

# Metastability for Ginzburg-Landau-type SPDEs with space-time white noise

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## Metastability in physics

Examples:

- Supercooled liquid
- Supersaturated gas
- Wrongly magnetised ferromagnet



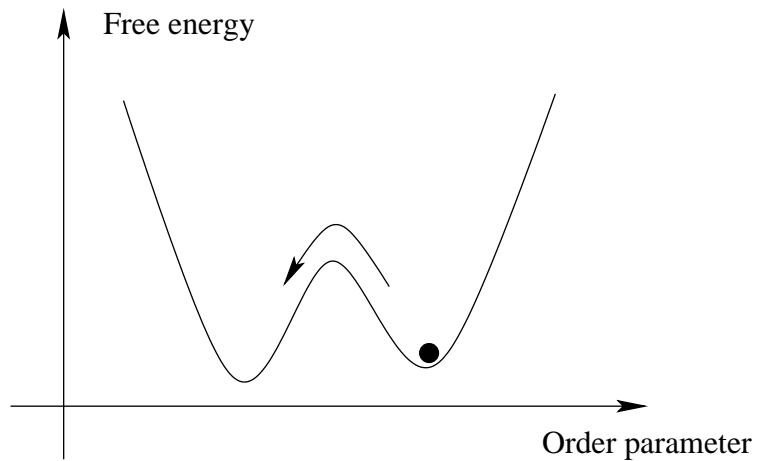
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Examples:

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- ▷ Near first-order phase transition
- ▷ Nucleation implies crossing energy barrier



## Metastability in stochastic lattice models

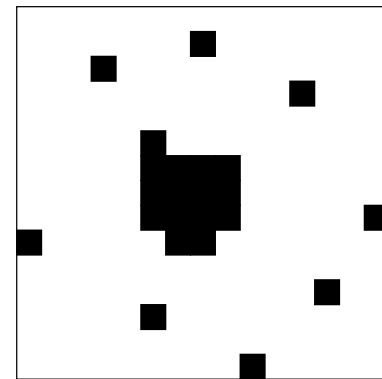
- ▷ Lattice:  $\Lambda \subset \mathbb{Z}^d$
- ▷ Configuration space:  $\mathcal{X} = S^\Lambda$ ,  $S$  finite set (e.g.  $\{-1, 1\}$ )
- ▷ Hamiltonian:  $H : \mathcal{X} \rightarrow \mathbb{R}$  (e.g. Ising or lattice gas)
- ▷ Gibbs measure:  $\mu_\beta(x) = e^{-\beta H(x)} / Z_\beta$
- ▷ Dynamics: Markov chain with invariant measure  $\mu_\beta$   
(e.g. Metropolis: Glauber or Kawasaki)

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Results (for  $\beta \gg 1$ ) on

- Transition time between  $+$  and  $-$  or empty and full configuration
- Transition path
- Shape of critical droplet



- ▷ Frank den Hollander, *Metastability under stochastic dynamics*, Stochastic Process. Appl. **114** (2004), 1–26.
- ▷ Enzo Olivieri and Maria Eulália Vares, *Large deviations and metastability*, Cambridge University Press, Cambridge, 2005.

## Reversible diffusion

$$dx_t = -\nabla V(x_t) dt + \sqrt{2\varepsilon} dW_t$$

- ▷  $V : \mathbb{R}^d \rightarrow \mathbb{R}$ : potential, growing at infinity
- ▷  $W_t$ :  $d$ -dim Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$

Reversible w.r.t.  
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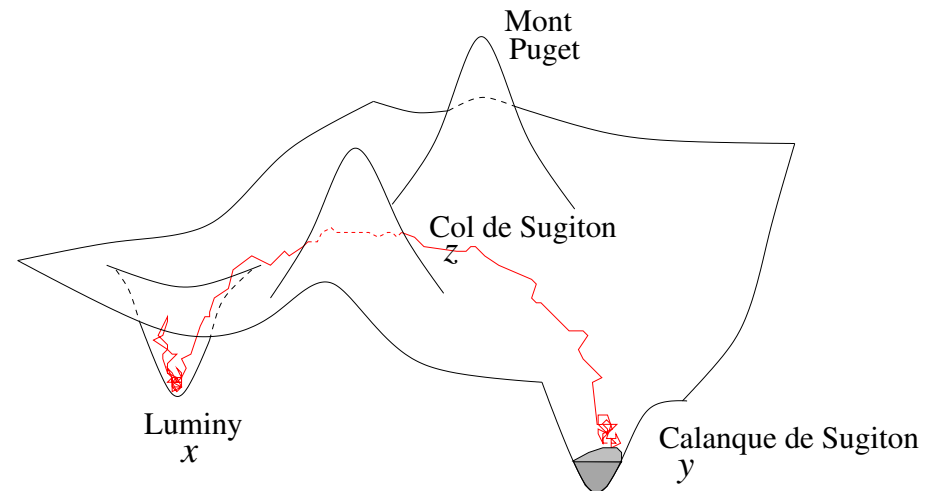
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$\tau_y^x$ : first-hitting time of small ball  $\mathcal{B}_\varepsilon(y)$ , starting in  $x$   
“Eyring–Kramers law” (Eyring 1935, Kramers 1940)

