2 \leq \left|X_{t_0}^1 - X_{t_0}^2\right|^2 e^{-2L(t-t_0)} \to 0 \qquad \text{as} \quad t \to \infty$$

i.e. all solutions converge pathwise together — but to what?

Special case : Ito SDE with linear drift $f(x) = -x$

$$dX_t = -X_t\,dt + \alpha\,dW_t$$

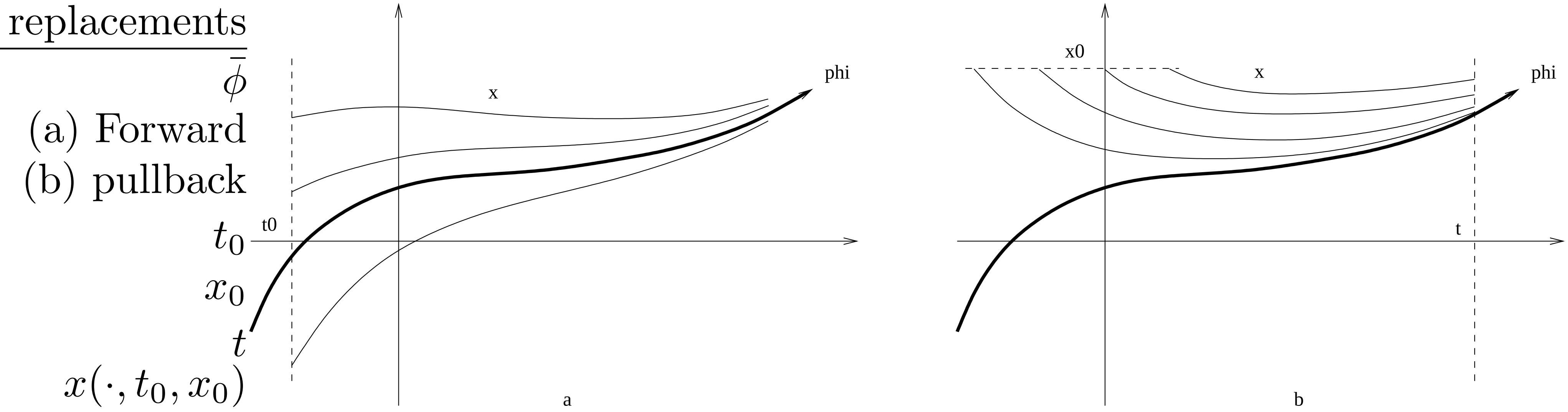
explicit solution

$$X_t = X_{t_0} e^{-(t-t_0)} + \alpha e^{-t} \int_{t_0}^{t} e^s\,dW_s$$

The forward limit as $t \to \infty$ does not exist — moving target!

But the pullback limit as $t_0 \to -\infty$ with t fixed does exist:

$$\lim_{t_0 \to -\infty} X_t = \bar{O}_t := \alpha e^{-t} \int_{-\infty}^{t} e^s\,dW_s \qquad \text{(pathwise)}$$



The Ornstein-Uhlenbeck stochastic stationary process $\bar{O}_t$ is a solution of the linear SDE and all other solutions converge pathwise to it in the forward sense

$$\left| X_t - \bar{O}_t \right| \to 0 \qquad \text{as} \quad t \to \infty \qquad \text{(pathwise)}$$

Random dynamical systems : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (X, d_X) a metric space.

A random dynamical system (θ, ϕ) on $\Omega \times X$ consists of

- a metric dynamical system θ on Ω, which models the noise,

- a cocycle mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$, which represents the dynamics on the state space X and satisfies

1).	$\phi(0, \omega, x_0) = x_0$	(initial condition)
2).	$\phi(s + t, \omega, x_0) = \phi(s, \theta_t\omega, \phi(t, \omega, x_0))$	(cocycle property)
3).	$(t, x_0) \mapsto \phi(t, \omega, x_0)$ is continuous	(continuity)
4).	$\omega \mapsto \phi(t, \omega, x_0)$ is $\mathcal{F}$-measurable	(measurability)

for all s, $t \geq 0$, $x_0 \in X$ and $\omega \in \Omega$.

