Convergence of the stochastic Euler scheme for locally Lipschitz coefficients

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Overview

Overview

Consider
- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $T > 0$
- a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion $W : [0,T] \times \Omega \to \mathbb{R}$
- continuous functions $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ and
- a $\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable mapping $\xi : \Omega \to \mathbb{R}$.

Then let $X : [0,T] \times \Omega \to \mathbb{R}$ be an adapted stochastic process with continuous sample paths which fulfills

$$X_t = \xi + \int_0^t \mu(X_s)\,ds + \int_o^t \sigma(X_s)\,dW_s \qquad \mathbb{P}\text{-a.s.}$$

for all $t \in [0,T]$. Short form:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad X_0 = \xi, \qquad t \in [0,T].$$

Examples of SDEs I

Black-Scholes model with $\bar{\mu}, \bar{\sigma}, x_0 \in (0, \infty)$:

$$dX_t = \bar{\mu}\, X_t\, dt + \bar{\sigma}\, X_t\, dW_t\,, \qquad X_0 = x_0, \qquad t \in [0, T]$$

A SDE with a cubic drift and additive noise:

$$dX_t = -X_t^3 dt + dW_t\,, \qquad X_0 = 0, \qquad t \in [0, 1]$$

A SDE with a cubic drift and multiplicative noise:

$$dX_t = -X_t^3 dt + 6\, X_t \circ dW_t\,, \qquad X_0 = 1, \qquad t \in [0, 3]$$

Examples of SDEs II

A stochastic Verhulst equation with $\eta, x_0 \in (0, \infty)$:

$$dX_t = X_t \left(\eta - X_t\right) dt + X_t \, dW_t, \quad X_0 = x_0, \quad t \in [0, T]$$

A Feller diffusion with logistic growth with $\eta, x_0 \in (0, \infty)$:

$$dX_t = X_t \left(\eta - X_t\right) dt + \sqrt{X_t} \, dW_t, \qquad X_0 = x_0, \qquad t \in [0, T]$$

Overview

Weak approximation problem of the SDE (see e.g. Kloeden & Platen (1992))

Suppose we want to compute

$$\mathbb{E}\Big[f(X_T)\Big]$$

for a given smooth function $f : \mathbb{R} \to \mathbb{R}$ whose derivatives grow at most polynomially.

For instance, $f(x) = x^2$ for all $x \in \mathbb{R}$ and we want to compute

$$\mathbb{E}\Big[(X_T)^2\Big]$$

the second moment of the SDE.

Approximation of $\mathbb{E}\big[f(X_T)\big]$

The **stochastic Euler scheme** $Y_k^N \colon \Omega \to \mathbb{R}$, $k \in \{0, 1, \dots, N\}$, $N \in \mathbb{N}$, is given by $Y_0^N(\omega) = \xi(\omega)$ and

$$
\begin{aligned}
& Y_{k+1}^N(\omega) \\
& = Y_k^N(\omega) + \frac{T}{N} \cdot \mu\big(Y_k^N(\omega)\big) + \sigma\big(Y_k^N(\omega)\big) \cdot \Big(W_{\frac{(k+1)T}{N}}(\omega) - W_{\frac{kT}{N}}(\omega)\Big)
\end{aligned}
$$

for all $\omega \in \Omega$, $k \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$. Let $Y_k^{N,m} \colon \Omega \to \mathbb{R}$, $k \in \{0, 1, \dots, N\}$, $N \in \mathbb{N}$, for $m \in \mathbb{N}$ be independent copies of the Euler approximations. The **Monte Carlo Euler approximation** is then given by

$$
\frac{1}{M} \left(\sum_{m=1}^{M} f(Y_N^{N,m}) \right) \approx \mathbb{E}\Big[f(X_T)\Big]
$$

with $N \in \mathbb{N}$ time steps and $M \in \mathbb{N}$ Monte Carlo runs.

Approximation of $\mathbb{E}\big[f(X_T)\big]$

The **stochastic Euler scheme** $Y_k^N \colon \Omega \to \mathbb{R}$, $k \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, is given by $Y_0^N(\omega) = \xi(\omega)$ and

$$\begin{aligned} &Y_{k+1}^N(\omega) \\ &= Y_k^N(\omega) + \frac{T}{N} \cdot \mu\big(Y_k^N(\omega)\big) + \sigma\big(Y_k^N(\omega)\big) \cdot \Big(W_{\frac{(k+1)T}{N}}(\omega) - W_{\frac{kT}{N}}(\omega)\Big) \end{aligned}$$

for all $\omega \in \Omega$, $k \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Let $Y_k^{N,m} \colon \Omega \to \mathbb{R}$, $k \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, for $m \in \mathbb{N}$ be independent copies of the Euler approximations. The **Monte Carlo Euler approximation** is then given by

$$\frac{1}{N^2}\left(\sum_{m=1}^{N^2} f(Y_N^{N,m})\right) \approx \mathbb{E}\big[f(X_T)\big]$$

with $N \in \mathbb{N}$ time steps and $N^2 \in \mathbb{N}$ Monte Carlo runs.

