Simple dynamical models interpreting climate data and their metastability

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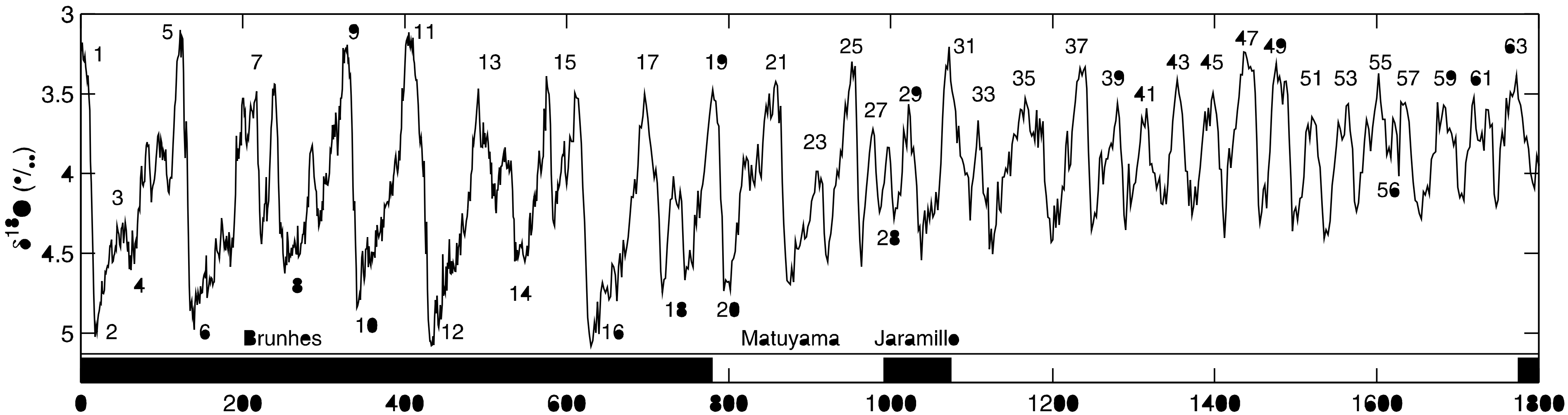
1. Paleoclimatic time series



Lisiecki, Raymo, Paleoceanography 2005 concentration variation of ^{18}O to ^{16}O taken from marine sediments at 57 globally distributed sites (e.g. Brunhes, Matuyama, Jaramillo):

global average temperature time series

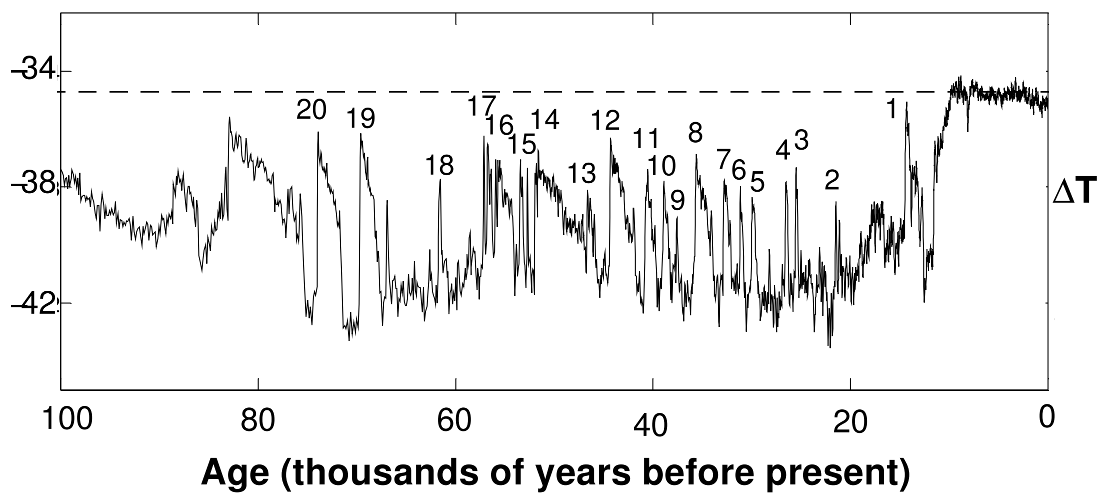
basic feature:

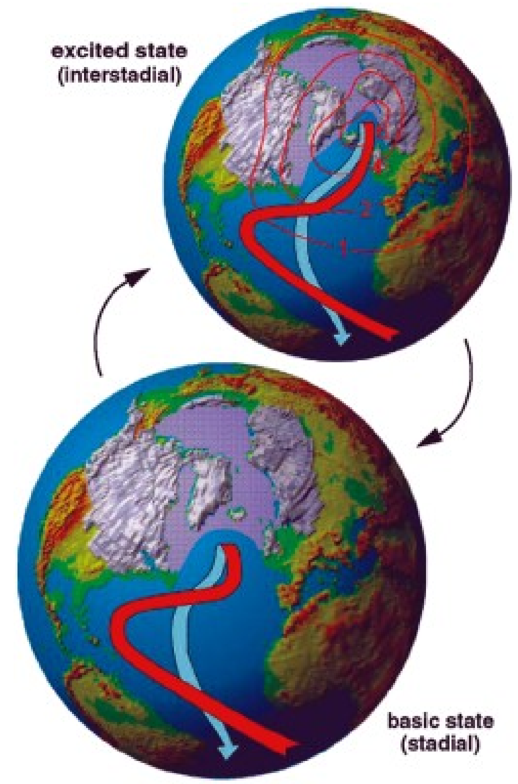
- from 0 to -1 Myr periodicity $\sim$ 100 000 y
- from -1 Myr to -1.8 Myr periodicity $\sim$ 44 000 y
- Milankovich cycles: axial tilt (41 000 y) and eccentricity (100 000 y)

2. Dansgaard-Oeschger events

temperature indicators: ^{18}O, ^{16}O, methane, calcium etc.