- Dim 1:  $\mathbb{E}[\tau_y^x] \simeq \frac{2\pi}{\sqrt{|V''(x)||V''(z)|}} e^{[V(z)-V(x)]/\varepsilon}$
- Dim  $\geq 2$ :  $\mathbb{E}[\tau_y^x] \simeq \frac{2\pi}{|\lambda_1(z)|} \sqrt{\frac{|\det(\nabla^2 V(z))|}{|\det(\nabla^2 V(x))|}} e^{[V(z)-V(x)]/\varepsilon}$

## Towards a proof of Kramers' law

- Large deviations (Wentzell & Freidlin 1969):

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log(\mathbb{E}[\tau_y^x]) = V(z) - V(x)$$

- Analytic (Helffer, Sjöstrand 85, Miclo 95, Mathieu 95, Kolokol'tsov 96, . . .):  
low-lying spectrum of generator
- Potential theory/variational (Bovier, Eckhoff, Gaynard, Klein 2004):

$$\mathbb{E}[\tau_y^x] = \frac{2\pi}{|\lambda_1(z)|} \sqrt{\frac{|\det(\nabla^2 V(z))|}{\det(\nabla^2 V(x))}} e^{[V(z)-V(x)]/\varepsilon} \left[ 1 + \mathcal{O}(\varepsilon^{1/2} |\log \varepsilon|^{1/2}) \right]$$

and similar asymptotics for eigenvalues of generator

- Witten complex (Helffer, Klein, Nier 2004):  
full asymptotic expansion of prefactor
- Distribution of  $\tau_y^x$  (Day 1983, Bovier *et al* 2005):

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\left\{ \tau_y^x > t \mathbb{E}[\tau_y^x] \right\} = e^{-t}$$



## Ginzburg–Landau equation

$$\partial_t u(x, t) = \partial_{xx} u(x, t) + u(x, t) - u(x, t)^3 + \text{noise}$$

$x \in [0, L]$ ,  $u(x, t) \in \mathbb{R}$  represents e.g. magnetisation

- Periodic b.c.
- Neumann b.c.  $\partial_x u(0, t) = \partial_x u(L, t) = 0$

**Noise:** weak, white in space and time

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Deterministic system is gradient

$$\partial_{xx} u + u - u^3 = -\frac{\delta V}{\delta u}$$
$$V[u] = \int_0^L \left[ \frac{1}{2} u'(x)^2 - \frac{1}{2} u(x)^2 + \frac{1}{4} u(x)^4 \right] dx \quad \rightarrow \quad \min$$

## Stationary solutions

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- $u_{\pm}(x) \equiv \pm 1$ : global minima of  $V$ , stable
- $u_0(x) \equiv 0$ : unstable
- **Neumann b.c**: for  $k = 1, 2, \dots$ , if  $L > \pi k$ ,  

