Numerical Stability of Stochastic Differential Equations with Additive Noise

Peter Kloeden
Institut für Mathematik
Goethe Universität
Frankfurt am Main

$\star$ joint work with Evelyn Buckwar and Martin Riedler (Edinburg)

Deterministic case The propagation of roundoff error in a numerical scheme is often investigated for a class of complex-valued linear "test" ODEs

$$\frac{dz}{dt} = \lambda z$$

where $\lambda = \alpha + \imath\beta \in \mathbb{C}$, $z = x + \imath y \in \mathbb{C}$ and $\imath = \sqrt{-1}$.

Its solution for the initial value $z(0) = z_0$ is

$$z(t) = e^{\lambda t} z_0 = e^{\alpha t}\ e^{\imath\beta t} z_0$$

Thus

$$|z(t)| = e^{\alpha t}|z_0| \to 0 \quad \text{for} \quad t \to \infty \quad \forall z_0$$

if and only if

$$\boxed{\alpha = \mathrm{Re}(\lambda) < 0}.$$

in which case the zero solution $z(t) \equiv 0$ is called asymptotically stable.

A Runge-Kutta scheme with the Butcher Tableau

$$\begin{array}{c|c} c & A \\ \hline & b \end{array}$$

for the test ODE function $f(z) = \lambda z$ and constant step size h simplifies to

$$\boxed{z_{n+1} = R(h\lambda) z_n}$$

where

$$\boxed{R(z) = 1 + zb(I - zA)^{-1}\mathbf{1}}$$

i.e., a complex-valued mapping $R: \mathbb{C} \to \mathbb{C}$.

$\Rightarrow \quad |z_n| = |R(h\lambda)|^n \, |z_0| \to 0 \quad \text{for } n \to \infty \quad \text{if and only if} \quad |R(h\lambda)| < 1.$

Thus the zero solution $z_n \equiv 0 + \imath 0 \in \mathbb{C}$ of the Runge-Kutta scheme is also asymptotically stable and approximates the behaviour of a test ODE if and only if the step size $h > 0$ satisfies the inequality

$$\boxed{|R(h\lambda)| < 1}.$$

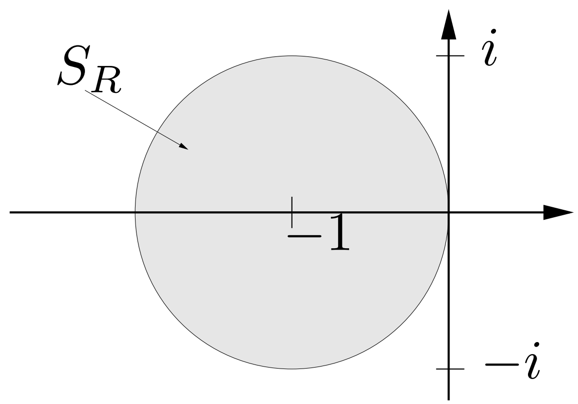
Definition *The set*

$$S_R := \{z \in \mathbb{C} : \ |R(z)| < 1\}$$

is called the stability region of the Runge-Kutta scheme with the mapping R.

Definition *A Runge-Kutta scheme with mapping R is said to be A-stable when*

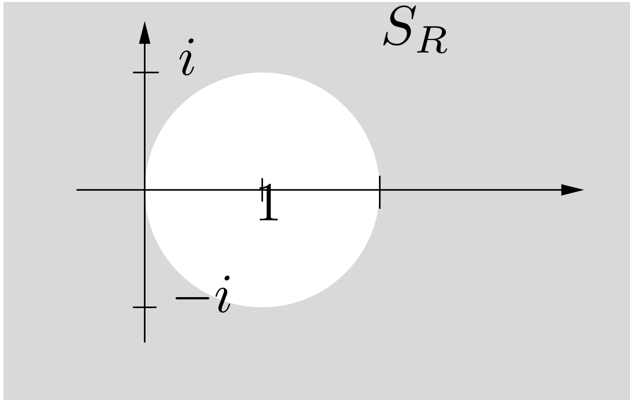
$$\mathbb{C}^- := \{z \in \mathbb{C} : \ \mathrm{Re}(z) < 0\} \subset S_R.$$



The explicit Euler scheme $\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$, $\qquad R(z) = 1 + z.$

$|R(z)| = |1 + z| < 1$ is the interior of the unit circle with centre $-1 + \imath 0$.

$\Rightarrow$ the explicit Euler scheme is not A-stable.



The implicit Euler scheme $\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$, $\qquad R(z) = \dfrac{1}{1-z}$

$|R(z)| < 1 \iff |1-z| > 1$ and S_R is the exterior of the unit circle with centre $1 + \imath 0$.