GRIP ice core data: 20 abrupt changes in climate of Greenland during last ice age (-91 000 to -11 000 y) (D/O events).





- rapid warming by 5-10°C within one decade
- subsequent slower cooling
  within a few centuries
- fast return to stable cold ground state

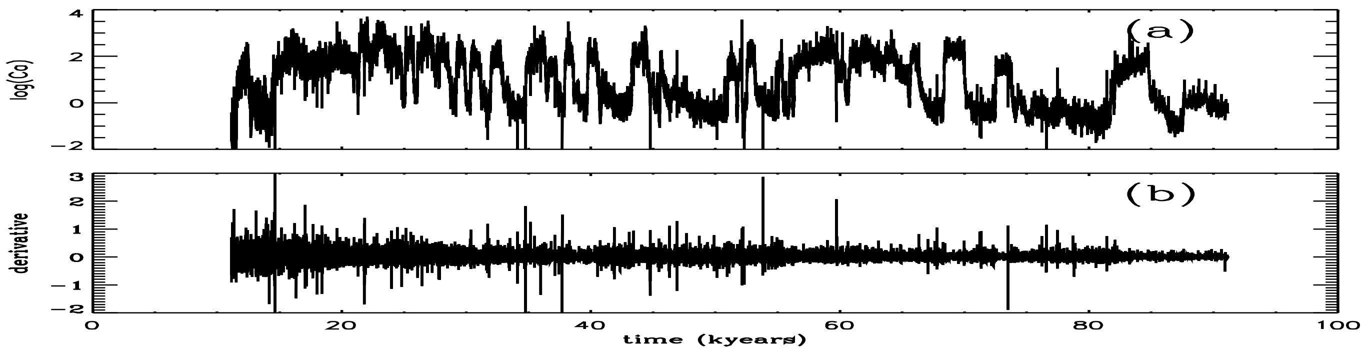
simulations: Ganopolsky/Rahmstorf,

Potsdam Institute for Climate Impact

Research

2. Dansgaard-Oeschger events. Statistical analysis

Calcium signal from GRIP: about 80 000 samples for 80 000 y



typical waiting time between D/O events: 1000 – 2000 y,
waiting times between D/O events: multiples of $\sim$1470 years.

What triggers the transitions?

modeling by Langevin equation:

$$dX(t) = -U'(t, X(t))dt + \text{NOISE}$$

U — multi well potential, wells correspond to climate states

P. Ditlevsen (*Geophys. Res. Lett. 1999*): power spectrum analysis of time series:
NOISE contains strong α-stable component with $\alpha \approx 1.75$.

3. p-Variation as test statistic

Which model of noise fits best with time series: parametrized test.

Ditlevsen's analysis: power spectrum of residua of time series

Problem: Stationarity?

Aim: better test statistics than peaks of power spectrum.

Model assumption: with some U interpret data as

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))ds + \varepsilon L(t) = Y^\varepsilon(t) + L^\varepsilon(t)$$

L Lévy process containing α-stable component with unknown α, Y^ε of bounded variation; test α

Idea: p-variation characteristic for fluctuation behavior of noise processes.

$$V_t^{p,n}(X) = \sum_{i=1}^{[nt]} |X(\frac{i}{n}) - X(\frac{i-1}{n})|^p, \quad V_t^p = \lim_{n\to\infty} V_t^{p,n}$$

4. p-Variation and the Blumenthal-Getoor Index

L α-stable process with jump measure ν; then p-variation identified by

Blumenthal-Getoor index

$$\beta_L = \inf\{s \geq 0 : \int_{\{|y|\leq 1\}} |y|^s \nu(dy) < \infty\}$$

$$\gamma_L = \inf\{p > 0 : V_1^p(L) < \infty\}$$

Thm 1
L symmetric α-stable. Then

$$\gamma_L = \beta_L = \alpha.$$

Problem: How to read $\gamma_L = \alpha$ off the sequence $(V_t^{p,n}(L))_{n\in\mathbf{N}}$?

Calls for results about the asymptotic behavior of the sequence.

