From PET to SPLIT

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PET: Polynomial Ergodic Theorem (Bergelson)

$$\lim_{N\to\infty} 1/N \sum_{n=1}^{N} T^{q_1(n)} f_1 \cdots T^{q_\ell(n)} f_\ell$$
$$= \prod_{i=1}^{\ell} \int f_i d\mu \qquad \text{in} \quad L^2$$

T is μ preserving and weakly mixing

f_i bounded measurable functions

q_i polynomials, integer valued on the integers

and $q_{i+1}(n) - q_i(n) \to \infty$ as $n \to \infty$

SPLIT: sum product limit theorem

WANT TO PROVE that

$$\frac{1}{\sqrt{N}}\left(\sum_{n=1}^{N}(T^{q_1(n)}f_1\cdots T^{q_\ell(n)}f_\ell-\prod_{i=1}^{\ell}\int f_i\,d\mu)\right)$$

Is asymptotically normal: strong mixing conditions are needed

If $f_i=\chi_{A_i}$ then

$$\sum_{n=1}^{N}T^{q_1(n)}f_1(x)\cdots T^{q_\ell(n)}f_\ell(x)=\#\left\{k:\bigcap_{i=1}^{\ell}\left\{T^{q_i(k)}x\in A_i\right\}\right\}$$

Application to multiple recurrence

setup:

- $X_1,\ldots,X_\ell; |X_i| \le D < \infty$ bounded stationary processes
- $\mathcal{F}_{kl} \subset \mathcal{F}_{k'l'}$ if $k` \le k$ and $l \le l`$ $\sigma -$ algebras
- $\alpha -$ mixing coefficient

$$\alpha(n) = \sup_{k\ge 0} \sup_{A\in\mathcal{F}_{-\infty,k}, B\in\mathcal{F}_{k+n,\infty}} |P(A\cap B) - P(A)P(B)|,\ n \ge 0.$$

- $\beta -$ approximation coefficient

$$\beta_j(n) = \sup_{m\ge 0} E|X_j(m) - E(X_j(m)|\mathcal{F}_{m-n,m+n})|.$$

- Assumption:

$$\alpha(n) + \max_{1\le j\le \ell} \beta_j(n) \le \kappa^{-1} e^{-\kappa n}.$$

Functions q:

- $q_1(n),\ldots,q_\ell(n)$ nonnegative integer valued on integers functions, satisfying

$$q_1(n) = rn + p \text{ for integer } r > 0, p \geq 0$$

$$q_j(n+1) \geq q_j(n) + n^\gamma,\ \gamma \in (0,1), \forall \mathrm{n} \geq \mathrm{n}_0 > 1$$

$$q_{j+1}([n^{1-\gamma}]) \geq q_j(n) n^\gamma$$

Result:

Set $a_j = EX_j(0)$ then

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N}\Big(\prod_{j=1}^{\ell} X_j(q_j(n)) - \prod_{j=1}^{\ell} a_j\Big),$$

Is asymptotically normal with zero mean and The the variance

$$\sigma^2 = \sigma_\ell^2 = EX_1^2(0)\Big(\prod_{j=2}^{\ell} EX_j^2(0) - \prod_{j=2}^{\ell} a_j^2\Big) + \sigma_1^2 \prod_{j=2}^{\ell} a_j^2$$

where

$$\sigma_1^2 = \lim_{N\to\infty} \tfrac{1}{N} E\big(\textstyle\sum_{n=1}^{N}(X_1(q_1(n)) - a_1)\big)^2$$

$$= EX_1^2(0) - a_1^2 + 2\textstyle\sum_{n=1}^{\infty} E\big((X_1(rn) - a_1)(X_1(0) - a_1)\big)$$

when variance is zero?

$$\sigma = 0 \Leftrightarrow$$

either $X_j(0) = 0$ almost surely for some j

or $X_j(0) = a_j$ for all $j \geq 2$ and

$$X_1(rm+p) - a_1 = U^{m+1}X - U^m X \quad a.s.$$

U unitary on , X random variable from

$$L^2(\Omega, \sigma\{X_1\}, P)$$

Functional CLT

- For $u \in [0,1]$ set

$$W_N(u) = N^{-1/2} \sum_{n=0}^{[uN]} \Big(\prod_{j=1}^{\ell} X_j(q_j(n)) - \prod_{j=1}^{\ell} a_j \Big).$$