$\mathbb{C}^- \subset S_R \Rightarrow$ the implicit Euler scheme is A-stable.

Generalizations

A-stability is very strong (i.e. no restriction on the step size) and also very restrictive.

To investigate the preservation of asymptotic stability under discretisation for nonlinear systems test ODES are considered of the form

$$\frac{dx}{dt} = f(x),$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfies a one-sided dissipative Lipschitz condition, i.e. there exists a constant $L > 0$ such that

$$\boxed{\langle x - y, f(x) - f(y) \rangle \leq -L|x - y|^2} \qquad \forall\ x, y \in \mathbb{R}^d.$$

$\Rightarrow$ the ODE has a globally asymptotically stable steady state solution.

Stochastic differential equations with additive noise

Consider a linear SDE with additive noise

$$\boxed{dX_t = \lambda X_t \, dt + \sigma \, dW_t,} \tag{1}$$

where W_t is a standard two-sided Wiener process and constants λ, σ.

A 1-step numerical scheme with constant step size h for the linear SDE (1) has the form

$$\boxed{Y_{n+1} = R(h\lambda)Y_n + Z_n,} \tag{2}$$

where the Z_n are random variables that do not depend on Y_0, Y_1, $\cdots$ or λ, e.g. the Euler-Maruyama scheme

$$Y_{n+1} = (1 + h\lambda) \, Y_n + \Delta W_n, \quad \text{i.e.} \;\; R(z) \equiv z$$

Results in the literature

(I) Milstein & Tretyakov (2004) propose the following line of thought:

• If for $\sigma = 0$ the trivial solution of (1) is asymptotically stable, then for any $\sigma \neq 0$ a solution y_n of a numerical scheme (2) applied to the SDE (1) with $\mathbb{E}|X_0|^2 < \infty$ has second order moments that are uniformly bounded in n.

• Otherwise, if for $\sigma = 0$ the trivial solution of (1) is unstable, then the second order moments of the numerical iterates tend to infinity.

They conclude that the stability properties of a numerical scheme (2) applied to the SDE (1) can be deduced from the corresponding results on the scheme applied to the ODE obtained from the SDE (1) with $\sigma = 0$. Thus the stability concepts of deterministic numerical analysis can be transferred without modifications to the stochastic case.

But they do not say what they mean by an equilibrium solution.

(II) Kloeden & Platen (1992) defined A-stability of the numerical scheme (2) for a complex valued version of the test SDE (1) in terms of $|R(h\lambda)| < 1$ for a complex λ.

What is the equilibrium solution of the SDE (1) ?

What is the equilibrium solution of the numerical scheme (2) ?

(III) Hernández & Spigler (1992) show for the SDE (1) with $\Re\lambda < 0$ that there is a unique stationary complex Gaussian random process Z with zero mean and variance $\sigma^2/2|\Re\lambda|$ to which all other solutions decay exponentially (but they did not state in which sense).

They showed that a similar situation holds for the numerical scheme (2).

Essentially they consider an invariant measure of the numerical stationary process since the mean and the covariance may not provide enough information when the scheme does not have Gaussian solutions.

But they do not consider the question of convergence of the numerical stationary solution to that of the continuous time system.

(IV) Artemiev & Averina (1997) say that a numerical scheme is *asymptotically unbiased with step size* $h > 0$ if when applied to the linear test SDE (1) with $\lambda < 0$, the distribution of the numerical solution y_n converges as $n \to \infty$ to the normal distribution with zero mean and variance $\sigma^2/2|\lambda|$.

They note that the solution of SDE (1) is a Gaussian process whenever X_0 is Gaussian (or deterministic) and thus determined by its mean and variance. The property of asymptotic unbiasedness of a numerical scheme (assuming that it produces Gaussian iterates) can then be deduced from the mean and variance of the iterates:

$$\lim_{n\to\infty} \mathbb{E}y_n = 0, \qquad \lim_{n\to\infty} \mathbb{E}y_n^2 = \frac{\sigma^2}{2|\lambda|}.$$

Thus the asymptotic behaviour of a scheme described by the definition of asymptotically unbiasedness is consistent with the asymptotic behaviour of the distribution of the exact solution.

(V) Saito & Mitsui (1996, 2007) say that a numerical scheme with step size h is *numerically stable in mean* if the numerical solution $y_n^{(h)}$ applied to the SDE (1) satisfies $\mathbb{E}y_n^{(h)} \to 0$ for $n \to \infty$.

They show that the second moment of the Euler-Maruyama scheme schemes satisfies

$$\mathbb{E}|y_n^{(h)}|^2 \to \frac{\sigma^2}{2|\Re\lambda| + |\lambda|^2 h} \quad \text{for } n \to \infty$$

They note that the equilibrium value in mean square sense is different from the true value $\dfrac{\sigma^2}{2|\Re\lambda|}$, but converges to this value for $h \to 0$.