Overview

The triangle inequality shows

$$\left| \mathbb{E}\Big[f\big(X_T\big) \Big] - \frac{1}{N^2} \sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big) \right| \\ \leq \underbrace{\left| \mathbb{E}\Big[f\big(X_T\big) \Big] - \mathbb{E}\Big[f\big(Y_N^N\big) \Big] \right|}_{\text{time discretization error}} + \underbrace{\left| \mathbb{E}\Big[f\big(Y_N^N\big) \Big] - \frac{1}{N^2} \sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big) \right|}_{\text{statistical error}} \tag{1}$$

for all $N \in \mathbb{N}$.

The stochastic Euler scheme converges in the **numerically weak sense** if

$$\lim_{N \to \infty} \left| \mathbb{E}\Big[f\big(X_T\big) \Big] - \mathbb{E}\Big[f\big(Y_N^N\big) \Big] \right| = 0 \tag{2}$$

holds for every smooth function $f : \mathbb{R} \to \mathbb{R}$ whose derivatives have at most polynomial growth (see e.g. Kloeden & Platen (1992), Milstein (1995), Talay (1996), Higham (2001), Rössler (2003)).

Numerically weak convergence

Theorem (see e.g. Kloeden & Platen (1992))

*Let $\mu, \sigma, f : \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable with at most polynomially growing derivatives. Moreover, let $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ be **globally Lipschitz continuous**. Then there is a real number $C > 0$ such that*

$$\left| \mathbb{E}\Big[f(X_T) \Big] - \mathbb{E}\Big[f(Y_N^N) \Big] \right| \leq C \cdot \frac{1}{N}$$

holds for all $N \in \mathbb{N}$.

The stochastic Euler scheme converges in the numerically weak sense if the coefficients of the SDE are smooth and globally Lipschitz continuous.

Numerically weak convergence yields

$$\left| \mathbb{E}\Big[f\big(X_T\big) \Big] - \frac{1}{N^2} \sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big) \right|$$

$$\leq \left| \mathbb{E}\Big[f\big(X_T\big) \Big] - \mathbb{E}\Big[f\big(Y_N^N\big) \Big] \right| + \left| \mathbb{E}\Big[f\big(Y_N^N\big) \Big] - \frac{1}{N^2} \sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big) \right|$$

$$\leq C \cdot \frac{1}{N} + C_\varepsilon \cdot \frac{1}{N^{(1-\varepsilon)}} \leq \big(C + C_\varepsilon\big) \cdot \frac{1}{N^{(1-\varepsilon)}} \qquad \mathbb{P}-\text{a.s.}$$

for all $N \in \mathbb{N}$ and all $\varepsilon \in (0, 1)$ with an appropriate constant $C \in (0, \infty)$ and appropriate random variables $C_\varepsilon : \Omega \to [0, \infty)$, $\varepsilon \in (0, 1)$.

The Monte Carlo Euler method converges if the coefficients of the SDE are smooth and globally Lipschitz continuous.

Examples of SDEs I

The global Lipschitz assumption on the coefficients of the SDE is a serious shortcoming:

Black-Scholes model with $\bar{\mu}, \bar{\sigma}, x_0 \in (0, \infty)$:

$$dX_t = \bar{\mu}\, X_t\, dt + \bar{\sigma}\, X_t\, dW_t\,, \qquad X_0 = x_0, \qquad t \in [0, T]$$

Overview

Theorem (Hutzenthaler & J (2009))

Suppose $\mathbb{P}\big[\sigma(\xi) \neq 0\big] > 0$ *and let* $\alpha, C > 1$ *be such that*

$$|\mu(x)| \geq \frac{|x|^{\alpha}}{C} \qquad \text{and} \qquad |\sigma(x)| \leq C|x|$$

holds for all $|x| \geq C$*. If the exact solution of the SDE satisfies* $\mathbb{E}\Big[|X_T|^p\Big] < \infty$ *for one* $p \in [1, \infty)$*, then*

$$\lim_{N\to\infty} \mathbb{E}\Big[\big|X_T - Y_N^N\big|^p\Big] = \infty, \quad \lim_{N\to\infty} \Big|\mathbb{E}\Big[|X_T|^p\Big] - \mathbb{E}\Big[\big|Y_N^N\big|^p\Big]\Big| = \infty$$

holds.

