Some applications of quasistationary distributions to random Poincaré maps

Nils Berglund
MAPMO, Université d'Orléans
CNRS, UMR 7349 & Fédération Denis Poisson
www.univ-orleans.fr/mapmo/membres/berglund
nils.berglund@math.cnrs.fr

Collaborators: Barbara Gentz (Bielefeld), Damien Landon (Dijon)
ANR project MANDy, Mathematical Analysis of Neuronal Dynamics

5th Workshop on Random Dynamical Systems
Bielefeld, 5 October 2012

Random Poincaré map

Stochastic differential equation (SDE)

$$\mathrm{d}\varphi_t = f(\varphi_t, x_t)\,\mathrm{d}t + \sigma F(\varphi_t, x_t)\,\mathrm{d}W_t \qquad \varphi \in \mathbb{R}$$
$$\mathrm{d}x_t = g(\varphi_t, x_t)\,\mathrm{d}t + \sigma G(\varphi_t, x_t)\,\mathrm{d}W_t \qquad x \in \mathbb{R}\,(\mathbb{R}^n)$$

- ▷ all functions periodic in φ (say period 1)
- ▷ $f \geqslant c > 0$ and σ small $\Rightarrow$ φ_t likely to increase
- ▷ process may be killed when x leaves (a, b)

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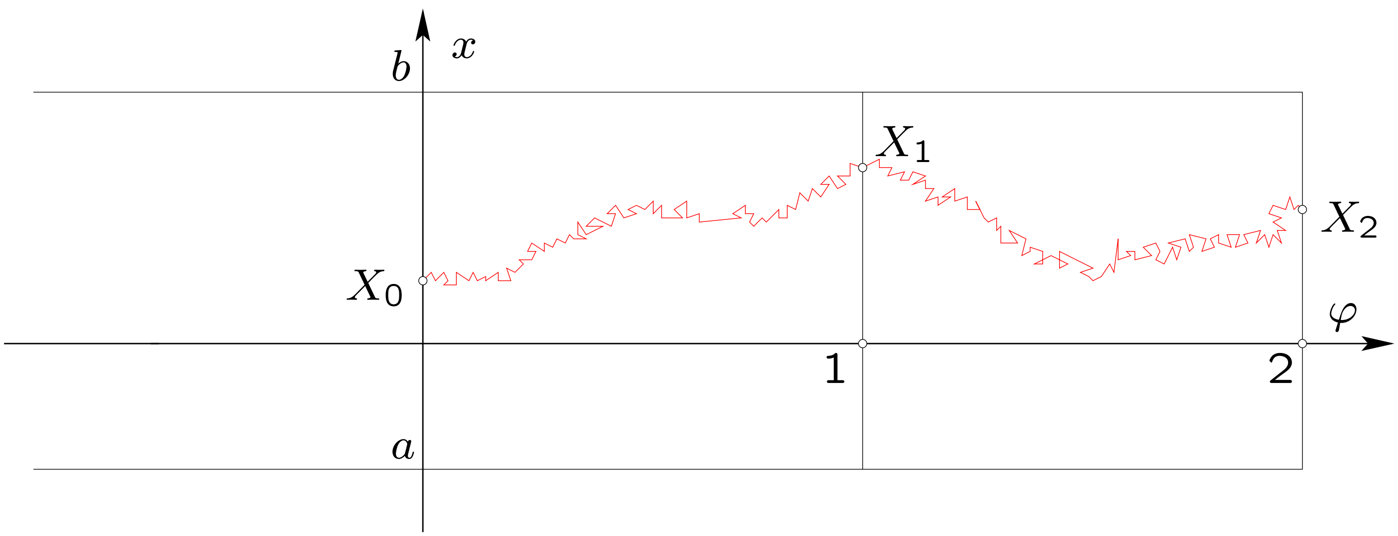
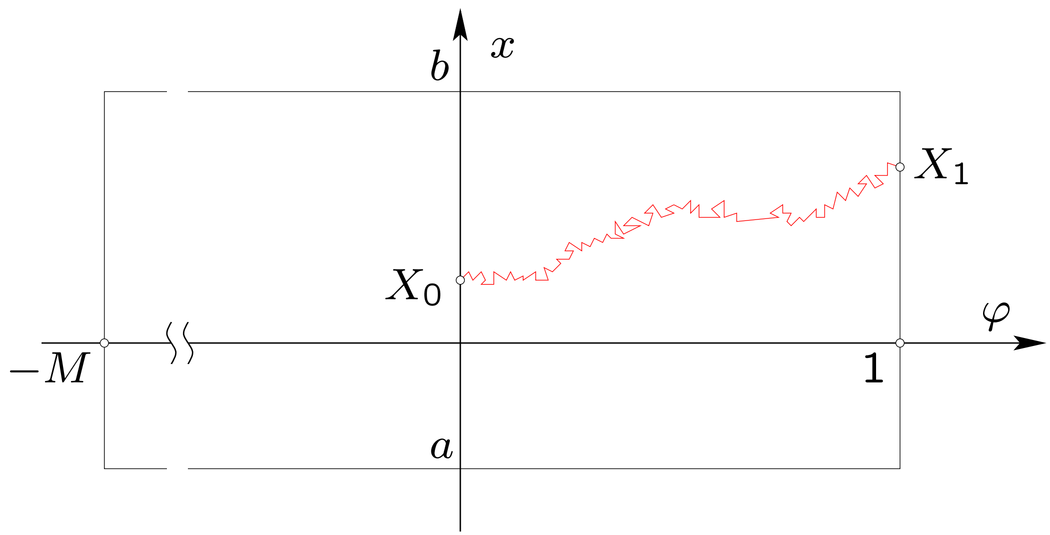