They proposes the following definition:

Definition: *A numerical scheme asymptotically consistent in mean square if the numerical solution* $y_n^{(h)}$ *for the test SDE* (1) *satisfies*

$$\lim_{h\to 0}\left(\lim_{n\to\infty}\mathbb{E}|y_n^{(h)}|^2\right)=\frac{\sigma^2}{2|\Re\lambda|}\,.$$

This is obviously true for the Euler-Maruyama scheme.

BUT note the double limit!

SHORTCOMING

None of the above references above provides

i) a rigorous justification that such a test equation that is appropriate in the stochastic setting,

ii) an equilibrium solution, whose stability properties can be discussed,

iii) a precise stability notion.

Explanation through Random Dynamical Systems

The linear scalar test SDE (1) has the explicit solution

$$X_t = e^{\lambda(t-t_0)} X_{t_0} + \sigma e^{\lambda t} \int_{t_0}^{t} e^{-\lambda s}\, dW_s \tag{3}$$

for all $t \geq t_0$.

Henceforth let $\lambda < 0$.

The expression (3) has no forward limit, but the pathwise pullback limit (i.e., as $t_0 \to -\infty$ with t fixed) exists and is given by

$$\widehat{O}_t := \sigma e^{\lambda t} \int_{-\infty}^{t} e^{-\lambda s}\, dW_s = \sigma e^{-|\lambda| t} \int_{-\infty}^{t} e^{|\lambda| s}\, dW_s. \tag{4}$$

This is known as the scalar Ornstein-Uhlenbeck stochastic stationary process. It is Gaussian with zero mean and variance $\sigma^2/2|\lambda|$.

The Ornstein-Uhlenbeck process (4) is a stochastic stationary solution of the linear test SDE (1).

Moreover, it also attracts all other solutions of the SDE forwards in time in the pathwise sense. To see this simply subtract one solution of (1) from another to obtain

$$X_t^1 - X_t^2 = e^{-\lambda(t-t_0)}\left(X_{t_0}^1 - X_{t_0}^2\right)$$

and then replace X_t^2 by the Ornstein-Uhlenbeck process $\widehat{O}_t$.

The Ornstein-Uhlenbeck process is thus the *equilibrium solution* alluded to in the papers discussed above.

A numerical scheme applied to the linear test SDE (1) also has a discrete time analog of the Ornstein-Uhlenbeck process, which pathwise attracts all other numerical solutions.

This is easily illustrated with the explicit Euler-Maruyama scheme with constant step size h,

$$X_{n+1} = (1 - |\lambda| h)\, X_n + \sigma \Delta W_n \,,$$

which has the explicit solution

$$X_n = (1 - |\lambda| h)^{n-n_0}\, X_{n_0} + \sigma \sum_{j=n_0}^{n-1} (1 - |\lambda| h)^{n-1-j}\, \Delta W_j .$$

The pathwise pullback limit (taking $n_0 \to -\infty$ with n held fixed and $X_{n_0} \equiv X_0$ for all n_0 and constant step size h) exists, provided that $0 < h < 2/|\lambda|$, and is given by

$$\widehat{O}_n^{(h)} := \sigma \sum_{j=-\infty}^{n-1} (1 - |\lambda| h)^{n-1-j} \, \Delta W_j \, . \tag{5}$$

This is a stochastic stationary solution and attracts all other solutions of the Euler-Maruyama scheme forwards in time in the pathwise sense, since the difference of two solutions satisfies

$$X_n^1 - X_n^2 = (1 - \lambda h)^{n-n_0} \left(X_{n_0}^1 - X_{n_0}^2 \right) \, .$$

Moreover, one can show that

$$\boxed{\widehat{O}_0^{(h)} \to \widehat{O}_0 \qquad \text{as } \; h \to 0}$$

in the pathwise sense, and hence for any other time instant.

The Ornstein-Uhlenbeck stationary process $\widehat{O}(t)$ and its numerical counterpart $\widehat{O}_n^{(h)}$ provide the appropriate equilibrium solutions for the test SDE (1) **and the explicit Euler-Maruyama scheme.**

These Ornstein-Uhlenbeck solutions were, in fact, used implicitly in the literature discussed above.

The above expressions for the differences of two solutions of the SDE or the explicit Euler-Maruyama scheme show that the factor $1 - |\lambda|h$ in the drift alone determines the stability of the equilibrium solution under discussion, as indicated by Milstein & Tretyakov.

A similar situation holds for other numerical schemes applied to the test SDE (1).

Justification of the linear test equation

In deterministic numerical analysis the scalar linear test ODE

$$\frac{dx}{dt} = \lambda x$$

is derived via linearisation of a nonlinear ODE system around an equilibrium solution, centering and diagonalising the resulting linear system.