5. The case $\alpha = 2$: Brownian motion

For $n \in \mathbf{N}$ $V_1^{p,n}(W)$ consists of n independent increments and

$$E(V_1^{p,n}(W)) = n^{1-\frac{p}{2}} E(|W(1)|^p)$$

Thm 2 (LLN type)

$$n^{-1+\frac{p}{2}} V_t^{p,n}(W) \to tE(|W(1)|^p) \quad \text{in probability,}$$

Y of bounded variation. Then also

$$n^{-1+\frac{p}{2}} V_t^{p,n}(W+Y) \to tE(|W(1)|^p) \quad \text{in probability.}$$

Thm 3 (CLT type)

$$\begin{aligned}(n^{\frac{1}{2}}[n^{-1+\frac{p}{2}} V_t^{p,n}(W) - tE(|W(1)|^p)])_{t\geq 0} &= (n^{-\frac{1}{2}+\frac{p}{2}} V_t^{p,n}(W) - n^{\frac{1}{2}} tE(|W(1)|^p))_{t\geq 0} \\ &\to ((var(|W(1)|^p))^{\frac{1}{2}} \tilde{W}(t))_{t\geq 0}\end{aligned}$$

weakly with respect to the Skorokhod metric, and an independent Brownian motion $\tilde{W}$.

6. The case $\alpha < 2$

(Lit: Corcuera, Nualart, Wörner '07; case $p < \alpha$ for LLN type, $p < \frac{\alpha}{2}$ for CLT type)

Problem: $p < \frac{\alpha}{2} < 1$ not satisfactory for paleo-climatic data! Beyond $\frac{\alpha}{2}$ no CLT type result available, no asymptotic normality, but asymptotically of different type.

Thm 4 (LT type)
L α-stable with $\alpha \in]0, 2[$. Then

$$(V_t^{p,n}(L) - B_t^n(\alpha, p))_{t \geq 0} \to \tilde{L}$$

weakly with respect to the Skorokhod metric, and an independent $\frac{\alpha}{p}$-stable process $\tilde{L}$. Here

$$B_t^n(\alpha, p) = \begin{cases} n^{1-\frac{p}{\alpha}} t E(|L(1)|^p), & \frac{\alpha}{2} < p < \alpha, \\ n t^2 E(\sin((nt)^{-1} |L(1)|^p)), & p = \alpha, \\ 0, & \alpha < p. \end{cases}$$

Same result with $L + Y$ instead of L if Y is of finite p-variation and $\frac{\alpha}{2} < p < 1$ or $p > \alpha$.

7. Methods of Proof

- show that $(|L(n) - L(n-1)|^p)_{n\in\mathbf{N}}$ is in domain of attraction of $\frac{\alpha}{p}$-stable law; use tails and characteristic functions

- use Aldous' criterion for tightness of sequence $(X^n)_{n\in\mathbf{N}}$ in Skorokhod metric:

$$\text{(i)}\ \lim_{K\to\infty}\sup_{n\in\mathbf{N}} P(\sup_{t\geq N}|X^n(t)|>K)=0 \quad \text{for all } N\geq 0,$$

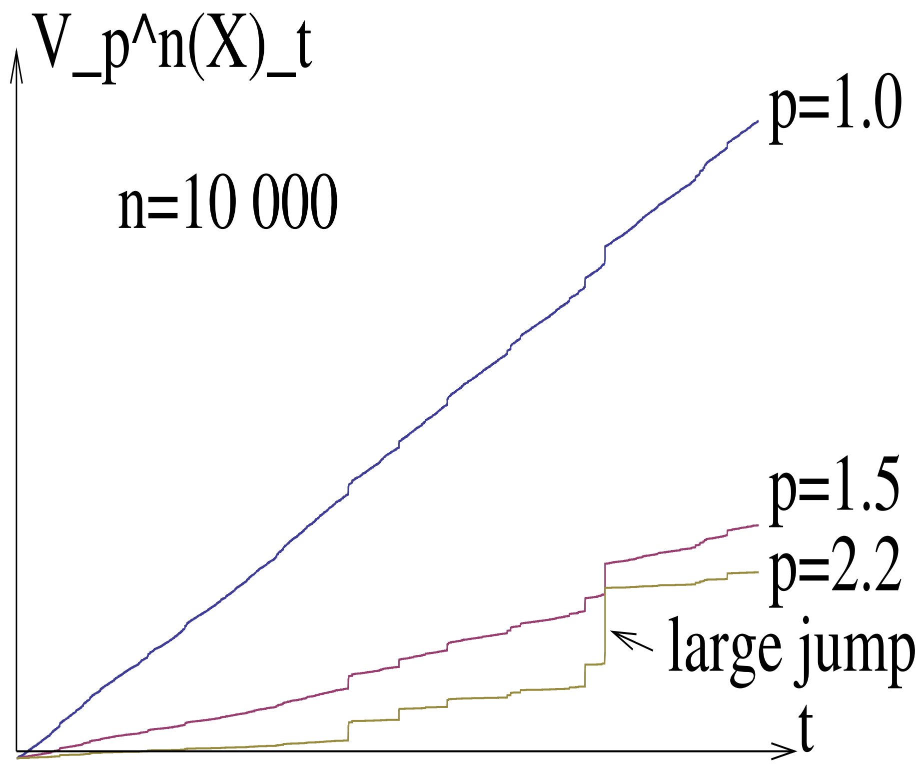
$$(ii)\ \lim_{\theta\downarrow 0}\limsup_{n\to\infty}\sup_{S\leq T\leq S+\theta} P(|X^n(T)-X^n(S)|\geq\varepsilon)=0, \quad \text{for all } N\geq 0, \varepsilon>0$$

  S, T stopping times

- adding processes of smaller variation: new notion of Lipschitz continuity on large sets; comparison of small and large jumps