Strong and numerically weak convergence fails to hold if the diffusion coefficient grows at most linearly and the drift coefficient grows superlinearly.

Examples of SDEs I

Divergence of Euler's method

$$\lim_{N\to\infty} \mathbb{E}\left|X_T - Y_N^N\right| = \infty, \qquad \lim_{N\to\infty} \left|\mathbb{E}\left[(X_T)^2\right] - \mathbb{E}\left[\left(Y_N^N\right)^2\right]\right| = \infty$$

holds for:

A SDE with a cubic drift and additive noise:

$$dX_t = -X_t^3 dt + dW_t, \qquad X_0 = 0, \qquad t \in [0,1]$$

A SDE with a cubic drift and multiplicative noise:

$$dX_t = -X_t^3 dt + 6\,X_t \circ dW_t, \qquad X_0 = 1, \qquad t \in [0,3]$$

Examples of SDEs II

Divergence of Euler's method

$$\lim_{N\to\infty} \mathbb{E}\left|X_T - Y_N^N\right| = \infty, \quad \lim_{N\to\infty} \left|\mathbb{E}\left[\left(X_T\right)^2\right] - \mathbb{E}\left[\left(Y_N^N\right)^2\right]\right| = \infty$$

holds for:

A stochastic Verhulst equation with $\eta, x_0 \in (0,\infty)$:

$$dX_t = X_t\left(\eta - X_t\right) dt + X_t\, dW_t, \quad X_0 = x_0, \quad t \in [0,T]$$

A Feller diffusion with logistic growth with $\eta, x_0 \in (0,\infty)$:

$$dX_t = X_t\left(\eta - X_t\right) dt + \sqrt{X_t}\, dW_t, \quad X_0 = x_0, \quad t \in [0,T]$$

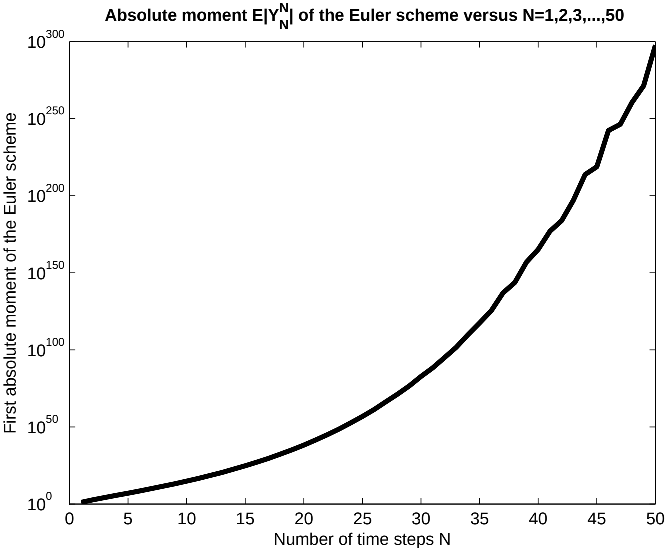
Simulations of the first absolute moment of the solution of a SDE

Consider the SDE

$$dX_t = -10 \operatorname{sgn}(X_t) \left|X_t\right|^{1.1} dt + 4 \, dW_t, \qquad X_0 = 0, \qquad t \in [0, 10].$$

The first absolute moment of X_T with $T = 10$ satisfies

$$\mathbb{E}\Big[\big|X_{10}\big|\Big] \approx 0.7141 \, .$$


Absolute moment E|Y_N^N| of the Euler scheme versus N=1,2,3,...,50
10^300
10^250
10^200
10^150
10^100
10^50
10^0
First absolute moment of the Euler scheme
0
5
10
15
20
25
30
35
40
45
50
Number of time steps N

Simulations for a SDE with a cubic drift and multiplicative noise

Consider the SDE

$$dX_t = -X_t^3 \, dt + 6 \, X_t \circ dW_t \,, \qquad X_0 = 1, \qquad t \in [0, 3].$$