$X_0, X_1, \dots$ form substochastic Markov chain

Random Poincaré map and harmonic measure



- τ: first-exit time of $z_t = (\varphi_t, x_t)$ from $\mathcal{D} = (-M, 1) \times (a, b)$
- $\mu_z(A) = \mathbb{P}^z\{z_\tau \in A\}$: harmonic measure (wrt generator $\mathcal{L}$)
- [Ben Arous, Kusuoka, Stroock '84]: under hypoellipticity cond, μ_z admits (smooth) density $h(z, y)$ wrt arclength on $\partial\mathcal{D}$

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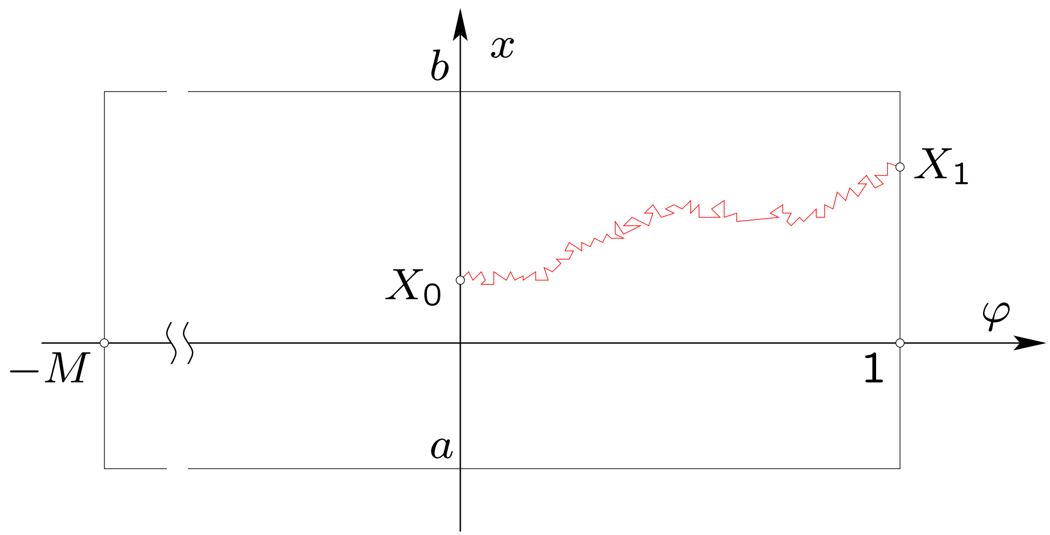

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- [Ben Arous, Kusuoka, Stroock '84]: under hypoellipticity cond, μ_z admits (smooth) density $h(z, y)$ wrt arclength on $\partial\mathcal{D}$
- Remark: $\mathcal{L}_z h(z, y) = 0$ (kernel is harmonic)
- For $B \subset (a, b)$ Borel set

$$\mathbb{P}^{X_0}\{X_1 \in B\} = K(X_0, B) := \int_B K(X_0, \mathrm{d}y)$$

where $K(x, \mathrm{d}y) = h((0, x), y)\,\mathrm{d}y =: k(x, y)\,\mathrm{d}y$

Fredholm theory

Let $E = [a, b]$ and consider integral operator K acting

- ▷ on L^∞ via $f \mapsto (Kf)(x) = \int_E k(x,y) f(y) \, \mathrm{d}y = \mathbb{E}^x[f(X_1)]$
- ▷ on L^1 via $m \mapsto (mK)(A) = \int_E m(x) k(x,y) \, \mathrm{d}x = \mathbb{P}^\mu\{X_1 \in A\}$

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[Fredholm 1903]:

- ▷ If $k \in L^2$, then K has eigenvalues λ_n of finite multiplicity
- ▷ Eigenfcts $Kh_n = \lambda_n h_n$, $h_n^* K = \lambda_n h_n^*$ form complete ON basis

[Jentzsch 1912]:

- ▷ Principal eigenvalue λ_0 is real, simple, $|\lambda_n| < \lambda_0 \; \forall n \geqslant 1$, $h_0 > 0$

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$$\Rightarrow \mathbb{P}^x\{X_n \in \mathrm{d}y | X_n \in E\} = \pi_0(\mathrm{d}x) + \mathcal{O}((|\lambda_1|/\lambda_0)^n)$$

where $\pi_0 = h_0^* / \int_E h_0^*$ is quasistationary distribution (QSD)
[Yaglom '47, Bartlett '57, Vere-Jones '62, . . .]