8. The p-variation process of a diffusion

p-variation of diffusion perturbed by 1.5-stable Levy process

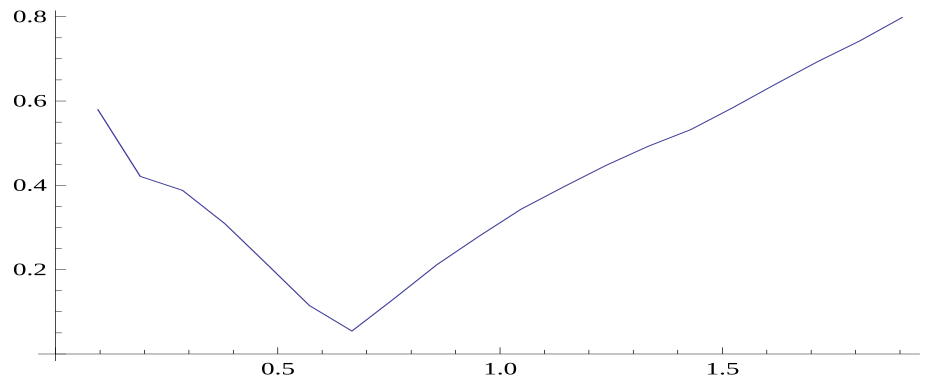


with increasing p, big jumps get stronger weight, continuous parts less

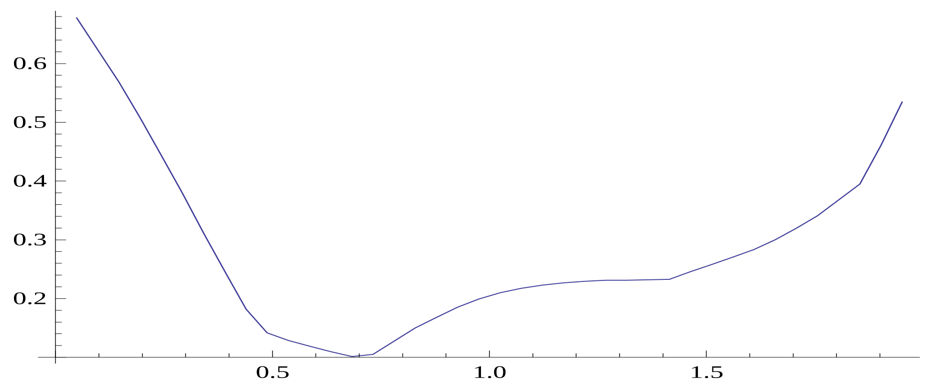
9. Test for α with real and simulated data

Thm 4: law of $V^{2\alpha,n}(X)$ converges to $\frac{1}{2}$-stable law if data of time series X have α-stable residuals

Kolmogorov-Smirnov statistics: distance between empirical law of $V^{2\alpha,n}(X)$ and $\frac{1}{2}$-stable law, as a function of α; minimum of curve: right α



simulated time series of a 0.6-stable Levy process, $n = 200$



real time series from the Greenland ice, $n = 200$

10. Simple system with Levy noise

consider SDE driven by α-stable Lévy noise of small intensity

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\,ds + \varepsilon L(t), \quad \varepsilon \downarrow 0.$$

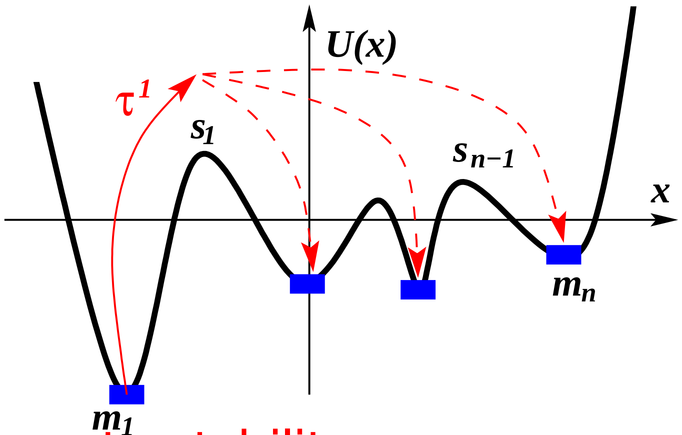
• L is α-stable symmetric Lévy process, $\alpha \in (0,2)$

multi well potential U

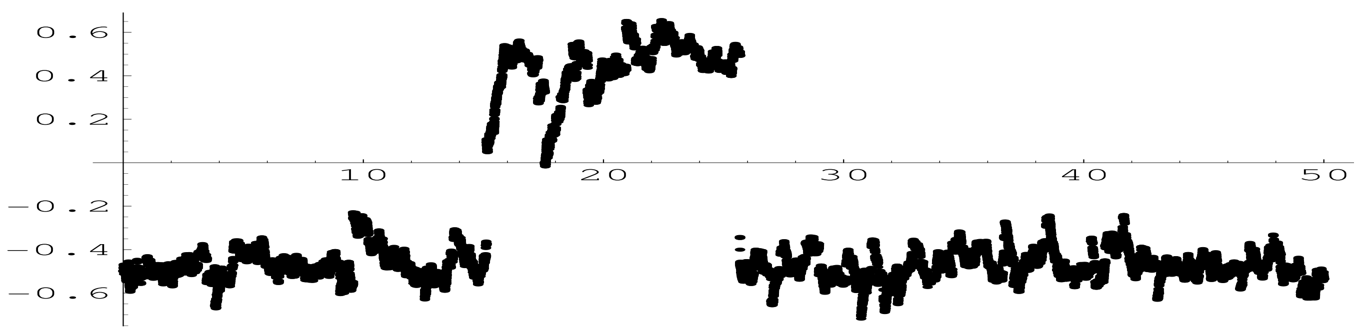
• n local minima m_i

• $n-1$ local maxima s_i

• $U''(m_i) > 0$, $U''(s_i) < 0$



aim: investigate exit and transition rates, meta-stability.