Random attractors

A random attractor is a family of nonempty measurable compact subsets of X

$$\widehat{A} = \{A(\omega) : \omega \in \Omega\}$$

which is

- ϕ-invariant $\quad \phi(t, \omega, A(\omega)) = A(\theta_t \omega) \qquad$ for all $t \geq 0$,

- pathwise pullback attracting in the sense that

$$\operatorname{dist}_X \left(\phi\left(t, \theta_{-t}\omega, D(\theta_{-t}\omega)\right), A(\omega)\right) \to 0 \qquad \text{for} \quad t \to +\infty$$

for all suitable families $\widehat{D} = \{D(\omega) : \omega \in \Omega\}$ of nonempty measurable bounded subsets of X.

Theorem (Crauel, Flandoli, Schmalfuß etc)

Let (θ, ϕ) be an RDS on $\Omega \times X$ such that $\phi(t, \omega, \cdot) : X \to X$ is a <u>*compact*</u> *operator for each fixed $t > 0$ and $\omega \in \Omega$.*

If there exists a <u>*pullback absorbing family*</u> $\widehat{B} = \{B(\omega) : \omega \in \Omega\}$ *of nonempty closed and bounded measurable subsets of X, i.e. there exists a $T_{\widehat{D},\omega} \geq 0$ such that*

$$\phi\left(t, \theta_{-t}\omega, D(\theta_{-t}\omega)\right) \subset B(\omega) \qquad \textit{for all} \quad t \geq T_{\widehat{D},\omega}$$

for all $\widehat{D} = \{D(\omega) : \omega \in \Omega\}$ in a given attracting universe.

<u>*Then*</u> *the RDS $\Omega \times X$ has a* <u>*random attractor*</u> $\widehat{A}$ *with component subsets given by*

$$A(\omega) = \bigcap_{s>0} \overline{\bigcup_{t \geq s} \phi\left(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\right)} \qquad \textit{for each} \quad \omega \in \Omega.$$

General case again

Substract the integral version of the linear SDE for $\bar{O}_t$ from the integral version of the nonlinear SDE

$$dX_t = f(X_t)\,dt + \alpha\,dW_t$$

to obtain

$$X_t - \bar{O}_t = X_{t_0} - \bar{O}_{t_0} + \int_{t_0}^{t} \left[f(X_s) + \bar{O}_s \right] \, ds$$

$\Rightarrow \quad V_t := X_t - \bar{O}_t$ is pathwise differentiable and satisfies the pathwise ODE

$$\frac{d}{dt} V_t = f(V_t + \bar{O}_t) + \bar{O}_t \qquad \text{(pathwise)}$$

- Apply the one-sided Lipschitz condition pathwise to

$$\frac{d}{dt}\left[X_t - \bar{O}_t\right] = \left[f(X_t) - f(\bar{O}_t)\right] + \left[f(\bar{O}_t) + \bar{O}_t\right] \qquad \text{(pathwise)}$$

  to obtain the pathwise estimate

$$|V_t|^2 \leq |V_{t_0}|^2 \, e^{-L(t-t_0)} + \frac{2}{L} e^{-Lt} \int_{t_0}^{t} e^{Ls} \left(|f(\bar{O}_s)|^2 + |\bar{O}_s|^2\right) ds$$

- Take pathwise pullback convergence as $t_0 \to -\infty$ to obtain

$$|X_t - \bar{O}_t| \;\leq\; \bar{R}_t := 1 + \frac{2}{L} e^{-Lt} \int_{-\infty}^{t} e^{Ls} \left(|f(\bar{O}_s)|^2 + |\bar{O}_s|^2\right) ds$$

  for $t \geq T$ depending on suitable bounded sets of initial values.

- i.e., there exists a family of compact pullback absorbing balls centered on $\bar{O}_t$ with random radius $\bar{R}_t$.

- Dynamical systems limit set ideas
  $\Rightarrow$ there exists a compact setvalued stochastic process A_t inside these absorbing balls which pathwise pullback attracts the solutions.

- BUT the solutions converge together pathwise in forwards sense, so the sets A_t are in fact all singleton sets

$\Rightarrow$ $\exists$ stochastic stationary solution $\bar{X}_t$.