How to estimate the principal eigenvalue

▷ "Trivial" bounds: $\forall A \subset E$ with Lebesgue$(A) > 0$,

$$\left[\inf_{x \in A} K \; (x, A)\right] \leqslant \lambda_0 \leqslant \left[\sup_{x \in E} K \; (x, E)\right]$$

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$$\lambda_0 \leqslant 1 - \frac{1}{\sup_{x \in E} \mathbb{E}^x[\tau_\Delta]} \qquad \text{where} \quad \tau_\Delta = \inf\{n > 0 \colon X_n \notin E\}$$

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- ▷ Bounds using Laplace transforms

Laplace transforms

Given $A \subset E$, $x \in E$ and $u \in \mathbb{C}$, define

$$\tau_A = \inf\{n \geqslant 1\colon X_n \in A\} \qquad G_A^u(x) = \mathbb{E}^x[\mathrm{e}^{u\tau_A} 1_{\{\tau_A<\infty\}}]$$
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▷ $G_A^u = H_A^u$ in $E \setminus A$ and $H_A^u = 1$ in A

▷ Feynman–Kac-type relation

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Proof:

$$\begin{aligned}
(KH_A^u)(x) &= \mathbb{E}^x\Big[\mathbb{E}^{X_1}\big[\mathrm{e}^{u\sigma_A} 1_{\{\sigma_A < \infty\}}\big]\Big] \\
&= \mathbb{E}^x\Big[1_{\{X_1 \in A\}} \mathbb{E}^{X_1}\big[\mathrm{e}^{u\sigma_A} 1_{\{\sigma_A < \infty\}}\big]\Big] + \mathbb{E}^x\Big[1_{\{X_1 \in E \setminus A\}} \mathbb{E}^{X_1}\big[\mathrm{e}^{u\sigma_A} 1_{\{\sigma_A < \infty\}}\big]\Big] \\
&= \mathbb{E}^x\big[1_{\{\tau_A = 1\}}\big] + \mathbb{E}^x\big[\mathrm{e}^{u(\tau_A - 1)} 1_{\{1 < \tau_A < \infty\}}\big] \\
&= \mathbb{E}^x\big[\mathrm{e}^{u(\tau_A - 1)} 1_{\{\tau_A < \infty\}}\big] = \mathrm{e}^{-u} G_A^u(x) .
\end{aligned}$$

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Consequences:

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- ▷ If $Kh = \mathrm{e}^{-u} h$ and $|\mathrm{e}^u| < \Big[\sup_{x \in E \setminus A} K(x, E \setminus A)\Big]^{-1}$ then

$$h(x) = \mathbb{E}^x\Big[\mathrm{e}^{u\tau_A} h(X_{\tau_A}) 1_{\{\tau_A < \infty\}}\Big] \qquad \forall x \in E$$

$\Rightarrow$ $h|_A$ determines $h|_{E \setminus A}$

- ▷ If $u \in \mathbb{R}$, $h > 0$ in closed connected A then $\exists x^* \in A$: $G_A^u(x^*) = 1$

How to estimate the spectral gap

Various approaches: coupling, Poincaré/log-Sobolev inequalities, Lyapunov functions, Laplace transform + Donsker–Varadhan, . . .

[Birkhoff '57] Under uniform positivity condition

$$s(x)\nu(A) \leqslant K(x, A) \leqslant Ls(x)\nu(A) \qquad \forall x \in E, \forall A \subset E$$

one has $|\lambda_1|/\lambda_0 \leqslant 1 - L^{-2}$

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Localised version: assume $\exists\, A \subset E$ and $m : A \to \mathbb{R}_+^*$ such that

$$m(y) \leqslant k(x,y) \leqslant Lm(y) \qquad \forall x,y \in A \tag{1}$$

Then

$$|\lambda_1| \leqslant L - 1 + \mathcal{O}\Big(\sup_{x\in E} K(x, E\setminus A)\Big) + \mathcal{O}\Big(\sup_{x\in A}[1 - K(x,E)]\Big)$$

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To prove the restricted positivity condition (1):