11. α-stable Lévy Processes

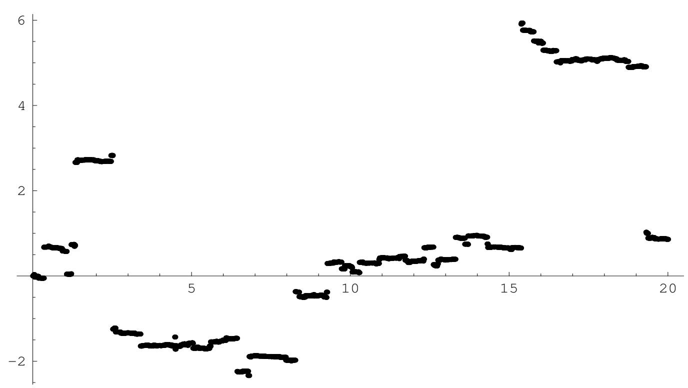
L Lévy process with characteristics (d, γ, ν) iff

$$E(\exp(iuL(t))) = \exp(t(-\frac{1}{2}du^2 + i\gamma u + \int_{\mathbf{R}}[e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}]\nu(dy))), \ \ u \in \mathbf{R}, t \geq 0,$$

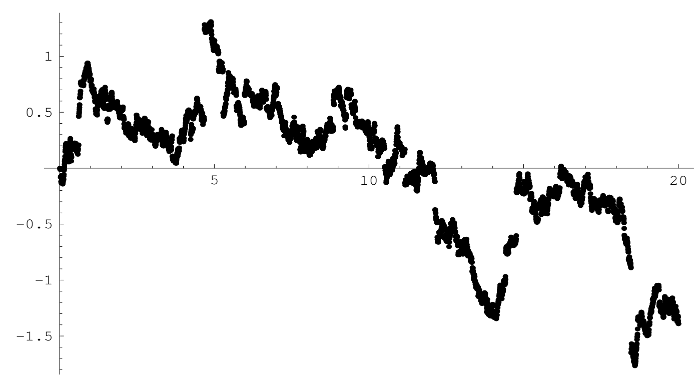
ν measure on Borel sets in $\mathbf{R}$ with $\nu(\{0\}) = 0$, $\int_{\mathbf{R}}[|y|^2 \wedge 1]\nu(dy) < \infty$.

L α-stable symmetric Lévy process if

$$E(\exp(iuL(t))) = \exp(-c(\alpha)t|u|^\alpha), \quad \nu(dy) = \frac{1}{|y|^{\alpha+1}}dy, \quad u, y \in \mathbf{R}.$$



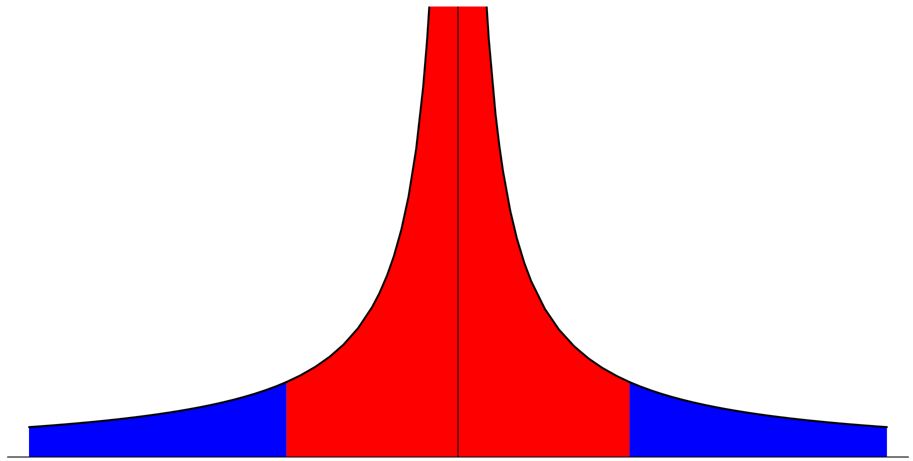
$\alpha = 0.75$



$\alpha = 1.75$

12. Probabilistic approach of exit times

$$L(t) = \xi^\varepsilon(t) + \eta^\varepsilon(t)$$



$$\nu_\xi^\varepsilon = \nu|_{[-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}]}, \qquad \nu_\eta^\varepsilon = \nu|_{[-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}]^c}$$

$$\nu_\xi^\varepsilon(\mathbb{R}) = \infty \qquad \nu_\eta^\varepsilon(\mathbb{R}) = \frac{2}{\alpha}\varepsilon^{\alpha/2} = \beta_\varepsilon$$

$\varepsilon\xi^\varepsilon$ sum of $\varepsilon\cdot$BM and small jump ($\leq \sqrt{\varepsilon}$) process