General Principles

- All regular Ito SDE in $\mathbb{R}^d$ can be transformed into pathwise ODE

[*Imkeller & Schmalfuß* (2001), *Imkeller & Lederer* (2001,2002)]

- and generate random dynamical systems

$\Rightarrow$ **pathwise theory and numerics for Ito SDE**

- Pullback convergence enables us to construct moving targets.

- Stochastic stationary solutions are a simple singleton set version of more general random attractors

⇒ **theory of random dynamical systems**

e.g., Ludwig Arnold (Bremen)

- parallel theory of **deterministic skew product flows**

e.g., almost periodic ODE : George Sell (Minneapolis)

⇒ **A theory of nonautonomous dynamical systems**

e.g., pullback attractors

Effects of discretization on synchronization

Numerical Ornstein-Uhlenbeck process

For the linear SDE with additive noise,

$$dX_t = -X_t\, dt + \alpha\, dW_t,$$

the drift-implicit Euler-Maruyama scheme with constant step size $h > 0$ is

$$X_{n+1} = X_n - hX_{n+1} + \alpha \Delta W_n, \qquad n = n_0, n_0 + 1, \ldots,$$

which simplifies algebraically to

$$X_{n+1} = \frac{1}{1+h} X_n + \frac{\alpha}{1+h}\, \Delta W_n,$$

Here the $\Delta W_n = W_{h(n+1)} - W_{hn}$ are mutually independent and $N(0, h)$ distributed

It follows that

$$X_n = \frac{1}{(1+h)^{n-n_0}} X_{n_0} + \frac{\alpha}{1+h} \sum_{j=n_0}^{n-1} \frac{1}{(1+h)^{n-1-j}} \Delta W_j$$

and the pathwise pullback limit, i.e. with n fixed and $n_0 \to -\infty$, gives the discrete time numerical Ornstein-Uhlenbeck process

$$\widehat{O}_n^{(h)} := \frac{\alpha}{1+h} \sum_{j=-\infty}^{n-1} \frac{1}{(1+h)^{n-1-j}} \Delta W_j, \qquad n \in \mathbb{Z}. \tag{1}$$

which is an entire solution of the numerical scheme and a discrete time stochastic stationary process.

One can show that it converges pathwise to the continuous time Ornstein-Uhlenbeck process in the sense that

$$\widehat{O}_0^{(h)} \to \widehat{O}_0 \qquad \text{as} \quad h \to 0.$$

Discretization of an isolated stochastic system

Consider the nonlinear SDE in $\mathbb{R}^d$with additive noise,

$$dX_t = f(X_t)\,dt + \alpha\,dW_t,$$

where the drift coefficient f is continuously differentiable and satisfies the one-sided dissipative Lipschitz condition with constant L.

The drift-implicit Euler-Maruyama scheme with constant step size $h > 0$ applied to this SDE is

$$X_{n+1} = X_n + hf(X_{n+1}) + \alpha\Delta W_n,$$

which is, in general, an implicit algebraic equation and must be solved numerically for X_{n+1} for each n.

The difference of any two solutions

$$X_{n+1} - X'_{n+1} = (X_n - X'_n) + h\left(f(X_{n+1}) - f(X'_{n+1})\right),$$

does not contain a driving noise term. Then

$$\begin{aligned}
\left|X_{n+1} - X'_{n+1}\right|^2 &= \left\langle X_{n+1} - X'_{n+1}, X_n - X'_n \right\rangle \\
&\quad + h\left\langle X_{n+1} - X'_{n+1}, f(X_{n+1}) - f(X'_{n+1}) \right\rangle \\
&\leq \left|X_{n+1} - X'_{n+1}\right| \left|X_n - X'_n\right| - hL\left|X_{n+1} - X'_{n+1}\right|^2,
\end{aligned}$$

$$\Rightarrow \qquad \left|X_{n+1} - X'_{n+1}\right| \leq \frac{1}{1+Lh}\left|X_n - X'_n\right|,$$

$$\Rightarrow \qquad \left|X_n - X'_n\right| \leq \frac{1}{(1+Lh)^n}\left|X_0 - X'_0\right| \to 0 \quad \text{as } n \to \infty.$$

i.e. all numerical solutions converge pathwise to each other forward in time.