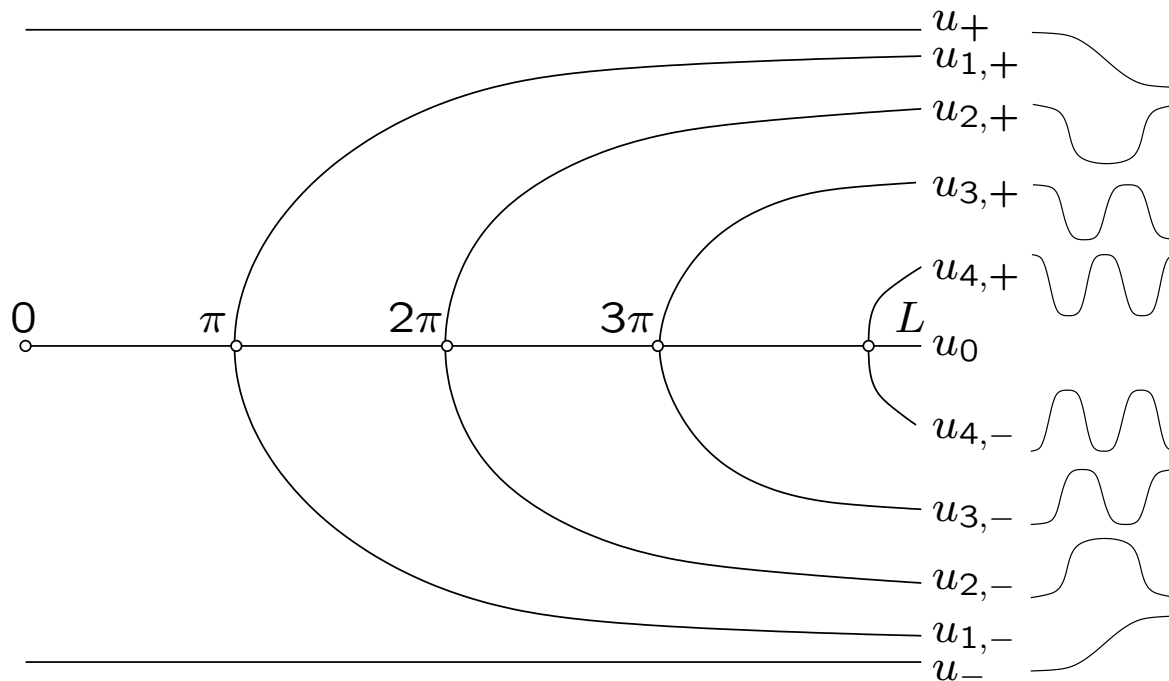
$$u_{k,\pm}(x) = \pm \sqrt{\frac{2m}{m+1}} \operatorname{sn}\left(\frac{kx}{\sqrt{m+1}} + K(m), m\right) \quad 2k\sqrt{m+1}K(m) = L$$

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## Stability of stationary solutions

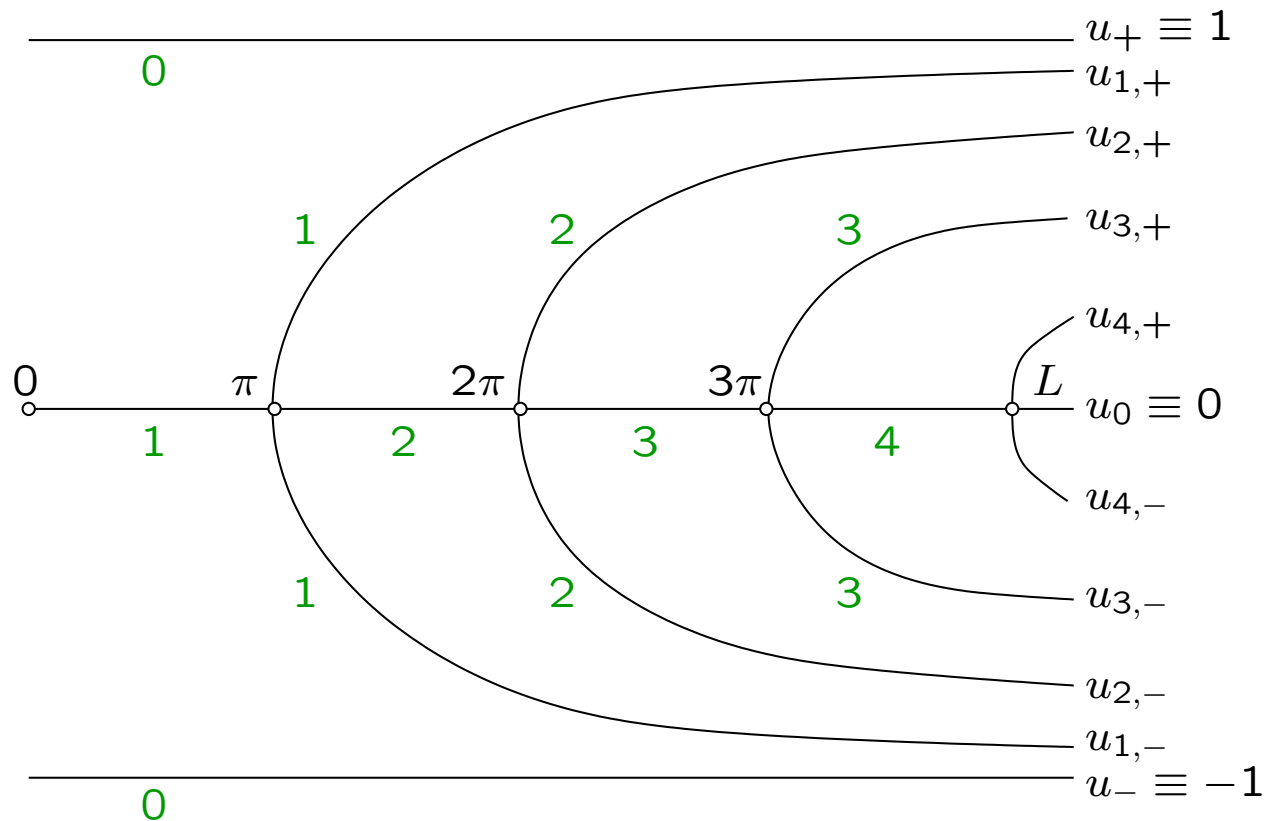
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- $u_{\pm}(x) \equiv \pm 1$ : eigenvalues  $-(2 + (\pi k/L)^2)$ ,  $k \in \mathbb{N}$
- $u_0(x) \equiv 0$ : eigenvalues  $1 - (\pi k/L)^2$ ,  $k \in \mathbb{N}$