$\varepsilon\eta^\varepsilon$ big jump ($\geq \sqrt{\varepsilon}$) compound Poisson process

big jumps at τ_k, inter-jump time T_k with exponential law

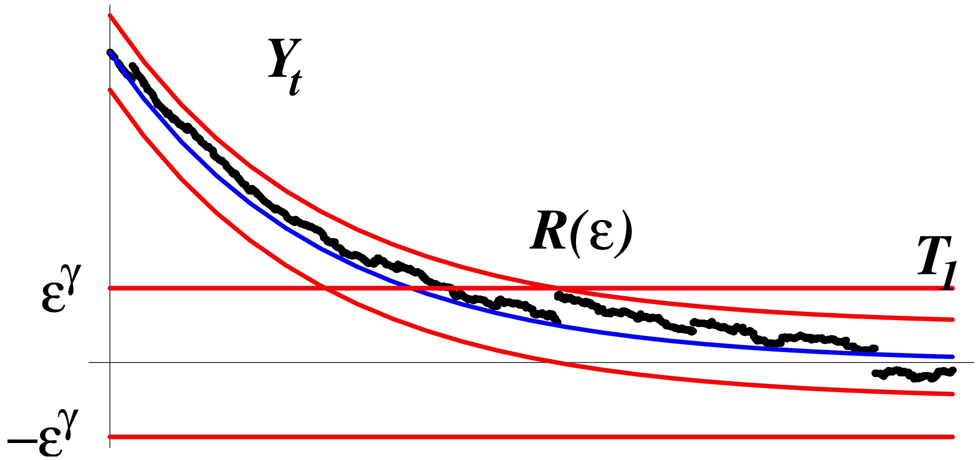
$$E(T_k) = (\beta^\varepsilon)^{-1} = \frac{\alpha}{2}\varepsilon^{-\alpha/2}$$

13. The small and large jump parts

U with stable state 0, exit from $[-b, a]$ for $a, b > 0$

between big jumps X^ε is Y perturbed by $\varepsilon\xi^\varepsilon$

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\,ds + \varepsilon\xi^\varepsilon(t), \quad t \in [0, T_1), \quad Y(t) = x - \int_0^t U'(Y(s))\,ds$$



deviation $$\mathbf{P}\left(\sup_{[0,T_1)} |X^\varepsilon(t) - Y(t)| \geq \frac{\varepsilon^\gamma}{2}\right) \leq \mathbf{P}\left(\sup_{[0,T_1)} |\varepsilon\xi^\varepsilon(t)| \geq \frac{\varepsilon^\gamma}{C}\right) \leq e^{-1/\varepsilon^\delta}$$

relaxation $$T(x,\varepsilon) = \int_{\varepsilon^\gamma/2}^{x} \frac{dy}{|U'(y)|} \approx \int_\delta^x \frac{dy}{|U'(y)|} + \int_{\varepsilon^\gamma/2}^{\delta} \frac{dy}{My}$$

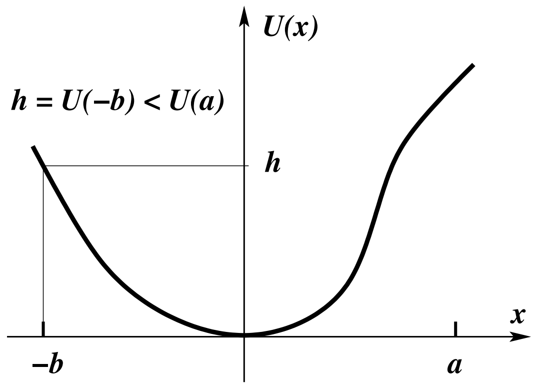
$$\approx \text{Const} + \frac{\gamma}{M}|\ln\varepsilon| \leq R(\varepsilon) = \mathcal{O}(|\ln\varepsilon|)$$

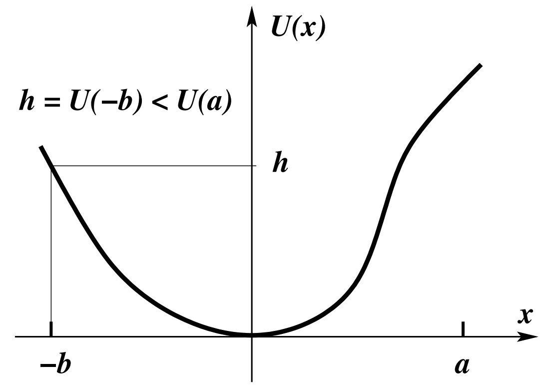
asymptotically, big jumps coincide with exits

15. 14: comparison of Gaussian and Lévy dynamics

$$\hat{\sigma} = \inf\{t \geq 0 \, : \, \hat{X}^\varepsilon(t) \notin [-b, a]\}$$

$$\sigma = \inf\{t \geq 0 \, : \, X^\varepsilon(t) \notin [-b, a]\}$$





$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(t))\, ds + \varepsilon W(t)$$

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\, ds + \varepsilon L(t)$$

Thm 5 (Freidlin-Wentzell):

$$\mathbf{P}_x(e^{(2h-\delta)/\varepsilon^2} < \hat{\sigma} < e^{(2h+\delta)/\varepsilon^2}) \to 1$$