Change variables to $U_n := X_n - \widehat{O}_n^{(h)}$, where $\widehat{O}_n^{(h)}$ is the numerical Ornstein-Uhlenbeck process, to obtain the numerical scheme

$$U_{n+1} = U_n + hf\left(U_{n+1} + \widehat{O}_{n+1}^{(h)}\right) + h\widehat{O}_n^{(h)}.$$

Taking the inner product of both sides with U_{n+1} we obtain

$$\begin{aligned} |U_{n+1}|^2 &= \langle U_{n+1}, U_n \rangle + h\left\langle U_{n+1}, f\left(U_{n+1} + \widehat{O}_{n+1}^{(h)}\right)\right\rangle + h\left\langle U_{n+1}, \widehat{O}_n^{(h)}\right\rangle \\ &\leq |U_{n+1}|\,|U_n| + h\left\langle U_{n+1}, f\left(U_{n+1} + \widehat{O}_{n+1}^{(h)}\right)\right\rangle + h\,|U_{n+1}|\left|\widehat{O}_n^{(h)}\right|. \end{aligned}$$

Rearranging, using the one-sided Lipschitz condition and simplifying gives

$$|U_{n+1}| \leq |U_n| - Lh\,|U_{n+1}| + h\left|f\left(\widehat{O}_{n+1}^{(h)}\right)\right| + h\left|\widehat{O}_n^{(h)}\right|.$$

$$\Rightarrow \qquad |U_{n+1}| \leq \frac{1}{1+Lh}\,|U_n| + \frac{h}{1+Lh} B_n^{(h)},$$

where

$$B_n(h) := \left| f\left(\widehat{O}_{n+1}^{(h)}\right)\right| + \left|\widehat{O}_n^{(h)}\right|,$$

$$\Rightarrow \qquad |U_n| \leq \frac{1}{(1+Lh)^{n-n_0}}\,|U_{n_0}| + \frac{h}{1+Lh} \sum_{j=n_0}^{n-1} \frac{1}{(1+h)^{n-1-j}}\, B_j^{(h)}.$$

Taking the pullback limit as $n_0 \to -\infty$ with n fixed, it follows that U_n is pathwise pullback absorbed into the ball $B_d[0, \bar{R}_n]$ in $\mathbb{R}^d$ centered on the origin with squared radius

$$\bar{R}_n^2 := 1 + \frac{h}{1+Lh} \sum_{j=-\infty}^{n-1} \frac{1}{(1+h)^{n-1-j}}\, B_j^{(h)}.$$

Note that $\bar{R}_n$ is random, but finite.

From the theory of random dynamical systems we conclude that the discrete time random dynamical system generated by drift-implicit Euler-Maruyama scheme has a random attractor with component sets in the corresponding balls $B_d[0, \bar{R}_n]$.

Since all of the trajectories converge together pathwise forward in time, the random attractor consists of a single stochastic stationary process which we shall denote by $\widehat{U}_n^{(h)}$.

Transforming back to the original variable, we have shown that the drift-implicit Euler-Maruyama scheme applied to the nonlinear SDE has a stochastic stationary solution

$$\widehat{X}_n^{(h)} := \widehat{U}_n^{(h)} + \widehat{O}_n^{(h)}, \qquad n \in \mathbb{Z},$$

taking values in the random balls $B_d[\widehat{O}_n^{(h)}, \bar{R}_n]$, which attracts all other solutions pathwise in both the forward and pullback senses.

Discretization of the coupled stochastic systems

Consider the coupled stochastic system in $\mathbb{R}^{2d}$ (now α, β are nonzero scalars)

$$
\begin{aligned}
dX_t &= (f(X_t) + \nu(Y_t - X_t))\, dt + \alpha\, dW_t^1, \\
dY_t &= (g(Y_t) + \nu(X_t - Y_t))\, dt + \beta\, dW_t^2,
\end{aligned}
$$