The second moment of X_T with $T = 3$ satisfies

$$\mathbb{E}\left[(X_3)^2\right] \approx 1.5423 \,.$$

Different simulation values of the Monte Carlo Euler method with 300 time steps and 10 000 Monte Carlo runs:

NaN	0.5097	NaN	0.5378	0.5197
0.5243	NaN	NaN	0.5475	NaN

Proof of divergence of Euler's method in the numerically weak sense

For simplicity we restrict our attention to the SDE

$$dX_t = -X_t^3 dt + dW_t, \qquad X_0 = 0, \qquad t \in [0,1]$$

and show

$$\lim_{N\to\infty} \mathbb{E}\Big[\big|X_T - Y_N^N\big|^p\Big] = \infty, \quad \lim_{N\to\infty} \Big|\mathbb{E}\Big[\big|X_T\big|^p\Big] - \mathbb{E}\Big[\big|Y_N^N\big|^p\Big]\Big| = \infty$$

for every $p \in [1,\infty)$. Of course, it remains to show

$$\lim_{N\to\infty} \mathbb{E}\big|Y_N^N\big| = \infty.$$

Proof: Define

$$\Omega_N := \left\{ \omega \in \Omega \,\middle|\, \sup_{k \in \{1,2,\dots,N-1\}} \left| W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega) \right| \leq 1, \right.$$
$$\left. \left| W_{\frac{1}{N}}(\omega) - W_0(\omega) \right| \geq 3N \right\}$$

for every $N \in \mathbb{N}$. Claim:

$$\left| \mathbf{Y}_{\mathbf{k}}^{\mathbf{N}}(\omega) \right| \geq (\mathbf{3N})^{\left(\mathbf{2}^{(\mathbf{k}-\mathbf{1})}\right)} \qquad \forall\, \mathbf{k} \in \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{N}\} \tag{3}$$

for every $\omega \in \Omega_N$ and every $N \in \mathbb{N}$.
We fix $N \in \mathbb{N}$, $\omega \in \Omega_N$ and show (3) by induction on $k \in \{1, 2, \dots, N\}$.

$$\begin{aligned} \left| Y_1^N(\omega) \right| &= \left| Y_0^N(\omega) - \frac{1}{N} \left(Y_0^N(\omega) \right)^3 + \left(W_{\frac{1}{N}}(\omega) - W_0(\omega) \right) \right| \\ &= \left| W_{\frac{1}{N}}(\omega) - W_0(\omega) \right| \geq 3N \end{aligned}$$

Induction hypothesis $|\mathbf{Y}_\mathbf{k}^\mathbf{N}(\omega)| \geq (\mathbf{3N})^{\left(\mathbf{2}^{(\mathbf{k}-\mathbf{1})}\right)}$ for one $k \in \{1, 2, \ldots, N\}$:

$$\begin{aligned}
\left|Y_{k+1}^N(\omega)\right| &= \left|Y_k^N(\omega) - \frac{1}{N}\left(Y_k^N(\omega)\right)^3 + \left(W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right)\right| \\
&\geq \left|\frac{1}{N}\left(Y_k^N(\omega)\right)^3\right| - \left|Y_k^N(\omega)\right| - \left|W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right| \\
&\geq \frac{1}{N}\left|Y_k^N(\omega)\right|^3 - \left|Y_k^N(\omega)\right| - 1 \\
&\geq \frac{1}{N}\left|Y_k^N(\omega)\right|^3 - 2\left|Y_k^N(\omega)\right|^2 \\
&\geq \left|Y_k^N(\omega)\right|^2\left(\frac{1}{N}\left|Y_k^N(\omega)\right| - 2\right) \\
&\geq \left|Y_k^N(\omega)\right|^2\left(\frac{1}{N}3N - 2\right) = \left|Y_k^N(\omega)\right|^2 \\
&\geq \left((3N)^{\left(2^{k-1}\right)}\right)^2 = (3N)^{\left(2^k\right)}
\end{aligned}$$

In particular, we obtain

$$\left|Y_N^N(\omega)\right| \geq (3N)^{\left(2^{(N-1)}\right)} \tag{4}$$

for all $\omega \in \Omega_N$ and all $N \in \mathbb{N}$. Recall that

$$\Omega_N = \Bigg\{ \omega \in \Omega \,\Bigg|\, \sup_{k \in \{1,\dots,N-1\}} \left| W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega) \right| \leq 1, \quad \left| W_{\frac{1}{N}}(\omega) - W_0(\omega) \right| \geq 3N \Bigg\}$$

holds and therefore

$$\mathbb{P}\big[\Omega_N\big] \geq e^{-cN^3} \tag{5}$$

for all $N \in \mathbb{N}$ with $c \in (0, \infty)$ appropriate. Combining (4) and (5) shows

$$\mathbb{E}\left|Y_N^N\right| \geq \mathbb{P}\big[\Omega_N\big] \cdot (3N)^{\left(2^{(N-1)}\right)} \geq e^{-cN^2} \cdot (3N)^{\left(2^{(N-1)}\right)} \xrightarrow{N \to \infty} \infty. \qquad \square$$

Do we need **new numerical methods** which converge in the numerically weak sense?