Theorems of the first approximation and the Hartman-Grobman theorem provide the background for relating the linear stability results back to the nonlinear problem.

In the case of the nonlinear SDE with additive noise

$$dX_t = f(X_t)\,dt + \sigma\,dWt,$$

which has a stochastic stationary solution $\bar{X}_t$ as an equilibrium, then linearisation about this equilibrium gives the random ODE

$$\frac{d}{dt}\,Y_t = \nabla f(\bar{X}_t)Y_t,$$

where $Y_t := X_t - \bar{X}_t$ and ∇f is the Jacobian of f.

There is no rigorous justification that studying the stability properties of numerical schemes applied to (1) **allows one to deduce properties of the schemes applied to nonlinear SDEs.**

However, the investigation of stability properties of the linear test SDE (1) is of interest in itself and in the context of SDEs arising as spatial discretisations of linear SPDEs.

Nonlinear test equation

To investigate the preservation of asymptotic stability under discretization for nonlinear SDEs with additive noise it is more appropriate to consider nonlinear test SDEs of, e.g., the form

$$dX_t = f(X_t)\, dt + \sigma\, dW_t,$$

where f satisfies a one-sided dissipative Lipschitz condition, e.g.

$$f(x) = -x - x^3.$$

These nonlinear test SDEs have a unique stochastic stationary solution that attracts all other solutions in the pathwise sense.

The numerical scheme should preserve this property.

Nonlinear test equation for SPDEs

For SDEs coming from Galerkin approximations of SPDEs it is more appropriate to use nonlinear test SDEs of the form

$$dX_t = [AX_t + f(X_t)]\, dt + \sigma\, dW_t,$$

where

i) f satisfies a global Lipschitz condition with constant K,

ii) A is a stable matrix in the sense that $\mu[A] \leq 0$, where $\mu[A]$ is the logarithmic norm of A defined by

$$\mu[A] = \lim_{\delta \to 0+} \frac{\|\mathrm{Id} + \delta A\| - 1}{\delta}.$$

Three numerical schemes

- the *linear implicit Euler-Maruyama scheme*

$$X_{n+1} = X_n + hAX_{n+1} + hf(X_n) + \sigma\Delta W_n\,, \tag{6}$$

- the *explicit exponential Euler scheme* (*Lord & Rougemont*)

$$Y_{n+1} = e^{hA}Y_n + he^{hA}f(Y_n) + e^{hA}\sigma\Delta W_n\,, \tag{7}$$

- the *explicit exponential Euler scheme* (*Jentzen & Kloeden*)

$$Z_{n+1} = e^{hA}Z_n + A^{-1}(e^{hA} - I)f(Z_n) + \sigma\int_{t_n}^{t_n+h} e^{(t_n+h-s)A}\,dW(s)\,. \tag{8}$$

In the last of these the matrix A must obviously be invertible.

Theorem 1 *Suppose that $\mu[A] \leq 0$ with respect to a suitable induced matrix norm and that f satisfies a global Lipschitz condition with constant K. Then each of the linear implicit Euler-Maruyama scheme and the exponential Euler scheme (Lord & Rougemont) has a unique stochastic stationary solution which is pathwise asymptotically stable for all step sizes $h > 0$ if*

$$K < -\mu[A]\,; \tag{9}$$

and there exists a $h^* > 0$ given by the positive solution of

$$1 + hK\|A^{-1}\|\,\|A\|\,e^{h\|A\|-h\mu[A]} = e^{-h\mu[A]}$$

such that the exponential Euler scheme (Jentzen & Kloeden) has a unique stochastic stationary solution which is pathwise asymptotically stable for all $h \in (0, h^*)$ if

$$K\|A\|\,\|A^{-1}\| < -\mu[A]. \tag{10}$$

Proof idea.

Subtract an Ornstein-Uhlenbeck stochastic stationary solution $\widehat{O}_t$ of the linear SDE

$$dX_t = AX_t\,dt + \sigma\,dW_t$$

from the solution of the nonlinear SDE

$$dX_t = \left[AX_t + f(X_t)\right]\,dt + \sigma\,dW_t$$

to get a <u>random ODE</u> in $z(t) := X_t - \widehat{O}_t$

$$\frac{d}{dt}z(t) = Az(t) + f\left(z(t) + \widehat{O}_t)\right).$$

Then use absorbing set estimates and the theory of random dynamical systems to show that the random ODE (and hence the nonlinear SDE) has a stochastic stationary solution $\widehat{z}(t)$, resp.

$$\widehat{X}_t := \widehat{z}(t) + \widehat{O}_t.$$

Similar approach for the numerical schemes with linear and nonlinear difference equations

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