The corresponding drift-implicit Euler-Maruyama scheme with constant step size,

$$
\begin{aligned}
X_{n+1} &= X_n + h\,(f(X_{n+1}) + \nu(Y_{n+1} - X_{n+1})) + \alpha\,\Delta W_n^1, \\
Y_{n+1} &= Y_n + h\,(g(Y_{n+1}) + \nu(X_{n+1} - Y_{n+1})) + \beta\,\Delta W_n^2,
\end{aligned}
$$

can be written as the $2d$-dimensional vector system

$$
\mathfrak{X}_{n+1} = \mathfrak{X}_n + h\,(\mathbf{F}(\mathfrak{X}_{n+1}) + \nu B \mathfrak{X}_{n+1}) + A\,\mathbf{\Delta W}_n
$$

with the $2d$-dimensional vectors

$$\mathfrak{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{W}_t = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}, \qquad \mathbf{F}(\mathfrak{X}) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix},$$

and the $2d \times 2d$-matrices

$$A = \begin{bmatrix} \alpha I_d & 0 \\ 0 & \beta I_d \end{bmatrix}, \qquad B = \begin{bmatrix} -I_d & I_d \\ I_d & -I_d \end{bmatrix},$$

where I_d is the $d \times d$ identity matrix.

The function $\mathbf{G} := \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ defined by

$$\mathbf{G}(\mathfrak{X}) := \mathbf{F}(\mathfrak{X}) + \nu B \mathfrak{X}$$

satisfies a dissipative one-sided Lipschitz condition with constant L.

i.e. the vector scheme has essentially the same structure as the scheme for the uncoupled nonlinear equation, but in a higher dimensional space.

The previous analysis can be repeated almost verbatim to give the existence of a unique stochastic stationary process

$$\widehat{\mathfrak{X}}_n^{(h,\nu)} = \left(\begin{array}{c} \widehat{X}_n^{(h,\nu)} \\ \widehat{Y}_n^{(h,\nu)} \end{array} \right), \qquad n \in \mathbb{Z},$$

which attracts all other solutions pathwise in both the forward and pullback senses.

Moreover, the $\widehat{\mathfrak{X}}_n^{(h,\nu)}$ take values in the random balls $B_{2d}[\widehat{\mathfrak{O}}_n^{(h)}, \hat{R}_n]$ for appropriately defined $\hat{R}_n$ (which are independent of ν), where $\widehat{\mathfrak{O}}_n^{(h)}$ is the discrete time Ornstein-Uhlenbeck stochastic stationary solution for the discrete time $2d$-dimensional linear system

$$\mathfrak{X}_{n+1} = \frac{1}{1+h}\mathfrak{X}_n + \frac{1}{1+h} A\, \Delta\mathbf{W}_n.$$

Theorem 1 *

$$\begin{pmatrix} \widehat{X}_n^{(h,\nu)} \\ \widehat{Y}_n^{(h,\nu)} \end{pmatrix} \to \begin{pmatrix} \widehat{Z}_n^{(h,\infty)} \\ \widehat{Z}_n^{(h,\infty)} \end{pmatrix}$$

pathwise uniformly on bounded integer intervals $[N_1, N_2]$ *as* $\nu \to \infty$*, where* $(\widehat{Z}_n^{(h,\infty)})_{n\in\mathbb{Z}}$ *is the discrete time stationary stochastic solution of the drift-implicit Euler-Maruyama scheme with constant step size*

$$Z_{n+1} = Z_n + \frac{1}{2}h\left(f(Z_{n+1}) + g(Z_{n+1})\right) + \frac{1}{2}\alpha\,\Delta W_n^1 + \frac{1}{2}\beta\,\Delta W_n^2$$

applied to the averaged SDE

$$dZ_t = \frac{1}{2}\left(f(Z_t) + g(Z_t)\right)\,dt + \frac{1}{2}\alpha dW_t^1 + \frac{1}{2}\beta dW_t^2.$$

Synchronization of SDE with linear noise

A Stratonovich stochastic differential equation with linear noise

$$dX_t = f(X_t)\,dt + \alpha\,X_t \circ dW_t$$

can be transformed to the pathwise random ordinary differential equation

$$\frac{dx}{dt} = F(x, \bar{O}_t(\omega)) := e^{-\bar{O}_t(\omega)} f\left(e^{\bar{O}_t(\omega)} x\right) + \bar{O}_t(\omega)\,x$$

using the transformation

$$x(t,\omega) = e^{-\bar{O}_t(\omega)}\,X_t(\omega).$$

with the Ornstein-Uhlenbeck process $\bar{O}_t := \alpha e^{-t} \int_{-\infty}^{t} e^s\,dW_s$.