The Monte Carlo Euler method works very well **in practice**!

Theorem (Hutzenthaler & J (2009))

Suppose that $\mu, \sigma, f \colon \mathbb{R} \to \mathbb{R}$ are four times continuously differentiable functions with at most polynomially growing derivatives. Moreover, let σ be globally Lipschitz continuous and let μ be ***globally one-sided Lipschitz continuous****, i.e.,*

$$(x - y) \cdot (\mu(x) - \mu(y)) \leq L\,(x - y)^2$$

holds for all $x, y \in \mathbb{R}$, where $L \in (0, \infty)$ is a fixed constant. Then there are $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $C_\varepsilon \colon \Omega \to [0, \infty)$, $\varepsilon \in (0,1)$, and a set $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}[\tilde{\Omega}] = 1$ such that

$$\left| \mathbb{E}\Big[f(X_T) \Big] - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f(Y_N^{N,m}(\omega)) \right) \right| \leq C_\varepsilon(\omega) \cdot \frac{1}{N^{(1-\varepsilon)}}$$

holds for every $\omega \in \tilde{\Omega}$, $N \in \mathbb{N}$ and every $\varepsilon \in (0,1)$.

The theorem applies to ...

Black-Scholes model with $\bar{\mu}, \bar{\sigma}, x_0 \in (0, \infty)$:

$$dX_t = \bar{\mu}\, X_t\, dt + \bar{\sigma}\, X_t\, dW_t, \qquad X_0 = x_0, \qquad t \in [0, T]$$

A SDE with a cubic drift and additive noise:

$$dX_t = -X_t^3 dt + dW_t, \qquad X_0 = 0, \qquad t \in [0, 1]$$

A SDE with a cubic drift and multiplicative noise:

$$dX_t = -X_t^3 dt + 6\, X_t \circ dW_t, \qquad X_0 = 1, \qquad t \in [0, 3]$$

A stochastic Verhulst equation with $\eta, x_0 \in (0, \infty)$:

$$dX_t = X_t\, (\eta - X_t)\, dt + X_t\, dW_t, \quad X_0 = x_0, \quad t \in [0, T]$$

Sketch of the proof:

For simplicity we restrict our attention again to the SDE

$$dX_t = -X_t^3 \, dt + dW_t , \qquad X_0 = 0$$

for $t \in [0, T]$ with $T = 1$.

Define the events $\Omega_N \in \mathcal{F}$, $N \in \mathbb{N}$, given by

$$\Omega_N := \left\{ \omega \in \Omega \,\middle|\, \sup_{0 \le t \le T} |W_t(\omega)| \le \sqrt{N/2} \right\}$$

for all $N \in \mathbb{N}$. Moreover, define $\tau_n^N : \Omega \to \{0, 1, \ldots, N\}$ by

$$\tau_n^N(\omega) := \max \left(\{0\} \cup \left\{ k \in \{1, 2, \ldots, n\} \,\middle|\, \operatorname{sgn}(Y_{k-1}^N(\omega)) \neq \operatorname{sgn}(Y_k^N(\omega)) \right\} \right)$$

for every $\omega \in \Omega$, $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$.

Then we obtain

$$
\begin{aligned}
Y_n^N(\omega) &= Y^N_{\tau_n^N(\omega)}(\omega) + \sum_{k=\tau_n^N(\omega)}^{n-1} \left(Y_{k+1}^N(\omega) - Y_k^N(\omega) \right) \\
&= Y^N_{\tau_n^N(\omega)}(\omega) + \sum_{k=\tau_n^N(\omega)}^{n-1} \left(-\frac{1}{N} \left(Y_k^N(\omega) \right)^3 + \left(W_{\frac{k+1}{N}} - W_{\frac{k}{N}} \right) \right) \\
&= Y^N_{\tau_n^N(\omega)}(\omega) - \frac{1}{N} \left(\sum_{k=\tau_n^N(\omega)}^{n-1} \left(Y_k^N(\omega) \right)^3 \right) + \left(W_{\frac{n}{N}} - W_{\frac{\tau_n^N(\omega)}{N}} \right)
\end{aligned}
$$

for every $\omega \in \Omega$, $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$. This implies

$$
\left| Y_n^N(\omega) \right| \leq \left| Y^N_{\tau_n^N(\omega)}(\omega) + \left(W_{\frac{n}{N}} - W_{\frac{\tau_n^N(\omega)}{N}} \right) \right|
$$

for every $\omega \in \Omega$, $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$.