Thm 6

$$\mathbf{P}_x(\tfrac{1}{\varepsilon^{\alpha-\delta}} < \sigma < \tfrac{1}{\varepsilon^{\alpha+\delta}}) \to 1$$

Kramers' law ('40, Williams, Bovier et al.):

$$\mathbf{E}_x\hat{\sigma} \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}} e^{2h/\varepsilon^2}$$

$$\mathbf{E}_x\sigma \approx \tfrac{1}{\varepsilon^\alpha}\Big(\int_{\mathbb{R}\setminus[-b,a]} \tfrac{dy}{|y|^{1+\alpha}}\Big)^{-1}$$

Exponential law (Day, Bovier et al.)

$$\mathbf{P}_x(\tfrac{\hat{\sigma}}{\mathbf{E}_x\hat{\sigma}} > u) \sim \exp(-u)$$

$$\mathbf{P}_x(\tfrac{\sigma}{\mathbf{E}_x\sigma} > u) \sim \exp(-u)$$

14. Gaussian versus α-stable revisited

W Wiener process

L symmetric α-stable Lévy process

Tail behavior

$$P(|W(1)| \geq x) \sim \exp(-cx^2)$$

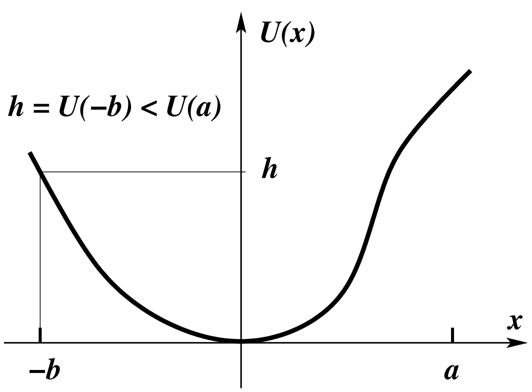
$$P(|L(1)| \geq x) \sim c\frac{1}{x^\alpha},\ x \to \infty$$

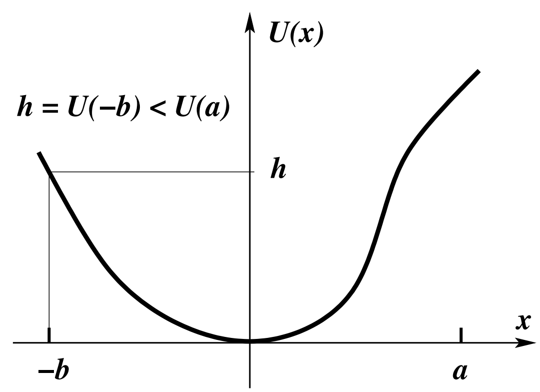
$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(t))\,ds + \varepsilon W(t)$$

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\,ds + \varepsilon L(t)$$

$$\hat{\sigma} = \inf\{t \geq 0\ :\ \hat{X}^\varepsilon(t) \notin [-b, a]\}$$

$$\sigma = \inf\{t \geq 0\ :\ X^\varepsilon(t) \notin [-b, a]\}$$





$$\mathbf{E}_x\hat{\sigma} \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}}\exp\left(\frac{2h}{\varepsilon^2}\right)$$

$$\mathbf{E}_x\sigma \approx \frac{1}{\varepsilon^\alpha}\left(\int_{\mathbb{R}\setminus[-b,a]}\frac{dy}{|y|^{1+\alpha}}\right)^{-1}$$

14. Gaussian versus α-stable revisited

Conjecture: make tails of Lévy process exponentially light to recover Gaussian exit behavior.

Tail behavior

L Lévy process with jump measure having tails

$$P(|L(1)| \geq x) \sim \exp(-cx^{\alpha}), \quad x \to \infty$$

sub-exponential tails: $\alpha < 1$ super-exponential tails: $\alpha > 1$

Consider

$$X^{\varepsilon}(t) = x - \int_0^t U'(X^{\varepsilon}(s-))\, ds + \varepsilon L(t)$$

$$\sigma(\varepsilon) = \inf\{t \geq 0 \,:\, X^{\varepsilon}(t) \notin [-1, 1]\}$$

Conjecture:

$$E_x(\sigma(\varepsilon)) \sim_{\varepsilon \to 0} \exp(\frac{c}{\varepsilon^2}) \quad \text{as} \quad \alpha \uparrow 2.$$

15. The phase transition at $\alpha = 1$

Thm 7 [sub-exponential tails] For $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, t \geq 0$:

$$(1-\delta)\exp(-C_\varepsilon^{1-\delta} t) \leq \sup_{|x|\leq 1} \mathbf{P}_x(\sigma(\varepsilon) > t) \leq \exp(-\tfrac{1}{2} C_\varepsilon t),$$

with $C_\varepsilon := 2\nu([\frac{1}{\varepsilon}, \infty))$. Hence for $|x| < 1$

$$\lim_{\varepsilon \to 0} \varepsilon^\alpha \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

Thm 8 [super-exponential tails] q_ε ε-quantile of jump measure ν. Then for $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, t \geq 0$:

$$(1-\delta)\exp(-D_\varepsilon^{1-\delta} t) \leq \sup_{|x|\leq 1} \mathbf{P}_x(\sigma(\varepsilon) > t) \leq (1+\delta)\exp(-D_\varepsilon^{1+\delta} t),$$

where $D_\varepsilon = \exp(-d_\alpha \frac{|\ln \varepsilon|}{\varepsilon q_\varepsilon})$ and $d_\alpha = \alpha(\alpha-1)^{\frac{1}{\alpha}-1}$. Hence for $|x| < 1$

$$d_\alpha^{-1} \lim_{\varepsilon \to 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha}-1} \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