NOTE: F satisfy the one-sided Lipschitz condition if f does.

Similar a pair of Stratonovich SDEs

$$dX_t = f(X_t)\,dt + \alpha\, X_t \circ dW_t^1,$$

$$dY_t = g(Y_t)\,dt + \beta\, Y_t \circ dW_t^2,$$

can be transformed to the RODEs

$$\frac{dx}{dt} = F(x, \bar{O}_t^1(\omega)) := e^{-\bar{O}_t^1(\omega)} f\left(e^{\bar{O}_t^1(\omega)} x\right) + \bar{O}_t^1(\omega)\, x,$$

$$\frac{dy}{dt} = G(y, \bar{O}_t^2(\omega)) := e^{-\bar{O}_t^2(\omega)} g\left(e^{\bar{O}_t^2(\omega)} y\right) + \bar{O}_t^2(\omega)\, y,$$

with the transformations

$$x(t,\omega) = e^{-\bar{O}_t^1(\omega)}\, X_t(\omega) \qquad \bar{O}_t^1 := \alpha e^{-t} \int_{-\infty}^t e^s\, dW_s^1$$

$$y(t,\omega) = e^{-\bar{O}_t^2(\omega)}\, Y_t(\omega). \qquad \bar{O}_t^2 := \beta e^{-t} \int_{-\infty}^t e^s\, dW_s^2$$

The coupled system of random ordinary differential eqautions (RODEs)

$$\frac{dx}{dt} = F(x, \bar{O}_t^1(\omega)) + \nu(y - x),$$

$$\frac{dy}{dt} = G(y, \bar{O}_t^2(\omega)) + \nu(x - y)$$

has a pathwise asymptotically stable random attractor consisting of single stochastic stationary process $(\bar{x}_\nu(t,\omega), \bar{y}_\nu(t,\omega))$ with

$$(\bar{x}_\nu(t,\omega), \bar{y}_\nu(t,\omega)) \rightarrow (\bar{z}(t,\omega), \bar{z}(t,\omega)) \qquad \text{as} \quad \nu \rightarrow \infty,$$

where $\bar{z}(t,\omega)$ is the pathwise asymptotically stable stochastic stationary process of the averaged RODE

$$\frac{dz}{dt} = \frac{1}{2}\left[F(z,\bar{O}_t^1) + G(z,\bar{O}_t^2)\right]$$

i.e.

$$\frac{dz}{dt} = \frac{1}{2}\left[e^{-\bar{O}_t^1} f\left(\bar{O}_t^1 z\right) + e^{-\bar{O}_t^2} f\left(\bar{O}_t^2 z\right)\right] + \frac{1}{2}\left[\bar{O}_t^1 + \bar{O}_t^2\right] z$$

or the equivalent Stratonovich SDE

$$dZ_t = \frac{1}{2}\left[e^{-\eta_t} f(e^{\eta_t} Z_t) + e^{\eta_t} g(e^{-\eta_t} Z_t)\right] dt + \frac{1}{2}\alpha \circ dW_t^1 + \frac{1}{2}\beta \circ dW_t^2.$$

where $\eta_t := \frac{1}{2}(\bar{O}_t^1 - \bar{O}_t^2)$.

Direct synchronization of the SDE

The corresponding system of coupled SDE is

$$dX_t = \left(f(X_t) + \nu\left(e^{\bar{O}_t^1 - \bar{O}_t^2} Y_t - X_t\right)\right) dt + \alpha\, X_t \circ dW_t^1,$$

$$dY_t = \left(g(Y_t) + \nu\left(e^{-\bar{O}_t^1 + \bar{O}_t^2} X_t - Y_t\right)\right) dt + \beta\, Y_t \circ dW_t^2.$$

has a unique stochastic stationary solution

$$\left(\bar{X}_t^\nu(\omega), \bar{Y}_t^\nu(\omega)\right)$$

which is pathwise globally asymptotically stable with

$$\left(\bar{X}_t^\nu(\omega), \bar{Y}_t^\nu(\omega)\right) \to \left(\bar{z}(t,\omega)e^{-\bar{O}_t^1(\omega)}, \bar{z}(t,\omega)e^{-\bar{O}_t^2(\omega)}\right) \quad \text{as} \quad \nu \to \infty,$$