We have

$$\left|Y_n^N(\omega)\right| \leq \left|Y^N_{\tau_n^N(\omega)}(\omega) + \left(W_{\frac{n}{N}} - W_{\frac{\tau_n^N(\omega)}{N}}\right)\right|$$

for every $\omega \in \Omega$, $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$.

This enables us to show **the domination of Euler's method by twice the supremum of the Brownian motion**

$$\sup_{k \in \{0,1,\ldots,N\}} \left|Y_k^N(\omega)\right| \leq 2 \left(\sup_{0 \leq t \leq T} |W_t(\omega)| \right)$$

for all $\omega \in \Omega_N$ and all $N \in \mathbb{N}$. The domination inequality can also be written as

$$\sup_{N \in \mathbb{N}} \sup_{k \in \{0,1,\ldots,N\}} \left(\mathbb{1}_{\Omega_N}(\omega) \cdot \left|Y_k^N(\omega)\right| \right) \leq 2 \left(\sup_{0 \leq t \leq T} |W_t(\omega)| \right)$$

for every $\omega \in \Omega$.

In particular, we obtain

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\mathbb{1}_{\Omega_N} \left|Y_N^N\right|^p\right] \leq 2^p \cdot \mathbb{E}\left[\sup_{0\leq t\leq T} \left|W_t\right|^p\right] < \infty$$

for all $p \in [1, \infty)$. This estimate complements the divergence

$$\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{(\Omega_N)^c} \left|Y_N^N\right|^p\right] = \infty$$

for all $p \in [1, \infty)$. Using now that

$$\mathbb{P}\left[(\Omega_N)^c\right] \leq e^{-cN}$$

holds for all $N \in \mathbb{N}$ with an appropriate constant $c \in (0, \infty)$, an adaption of the arguments in the global Lipschitz case yields the convergence of the Monte Carlo Euler method.

Simulations for a SDE with a cubic drift and additive noise

Consider the SDE

$$dX_t = -X_t^3 dt + dW_t, \qquad X_0 = 0, \qquad t \in [0, 1].$$

The second moment of X_T with $T = 1$ satisfies

$$\mathbb{E}\left[(X_3)^2\right] \approx 0.4529 \,.$$

Different simulation values of the Monte Carlo Euler method:

$N = 2^0$	$N = 2^1$	$N = 2^2$	$N = 2^3$	$N = 2^4$
1.4516	0.5166	0.4329	0.5308	0.4285

$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
0.4452	0.4602	0.4517	0.4548	0.4537

Summary

- **Counterexamples of numerically weak convergence** of the stochastic Euler scheme if the coefficients of the SDE grow superlinearly.
- **The Monte Carlo Euler method nevertheless converges** if the drift function is globally one-sided Lipschitz continuous, the diffusion function is globally Lipschitz continuous and both the drift and diffusion function are smooth with at most polynomially growing derivatives.

Remark: The situation is similar in the case of SPDEs and Multi-Level Monte Carlo.

Conclusion

Strong and numerically weak error estimates are convenient, since stochastic calculus is an L^2-calculus (Itô isometry, etc.).

But, if Euler's method is used to solve one of the nonlinear problems above, then one needs different concepts such as

$$\left| X_T - Y_N^N \right| \xrightarrow{N \to \infty} 0 \qquad \mathbb{P}\text{-a.s.}$$

for the strong approximation problem (Gyöngy (1998)) and

$$\left| \mathbb{E}\Big[f(X_T) \Big] - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f(Y_N^N) \right) \right| \xrightarrow{N \to \infty} 0 \qquad \mathbb{P}\text{-a.s.}$$

for the weak approximation problem (Hutzenthaler & J (2009)).

References

- Hutzenthaler and J (2009), Non-globally Lipschitz Counterexamples for the stochastic Euler scheme.
- Hutzenthaler and J (2009), Convergence of the stochastic Euler scheme for locally Lipschitz coefficients.