Then in distribution

$$W_N \Rightarrow \sigma W \quad \text{as} \quad N \to \infty$$

W – Brownian motion

Applications

Our results are applicable, for instance, to the case when $X_i(n) = f_i(\xi_n)$ for bounded measurable f_i's and a Markov chain ξ_n in a space M satisfying the Doeblin condition taken with its invariant measure μ which yields, in particular, that for any measurable sets $A_i \subset M$ with $\mu(A_i) > 0$, $i = 1, ..., \ell$ if $N(n)$ is the number of events $\cap_{i=1}^{\ell}\{\xi_{q_i(k)} \in A_i\}$ for k running between 1 and n then $n^{-1/2}(N(n) - \prod_{i=1}^{\ell} \mu(A_i))$ is asymptotically normal. Our SPLITs seem to be new even when $X_i(n)$, $n = 0, 1, 2, ...$ are independent identically distributed (i.i.d.) random variables though in this case the proof is much easier and the result holds true in more general circumstances. Another important class of processes satisfying our conditions comes from dynamical systems where $X_i(n) = f_i(T^n x)$ with T being a topologically mixing subshift of finite type or a C^2 expanding endomorphism or an Axiom A (in particular, Anosov) diffeomorphisms considered in a neighborhood of an attractor taken with a Gibbs invariant measure.

Basic (splitting) inequalities

Y and Z are $\mathcal{F}_{-\infty,k}$- and $\mathcal{F}_{k+n,\infty}$-measurable, respectively, then

$$|E(YZ) - EYEZ| \leq 4\alpha(n)\|Y\|_\infty \|Z\|_\infty$$

Let $Y(j), j = 0,1,...$ be bounded random variables and

$$\beta(n) = \sup_{j\geq 0} E\big|Y(j) - E(Y(j)|\mathcal{F}_{j-n,j+n})\big|$$

then for $0 \leq n_1 \leq ... \leq n_l < n_{l+1} \leq n_{l+2} \leq ... \leq n_m,$

$$\big|E\prod_{i=1}^m Y(n_i) - E\prod_{i=1}^l Y(n_i) E\prod_{i=l+1}^m Y(n_i)\big|$$
$$\leq (m\beta(k) + 4\alpha(k))\prod_{i=1}^m \max(1, \|Y(n_i)\|_\infty)$$

where $k = [(n_{l+1} - n_l)/3]$

variance:

- Set

$$R(n) = \prod_{j=1}^{\ell} X_j(q_j(n)) - \prod_{j=1}^{\ell} a_j$$

$$= \sum_{j=1}^{\ell} a_1 \cdots a_{j-1}(X_j(q_j(n)) - a_j)X_{j+1}(q_{j+1}(n)) \cdots X_\ell(q_\ell(n)).$$

Then

$$\lim_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N} R(n)\Big)^2 = \sigma^2$$

relying on estimates of $E\big(R(k_1)R(k_2)\big)$

Some estimates

we have

$$E\big(R(k_1)R(k_2)\big) = \sum_{j=1}^{\ell} a_1^2 \cdots a_{j-1}^2 EQ_{jj}(k_1,k_2)$$

$$+\sum_{\ell\geq j_2>j_1} a_1 \cdots a_{j_1-1} a_1 \cdots a_{j_2-1}\big(EQ_{j_1j_2}(k_1,k_2) + EQ_{j_1j_2}(k_2,k_1)\big)$$

and

$$Q_{j_1j_2}(k_1,k_2) = \prod_{i=1}^{2} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i}) X_{j_i+1}(q_{j_i+1}(k_i)) \cdots X_\ell(q_\ell(k_i))$$

Using splitting inequalities we get that $|EQ_{j_1j_2}(k_1,k_2)|$ is small if $j_1 \neq j_2$ or $|k_1 - k_2|$ is large and

$$|EQ_{jj}(k,k) - E\big(X_j(q_j(k)) - a_j\big)^2 E \prod_{i=1}^{\ell-j} X_{j+i}^2(q_{j+i}(k))|$$

is small and the limit of the previous slide follows.

more delicate Gaussian estimates

- Let $N \geq n > m \geq [N^{1-\gamma}] \geq n_0$. Then

$$|E\Big(\sum_{k=m+1}^{n} R(k)\Big)^2 - (n-m)\sigma^2| \leq \hat{C}$$

and

$$E\Big(\sum_{k=m+1}^{n} R(k)\Big)^4 \leq \tilde{C}(n-m)^2.$$

Block technique

- Set $\tau(N) = [N^{1-\varepsilon}], \ \theta(N) = [N^{1-L\varepsilon}], m(N) = [\frac{N}{\theta(N)+\tau(N)}]$ and

$$\Gamma_k(N) = \{n : \theta(N) + (k-1)(\theta(N) + \tau(N)) \le n \le k(\theta(N) + \tau(N))\}$$

and

$$\tilde{\Gamma}_k(N) = \{n : (k-1)(\theta(N) + \tau(N)) + 1 \le n \le \theta(N) + (k-1)(\theta(N) + \tau(N))\}.$$