pathwise on finite time intervals $[T_1, T_2]$ of $\mathbb{R}$

Stochastic reaction-diffusion system on a thin two-layer domain

Let $D_{1,\varepsilon}$ and $D_{2,\varepsilon}$ be thin bounded domains in $\mathbb{R}^{d+1}$, $d \geq 1$,

$$D_{1,\varepsilon} = \Gamma \times (0, \varepsilon), \qquad D_{2,\varepsilon} = \Gamma \times (-\varepsilon, 0),$$

with $0 < \varepsilon \leq 1$ and Γ a bounded C^2-domain in $\mathbb{R}^d$.

Write

$$x \in D_\varepsilon := D_{1,\varepsilon} \cup D_{2,\varepsilon} \quad \text{as} \quad x = (x', x_{d+1})$$

where

$$x' \in \Gamma \;\text{ and }\; x_{d+1} \in (-\varepsilon, 0) \cup (0, \varepsilon).$$

Consider the system of Ito stochastic PDE

$$\frac{\partial}{\partial t} U^i - \nu_i \Delta U^i + aU^i + f_i(U^i) + h_i(x) = \dot{W}(t, x'),$$

$$t > 0, \quad x \in D_{i,\varepsilon}, \quad i = 1, 2,$$

where $\dot{W}(t, x')$ white noise depending only $x' \in \Gamma$.

[Deterministic model: *Chueshov* & *Rekalo* (Matem. Sbornik, 2004)]

[Stochastic model: *Caraballo*, *Kloeden* & *Chueshov* (SIAM J. Math.Anal., 2007)]

Neumann boundary conditions

$$\big(\nabla U^i, n_i\big) = 0, \;\; x \in \partial D_{i,\varepsilon} \setminus \Gamma, \quad i = 1, 2,$$

on the external part of the boundary of the compound domain D_ε, where n is the outer normal to ∂D_ε

Matching condition on Γ

$$\left(-\nu_1 \frac{\partial U^1}{\partial x_{d+1}} + k(x', \varepsilon)(U^1 - U^2) \right)\Big|_\Gamma = 0,$$

$$\left(\nu_2 \frac{\partial U^2}{\partial x_{d+1}} + k(x', \varepsilon)(U^2 - U^1) \right)\Big|_\Gamma = 0.$$

Synchronization as $\varepsilon^{-1} \to \infty$ with the averaged system

$$\frac{\partial}{\partial t}U - \nu \Delta_\Gamma U + aU + f(U) + h(x') = \dot{W}(t, x'), \quad x' \in \Gamma,$$

on the spatial domain Γ with the Neumann boundary conditions on $\partial\Gamma$ and with

$$\nu = \frac{\nu_1 + \nu_2}{2},$$

$$f(U) = \frac{f_1(U) + f_2(U)}{2}, \qquad h(x') = \frac{h_1(x', 0) + h_2(x', 0)}{2}.$$

Method : Transform Ito SPDE into a pathwise random PDE.

Assumptions

- $f_i \in C^1(\mathbb{R})$ such that $f_i'(v) \geq -c$ for all $v \in \mathbb{R}$ and

$$v f_i(v) \geq a_0 |v|^{p+1} - c, \qquad |f_i'(v)| \leq a_1 |v|^{p-1} + c,$$

a_j and c positive constants and $1 \leq p < 3$;

- $h_i \in H^1(D_{i,1}), \quad i = 1, 2;$

- $k(\cdot, \varepsilon) \in L^\infty(\Gamma),\ k(x', \varepsilon) > 0,\ x' \in \Gamma,\ \varepsilon \in (0, 1],$

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} k(x', \varepsilon) = +\infty, \quad x' \in \Gamma;$$

- $W(t)$, $t \in \mathbb{R}$, two-sided $L_2(\Gamma)$-valued Wiener with covariance operator $K = K^* \geq 0$ such that for some $\quad \beta > \max\left\{1\ ,\ \frac{d}{4}\right\}$

$$\mathrm{tr}\left[K\,(-\Delta_N + 1)^{2\beta - 1}\right] < \infty\,,$$

Δ_N Laplacian in $L_2(\Gamma)$ with Neumann boundary conditions on $\partial\Gamma$.

