Stochastic Flows and Signed Measure Valued Stochastic Partial Differential Equations

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Abstract

We derive a class of quasi-linear stochastic partial differential equations (SPDEs) from the empirical distributions of stochastic flows of systems of stochastic ordinary differential equations (SODEs) on $\mathbb{R}^d$. We show that the solutions of the SODEs generate a.s. a homeomorphism from the initial conditions onto the solutions at time t. Generalizing results of Kotelenez (2008), Ch. 8,[1] from positive measure valued SPDEs to signed measure valued SPDEs, the solutions of the SPDEs may be represented as follows:

$$\mathcal{X}(t) = \mathcal{X}^+(t) - \mathcal{X}^-(t), \tag{$*$}$$

where $\mathcal{X}^+(t)$ and $\mathcal{X}^-(t)$ are positive measures with the following flow representation

$$\mathcal{X}^\pm(t) = \int_{\mathbb{R}^d} \delta_{(\bar{r}(t,\mathcal{X},q))}\mathcal{X}^\pm(0,dq). \tag{$**$}$$

$\mathcal{X}(0) = \mathcal{X}^+(0) - \mathcal{X}^-(0)$ is the initial distribution and $\mathcal{X}^\pm(0)$ is the Hahn-Jordan decomposition of $\mathcal{X}(0)$. Further, $\bar{r}(t,\mathcal{X},q)$ is the flow of solutions of the SODEs, depending on the "empirical distribution" $\mathcal{X}(\cdot)$ and the initial condition q. The flow properties imply that $\mathcal{X}^\pm(t)$ is the Hahn-Jordan decomposition of $\mathcal{X}(t)$. Smoothness and uniqueness hold for smooth initial conditions and smooth coefficients of the SODEs. This result has numerous applications in $2D$ fluid mechanics and other areas.

[1]Cf. Kotelenez, P. (2008) *Stochastic Ordinary and Stochastic Partial Differential Equations: Transition from Microscopic to Macroscopic Equations,* Springer-Verlag, Berlin-Heidelberg-New York.