Define for $k = 1, 2, ..., m(N)$:

$$Y_k = \sum_{n \in \Gamma_k(N)} R(n) \text{ and } Z_k = \sum_{n \in \tilde{\Gamma}_k(N)} R(n)$$

where $R(n) = \prod_{j=1}^{\ell} X_j(q_j(n)) - \prod_{j=1}^{\ell} a_j$

characteristic functions

- Set

$$\Phi_N(t) = E\exp\left(\frac{it}{\sqrt{N}}\sum_{n=0}^{N} R(n)\right)$$

  and

$$\Psi_N(t) = E\exp\left(\frac{it}{\sqrt{N}}\sum_{1\le n\le m(N)} Y_n\right)$$

  then

$$|\Phi_N(t) - \Psi_N(t)| \le \check{C}|t|(N^{-\varepsilon(\frac{L}{2}-1)} + N^{-\varepsilon/2})$$

  Set

$$\psi_N^{(k)}(t) = E\exp\left(\frac{it}{\sqrt{N}}Y_k\right),\ k \le m(N).$$

  Then (main estimate):

$$|\Psi_N(T) - \prod_{1\le k\le m(N)} \psi_N^{(k)}(t)| \le K_\varepsilon(t)N^{-\frac{\varepsilon}{2}\sqrt{N}}.$$

Main estimate: idea of the proof

1st step:

Proof. Set $\hat{Y}_k = Y_k + \tau(N)\prod_{j=1}^{\ell} a_j$,

$$\hat{\Psi}_N(t) = E\exp\Big(\frac{it}{\sqrt{N}}\sum_{1\le k\le m(N)}\hat{Y}_k\Big) \text{ and } \hat{\psi}_N^{(k)}(t) = E\exp\Big(\frac{it}{\sqrt{N}}\hat{Y}_k\Big).$$

Then, clearly,

$$|\Psi_N(t) - \prod_{1\le k\le m(N)}\psi_N^{(k)}(t)| = |\hat{\Psi}_N(t) - \prod_{1\le k\le m(N)}\hat{\psi}_N^{(k)}(t)|. \tag{4.3}$$

2nd step:

By the reminder formula for the Taylor expansion

$$|e^{iz} - \sum_{k=0}^{n} \frac{(iz)^k}{k!}| \le \frac{|z|^{n+1}}{(n+1)!}. \tag{4.4}$$

With the same $\varepsilon > 0$ as above set

$$n(N) = n_\varepsilon(N) = [N^{\frac{1}{2}+\varepsilon}] \tag{4.5}$$

and denote

$$I_N^{(k)}(t) = \sum_{l=0}^{n(N)} \frac{(it)^l}{N^{l/2} l!} \hat{Y}_k^l.$$

Then by (4.4),

$$|\exp\left(\frac{it}{\sqrt{N}} \hat{Y}_k\right) - I_N^{(k)}(t)| \le \frac{(|t| D\sqrt{N})^{n(N)+1}}{(n(N)+1)!} \le C_4^{n(N)} |t|^{n(N)} N^{-\varepsilon n(N)} \tag{4.6}$$

3-d step:

$$(4.7) \qquad |\hat{\Psi}_N(T) - \prod_{1\le k\le m(N)} \hat{\psi}_N^{(k)}(t)| \le J(t,N) + \delta(t,N)$$

where

$$J(t,N) = |E \prod_{1\le k\le m(N)} I_N^{(k)}(t) - \prod_{1\le k\le m(N)} EI_N^{(k)}(t)|$$

and

$$(4.8) \qquad \delta(t,N) = 2m(N)C_4^{n(N)}|t|^{n(N)}N^{-\varepsilon n(N)} \times (1 + C_4^{n(N)}|t|^{n(N)}N^{-\varepsilon n(N)})^{m(N)} \le C(\varepsilon,t)N^{-\frac{\varepsilon}{2}\sqrt{N}}$$

for some $C(\varepsilon,t) > 0$ independent of N.

It remains to estimate $J(t,N)$ which is the main point of the proof.