Number of positive eigenvalues:



## Ginzburg–Landau equation with noise

$$\dot{u}_t(x) = \Delta u_t(x) + f(u_t(x)) + \sqrt{2\varepsilon} \ddot{W}_{tx} \quad (\Delta \equiv \partial_{xx}, f(u) = u - u^3)$$

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Construction of mild solution:

$$1. \dot{u}_t = \Delta u_t \quad \Rightarrow \quad u_t = e^{\Delta t} u_0$$

$$\text{where } e^{\Delta t} \cos(k\pi x/L) = e^{-(k\pi/L)^2 t} \cos(k\pi x/L)$$

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$$\Rightarrow \quad u_t = e^{\Delta t} u_0 + \underbrace{\sqrt{2\varepsilon} \int_0^t e^{\Delta(t-s)} \dot{W}_x(ds)}_{\doteq w_t(x)}$$

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$$3. \dot{u}_t = \Delta u_t + \sqrt{2\varepsilon} \ddot{W}_{tx} + f(u_t)$$

$$\Rightarrow \quad u_t = e^{\Delta t} u_0 + \sqrt{2\varepsilon} w_t + \int_0^t e^{\Delta(t-s)} f(u_s) ds =: F_t[u]$$

$\Rightarrow$  Existence and a.s. uniqueness (Faris & Jona-Lasinio 1982)

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Fourier variables:  $u_t(x) = \sum_{k=-\infty}^{\infty} z_k(t) e^{i\pi kx/L}$

$$\Rightarrow \quad dz_k = -\lambda_k z_k dt - \sum_{k_1+k_2+k_3=k} z_{k_1} z_{k_2} z_{k_3} dt + \sqrt{2\varepsilon} dW_t^{(k)}$$

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Energy functional:

$$\frac{1}{L} V[u] = \frac{1}{2} \sum_{k=-\infty}^{\infty} \lambda_k |z_k|^2 + \frac{1}{4} \sum_{k_1+k_2+k_3+k_4=0} z_{k_1} z_{k_2} z_{k_3} z_{k_4}$$

## The question

How long does the system take to get from  $u_-(x) \equiv -1$  to (a neighbourhood of)  $u_+(x) \equiv 1$ ?

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Large deviations (Faris & Jona-Lasinio 1982)

$$\triangleright L \leq \pi : \Delta W = V[u_0] - V[u_-] = \frac{L}{4}$$

$$\triangleright L > \pi : \Delta W = V[u_{1,\pm}] - V[u_-] = \frac{1}{3\sqrt{1+m}} \left[ 8 E(m) - \frac{(1-m)(3m+5)}{1+m} K(m) \right]$$



Formal computation for Ginzburg–Landau (R.S. Maier, D. Stein, 01)

Case  $L < \pi$ :

$$\text{Kramers: } \Gamma_0 \simeq \frac{|\lambda_1(u_0)|}{2\pi} \sqrt{\frac{\det(\nabla^2 V[u_-])}{|\det(\nabla^2 V[u_0])|}}$$