15. α-stable towards Gaussian. The phase transition at $\alpha = 1$

Comparison of regimes for mean exit time

Power tails jump tails $\nu([x,\infty)) = x^{-r}$, $x \geq 1$ for some $r > 0$. Then

$$2 \lim_{\varepsilon \to 0} \varepsilon^r \mathbf{E}_x \sigma(\varepsilon) = 1.$$

Sub-exponential tails jump tails $\nu([u,\infty)) = \exp(-u^\alpha)$, $u \geq 1$, $\alpha < 1$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^\alpha \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

Super-exponential tails jump tails $\nu([u,\infty)) = \exp(-u^\alpha)$, $u \geq 1$, $\alpha > 1$. Then

$$d_\alpha^{-1} \lim_{\varepsilon \to 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha}-1} \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

Gaussian diffusion no jumps, L one-dimensional Brownian motion. Then

$$\tfrac{1}{2}(U(-1) \wedge U(1))^{-1} \lim_{\varepsilon \to 0} \varepsilon^2 \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

16. Heuristics of exits: climbing versus jumping

The Brownian case

LD theory: diffusion has to climb potential in order to exit at lowest saddle point

The power tail case

for $\varepsilon > 0$ split $L = \eta^\varepsilon + \xi^\varepsilon$, compound Poisson pure jump part η^ε with jumps of height larger than $\frac{1}{\sqrt{\varepsilon}}$; small jump and Gaussian part ξ^ε with jumps not exceeding $\frac{1}{\sqrt{\varepsilon}}$; exit asymptotically due to one big jump, as shown in first talk

The case of exponential tails

for $\varepsilon > 0$ split $L = \eta^\varepsilon + \xi^\varepsilon$, compound Poisson pure jump part η^ε with jumps of height larger than g_ε; small jump and Gaussian part ξ^ε with jumps not exceeding this bound;

choose g_ε individually according to sub- and super-exponential tails

show that exit before time T while not returning to an interval around stable fixed point 0 of radius $\delta > 0$ requires that either increments of ξ^ε exceed certain bounds (for which probability is small enough), or sum of large jumps before time T exceeds bound $1 - \delta$

in any case large jumps responsible for exits

16. Heuristics of exits: climbing versus jumping

N_T random number of large jumps before time T

W_i jump n^o i, $i \in \mathbf{N}$.

N_T Poisson with expectation $\beta_\varepsilon T$, where $\beta_\varepsilon = \nu([-g_\varepsilon, g_\varepsilon]^c) = 2\exp(-x^\alpha)$

For n fixed, probability that sum of large jumps exceeds bound $1-\delta$ estimated by

$$P(N_T > n) + \sum_{k=1}^{n} P(N_T = k) P(\sum_{i=1}^{k} |\varepsilon W_i| > 1 - \delta)$$

Idea for estimation:
for $n \in \mathbf{N}$

$$P(N_T > n) \leq (1+\delta)\exp(-n \ln n) \quad \text{(Stirling's formula)}$$

choose $n = n_\varepsilon$ suitably

16. Heuristics of exits: climbing versus jumping

essential term to estimate for $1 \leq k \leq n_\varepsilon$

$$P(\sum_{i=1}^{k} |\varepsilon W_i| > 1 - \delta)$$

law of i.i.d. random variables $(|W_i|)_{i \in \mathbf{N}}$: $\beta_\varepsilon^{-1} 2\nu|_{[g_\varepsilon, \infty[}$

hence

$$\begin{aligned} P(\sum_{i=1}^{k} |\varepsilon W_i| > 1 - \delta) &\leq \beta_\varepsilon^{-k} \exp(-\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i \geq \frac{1-\delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\}) \\ &= \beta_\varepsilon^{-k} \exp(-\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i = \frac{1-\delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\}) \end{aligned}$$

minimization problem in the exponent of this estimate causes phase transition

16. Heuristics of exits: climbing versus jumping

By suitable choice of g_ε: lower boundary for x_i in $\inf$ can be taken 0.

sub-exponential tails

$$\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i = 1, x_i \geq 0\} = 1$$

The minimum is taken on the boundary of the simplex, and $x_i = \frac{1}{n}, 1 \leq i \leq n,$ corresponds to maximum of the function

$$(x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_i^\alpha$$

16. Heuristics of exits: climbing versus jumping

Super-exponential tails

$$\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i = 1, x_i \geq 0\} = n(\frac{1}{n})^\alpha$$

The minimum is taken for $x_i = \frac{1}{n}, 1 \leq i \leq n$, the unique local minimum of the function

$$(x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_i^\alpha$$

Bifurcation in the asymptotic behavior:
phase transition due to switch from concavity to convexity at $\alpha = 1$ of

$$x \mapsto x^\alpha, \quad x \geq 0,$$

big jumps of the Lévy process govern asymptotic behavior

$\alpha < 1$: biggest jump responsible for exit

$\alpha > 1$: cumulative action of several large jumps responsible for exit