4^{th} step:

- Writing powers of sums as sums of products the estimate of $J(t,N)$ comes down to the estimate of

$$H_{l_1,\ldots,l_{m(N)}}(t,N) = |E \prod_{k=1}^{m(N)} \prod_{n\in\Gamma_k(N)} \prod_{j=1}^{\ell} X_j^{\sigma_n^{(k)}}(q_j(n))$$

$$- \prod_{k=1}^{m(N)} E \prod_{n\in\Gamma_k(N)} \prod_{j=1}^{\ell} X_j^{\sigma_n^{(k)}}(q_j(n))|.$$

Next, we change the order of products in the two expectations above so that the product $\prod_{j=1}^{\ell}$ appear immediately after the expectation and apply the "splitting" inequality ℓ times to the latter product for both expectations.

we obtain

$$|E\prod_{k=1}^{m(N)}\prod_{n\in\Gamma_k(N)}\prod_{j=1}^{\ell}X_j^{\sigma_n^{(k)}}(q_j(n))$$
$$-\prod_{j=1}^{\ell}E\prod_{k=1}^{m(N)}\prod_{n\in\Gamma_k(N)}X_j^{\sigma_n^{(k)}}(q_j(n))|$$
$$\leq \ell D^{\ell n(N)m(N)}\big(\ell n(N)m(N)\beta(\rho_6(N))+4\alpha(\rho_6(N))\big)$$

where

$$\rho_6(N)=[\frac{1}{3}(N^{1-L\varepsilon}-[N^{(1-\gamma)(1-L\varepsilon)}])]$$

and

$$|E\prod_{n\in\Gamma_k(N)}\prod_{j=1}^{\ell}X_j^{\sigma_n^{(k)}}(q_j(n))-\prod_{j=1}^{\ell}E\prod_{n\in\Gamma_k(N)}X_j^{\sigma_n^{(k)}}(q_j(n))|$$
$$\leq \ell D^{\ell n(N)}\big(\ell n(N)\beta(\rho_6(N))+4\alpha(\rho_6(N))\big).$$

Next,

- For each fixed j we apply the "splitting estimate" to the product $\prod_{k=1}^{m(N)}$ after the expectation and in view of the size of gaps Z_k between the
- blocks Y_k we obtain

$$|E\prod_{k=1}^{m(N)}\prod_{n\in\Gamma_k(N)}X_j^{\sigma_n^{(k)}}(q_j(n)) - \prod_{k=1}^{m(N)}E\prod_{n\in\Gamma_k(N)}X_j^{\sigma_n^{(k)}}(q_j(n))|$$
$$\leq m(N)D^{m(N)n(N)}\big(m(N)n(N)\beta([[N^{1-L\varepsilon}]/3]) + 4\alpha([[N^{1-L\varepsilon}]/3])\big).$$

Collecting the above estimates the main estimate follows.

Conclusion of proof:

- using

$$|e^{ix} - 1 - ix + \frac{x^2}{2}| \le |x|^3 \text{ and } |e^{-x} - 1 + x| \le x^2$$

- and Gaussian type estimates above we
- obtain

$$|\psi_N^{(k)}(t) - \exp\big(-\tfrac{\sigma^2 t^2 \tau(N)}{2N}\big)|$$
$$\le 2\ell D^\ell N^{\frac{1}{2}-\varepsilon}|t|\big(\ell\beta(\rho_6(N)) + 4\alpha(\rho_6(N))\big)$$
$$+\tilde{C}^{3/4}|t|^3 N^{-3\varepsilon/2} + \tfrac{\sigma^4 t^4}{4N^2}(\tau(N))^2$$

- and, finally,

$$|\prod_{1\le k\le m(N)} \psi_N^{(k)}(t) - \exp\big(-\tfrac{\sigma^2 t^2}{2}\big)| \le \tfrac{\sigma^2 t^2}{2}(1 - \tfrac{\tau(N)m(N)}{N})$$
$$+2\ell D^\ell N^{\frac{1}{2}-\varepsilon} m(N)|t|^3\big(\ell\beta(\rho_6(N)) + 4\alpha(\rho_6(N))\big)$$
$$+\tilde{C}^{3/4}|t|^3 N^{-3\varepsilon/2} m(N) + \tfrac{\sigma^4 t^4}{4N^2}(\tau(N))^2 m(N)$$

Concluding remark:

- the proof does not work when, for instance,

$$l = 2, q_1(n) = n, q_2(n) = 2n$$

Still, if $X(0), X(1), X(2), \ldots$ are i.i.d. then

$$N^{-1/2} \sum_{0 \le n \le N} \left(X(n)X(2n) - (EX(0))^2 \right)$$

is asymptotically normal!

This can be proved applying the standard clt for triangular series to $W_N = N^{-1/2} \sum_{0 \le \ell odd \le N} S_N(\ell)$

where

$$S_N(\ell) = \sum_{j: 1 \le 2^j \ell \le N} (X(2^j \ell) X(2^{j+1} \ell) - EX^2(0))$$