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**Problems:**

1. What happens when  $L \rightarrow \pi_-$ ? (bifurcation)
2. Is the formal computation correct in infinite dimension?

## Potential theory

Consider first Brownian motion  $W_t^x = x + W_t$

Let  $\tau_A^x = \inf\{t > 0 : W_t^x \in A\}$ ,  $A \subset \mathbb{R}^d$

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$$\Delta w_A(x) = 1 \quad x \in A^c$$

$$w_A(x) = 0 \quad x \in A$$

$$G_{A^c}(x, y) \text{ Green's function} \Rightarrow w_A(x) = \int_{A^c} G_{A^c}(x, y) \, dy$$

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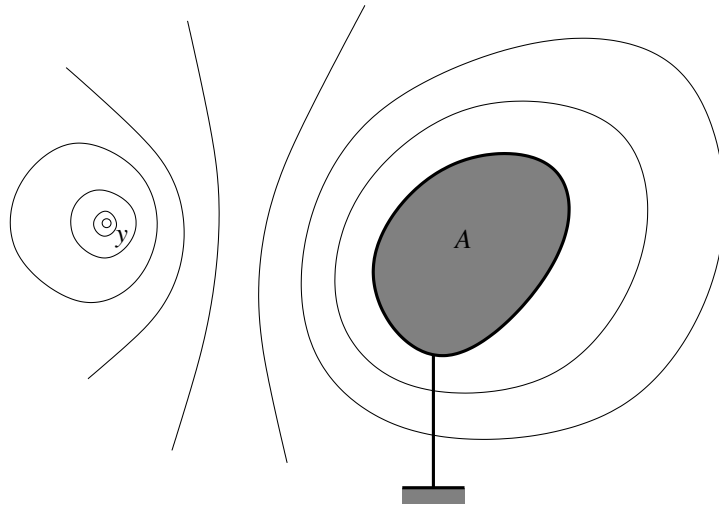
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## Potential theory

**Fact 2:**  $h_{A,B}(x) = \mathbb{P}[\tau_A^x < \tau_B^x]$  satisfies

$$\Delta h_{A,B}(x) = 0 \quad x \in (A \cup B)^c$$

$$h_{A,B}(x) = 1 \quad x \in A$$

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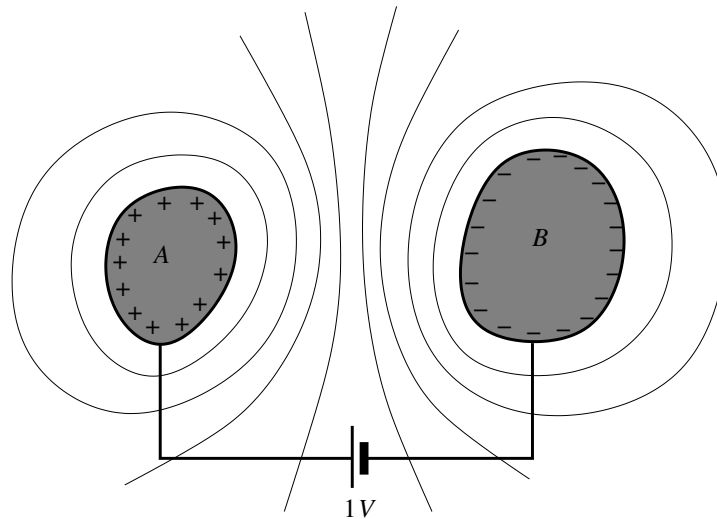
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$\rho_{A,B}$ : “surface charge density” on  $\partial A$





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Key observation: let  $C = \mathcal{B}_\varepsilon(x)$ , then (using  $G(y, z) = G(z, y)$ )

$$\begin{aligned} \int_{A^c} h_{C,A}(y) \, dy &= \int_{A^c} \int_{\partial C} G_{A^c}(y, z) \rho_{C,A}(dz) \, dy \\ &= \int_{\partial C} w_A(z) \rho_{C,A}(dz) \simeq w_A(x) \text{cap}_C(A) \end{aligned}$$

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Variational representation: Dirichlet form

$$\text{cap}_A(B) = \int_{(A \cup B)^c} \|\nabla h_{A,B}(x)\|^2 \, dx = \inf_{h \in \mathcal{H}_{A,B}} \int_{(A \cup B)^c} \|\nabla h(x)\|^2 \, dx$$

( $\mathcal{H}_{A,B}$ : set of sufficiently smooth functions satisfying b.c.)