- $(\Omega, \mathcal{F}, \mathbb{P})$ the corresponding probability space

Theorem 2 *Under the above Assumptions the following assertions hold.*

1. *The coupled SPDE generates an RDS* $(\theta, \bar{\phi}_\epsilon)$ *in the space*

$$\mathcal{H}_\epsilon = L_2(D_{1,\epsilon}) \oplus L_2(D_{2,\epsilon}) \sim L_2(D_\epsilon)$$

given by $\bar{\phi}_\epsilon(t,\omega)U_0 = U(t,\omega)$*, where* $U(t,\omega) = (U^1(t,\omega); U^2(t,\omega))$ *is a strong solution to the problem and* $U_0 = (U_0^1; U_0^2)$*.*

2. *Similarly, the averaged SPDE generates an RDS* $(\theta, \bar{\phi}_0)$ *in the space* $L_2(\Gamma)$*.*

3. *Cocycles* $\bar{\phi}_\epsilon$ *converge to* $\bar{\phi}_0$ *in the sense*

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \frac{1}{\epsilon} \int_{D_\epsilon} |\bar{\phi}_\epsilon(t,\omega)v - \bar{\phi}_0(t,\omega)v|^2 dx = 0, \quad \forall \omega,$$

for any $v(x) \in \mathcal{H}_\epsilon$ *independent of the variable* x_{d+1}*, and any* $T > 0$*.*

4. *These RDS* $(\theta, \bar{\phi}_\epsilon)$ *and* $(\theta, \bar{\phi}_0)$ *have random compact pullback attractors* $\{\bar{\mathfrak{A}}^\epsilon(\omega)\}$ *and* $\{\bar{\mathfrak{A}}^0(\omega)\}$ *in their corresponding state spaces. Moreover, if* K *is non-degenerate, then* $\{\bar{\mathfrak{A}}^0(\omega)\}$ *is a singleton, i.e.* $\bar{\mathfrak{A}}^0(\omega) = \{\bar{v}_0(\omega)\}$*, where* $\bar{v}_0(\omega)$ *is an* $L_2(\Gamma)$*-valued tempered random variable.*

5. *The attractors* $\{\bar{\mathfrak{A}}^\epsilon(\omega)\}$ *are upper semi-continuous as* $\epsilon \longrightarrow 0$ *in the sense that for all* $\omega \in \Omega$

$$\lim_{\epsilon\longrightarrow 0} \sup_{v\in\bar{\mathfrak{A}}^\epsilon(\omega)} \inf_{v_0\in\bar{\mathfrak{A}}^0(\omega)} \frac{1}{\epsilon}\int_{D_\epsilon} |v(x', x_{d+1}) - v_0(x')|^2 dx = 0.$$

6. *In addition, if*

$$\nu_1 = \nu_2 := \nu, \quad f_1(U) = f_2(U) := f(U),$$
$$h_1(x', x_{d+1}) = h(x') = h_2(x', x_{d+1});$$

$f(U)$ *is globally Lipschitz, i.e. there exists a constant* $L > 0$ *such that*

$$|f(U) - f(V)| \leq L|U - V|, \quad U, V \in \mathbb{R},$$

and also that

$$k(x', \epsilon) > k_\epsilon \text{ for } x' \in \Gamma,\ \epsilon \in (0,1] \quad \text{and} \quad \lim_{\epsilon\longrightarrow 0} \epsilon^{-1} k_\epsilon = +\infty,$$

then, there exists $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0]$ *the random pullback attractor* $\{\bar{\mathfrak{A}}^\epsilon(\omega)\}$ *for* $(\theta, \bar{\phi}_\epsilon)$ *has the form*

$$\bar{\mathfrak{A}}^\epsilon(\omega) \equiv \left\{ v(x', x_{d+1}) \equiv v_0(x') \ : \ v_0 \in \bar{\mathfrak{A}}^0(\omega) \right\},$$

where $\{\bar{\mathfrak{A}}^0(\omega)\}$ *is the random pullback attractor for the RDS* $(\theta, \bar{\phi}_0)$.

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