## Potential theory

General case:  $dx_t = -\nabla V(x_t) dt + \sqrt{2\varepsilon} dW_t$

Generator:  $\Delta \mapsto \varepsilon\Delta - \nabla V \cdot \nabla$

Then

$$\mathbb{E}[\tau_A^x] = w_A(x) \simeq \frac{\int_{A^c} h_{\mathcal{B}_\varepsilon(x), A}(y) e^{-V(y)/\varepsilon} dy}{\text{cap}_{\mathcal{B}_\varepsilon(x)}(A)}$$

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Rough a priori bounds on  $h$  show that if  $x$  potential minimum,

$$\int_{A^c} h_{\mathcal{B}_\varepsilon(x), A}(y) e^{-V(y)/\varepsilon} dy \simeq \frac{(2\pi\varepsilon)^{d/2} e^{-V(x)/\varepsilon}}{\sqrt{\det(\nabla^2 V(x))}}$$

## Estimation of capacity

Truncated energy functional: retain only modes with  $k \leq d$

$$\frac{1}{L}V[u] = -\frac{1}{2}z_0^2 + u_1(z_1) + \frac{1}{2} \sum_{k=2}^d \lambda_k |z_k|^2 + \dots$$

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**Theorem:** For all  $L < \pi$ ,

$$\text{cap}_{\mathcal{B}_\varepsilon(u_-)}(\mathcal{B}_\varepsilon(u_+)) = \varepsilon \frac{\int_{-\infty}^{\infty} e^{-u_1(z_1)/\varepsilon} dz_1}{\sqrt{2\pi\varepsilon}} \prod_{j=2}^d \sqrt{\frac{2\pi\varepsilon}{\lambda_j}} [1 + R(\varepsilon)]$$

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**Corollary:**

Expected first-passage time converges in the limit  $d \rightarrow \infty$

(Liu, 2003)

## Sketch of proof

### Upper bound:

$$\text{cap} = \inf_h \Phi(h) \leq \Phi(h_+) \quad \Phi(h) = \varepsilon \int \|\nabla h(z)\|^2 e^{-V(z)/\varepsilon} dz$$

Let  $\delta = \sqrt{c\varepsilon|\log \varepsilon|}$ , choose

$$h_+(z) = \begin{cases} 1 & \text{for } z_0 < -\delta \\ f(z_0) & \text{for } -\delta < z_0 < \delta \\ 0 & \text{for } z_0 > \delta \end{cases}$$

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### Lower bound:

Bound Dirichlet  $\Phi$  form below by restricting domain, taking only 1st component of gradient and use for b.c. a priori bound on  $h$

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$L < \pi$ :

$$\Gamma_0 = \frac{1}{2^{3/4}\pi} \sqrt{\frac{\sinh(\sqrt{2}L)}{\sin L}}$$

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where  $\lambda_1 = -1 + (\pi/L)^2$  and

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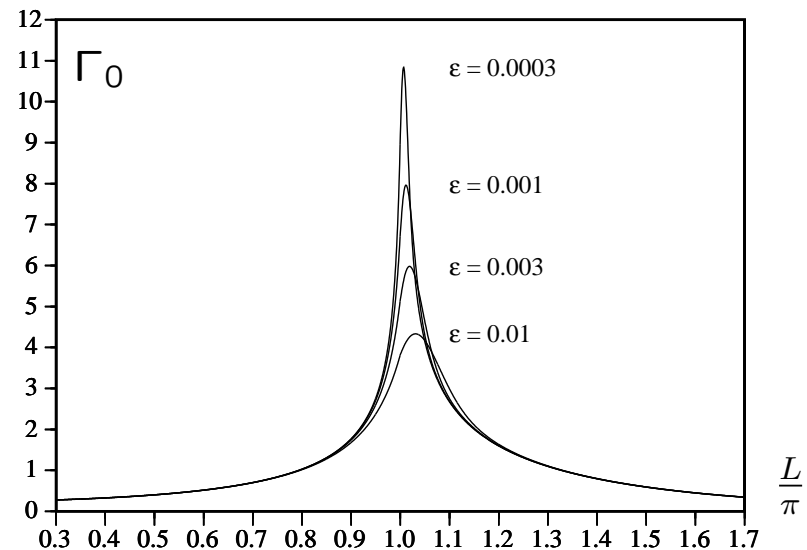
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Similar expression for  $L > \pi$

(product of eigenvalues computed using path-integral techniques, cf. Maier and Stein)



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