- ▷ Show that $|Y_n - X_n|$ likely to decrease exp for $X_0, Y_0 \in A$
- ▷ Use Harnack inequalities once $|Y_n - X_n| = \mathcal{O}(\sigma^2)$

Application 1: Exit through an unstable periodic orbit

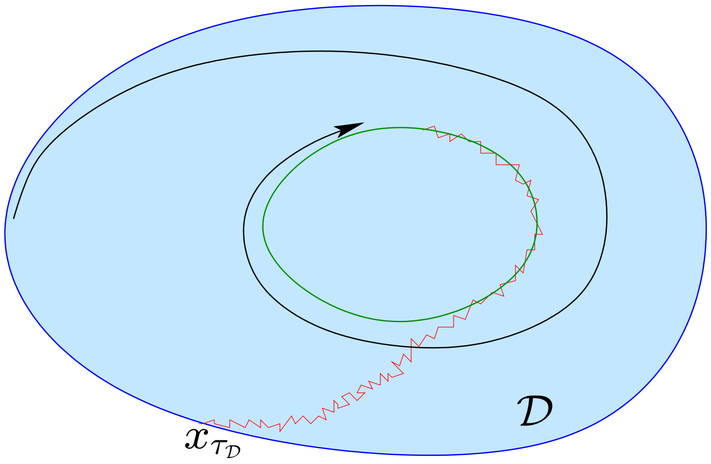
Planar SDE

$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t$

$\mathcal{D} \subset \mathbb{R}^2$: int of unstable periodic orbit

First-exit time: $\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$

Law of first-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$?



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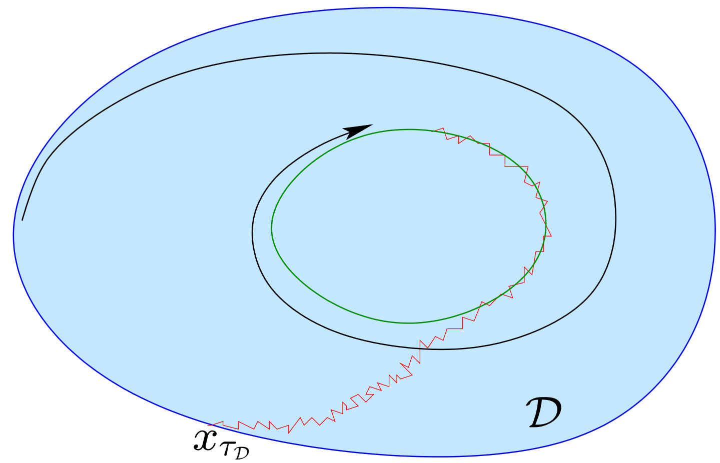

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Large-deviation principle with rate function

$$I(\gamma) = \frac{1}{2}\int_0^T (\dot\gamma_t - f(\gamma_t))^T D(\gamma_t)^{-1}(\dot\gamma_t - f(\gamma_t))\,\mathrm{d}t \qquad D = gg^T$$

Quasipotential:

$V(y) = \inf\{I(\gamma) \colon \gamma :$ stable orbit $\to y \in \partial\mathcal{D}$ in arbitrary time$\}$

Theorem [Freidlin, Wentzell '69]: If V reaches its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \to 0$

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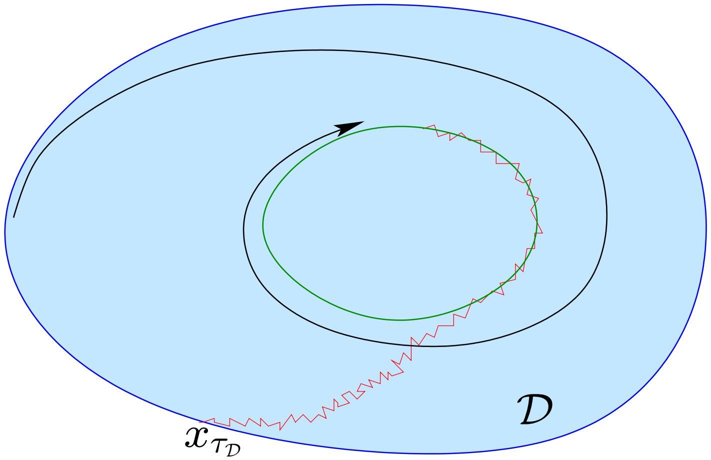

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Problem: V is constant on $\partial\mathcal{D}$!

Most probable exit paths

Minimisers of I obey Hamilton equations with Hamiltonian

$H(\gamma, \psi) = \frac{1}{2}\psi^T D(\gamma)\psi + f(\gamma)^T \psi$

where $\psi = D(\gamma)^{-1}(\dot{\gamma} - f(\gamma))$

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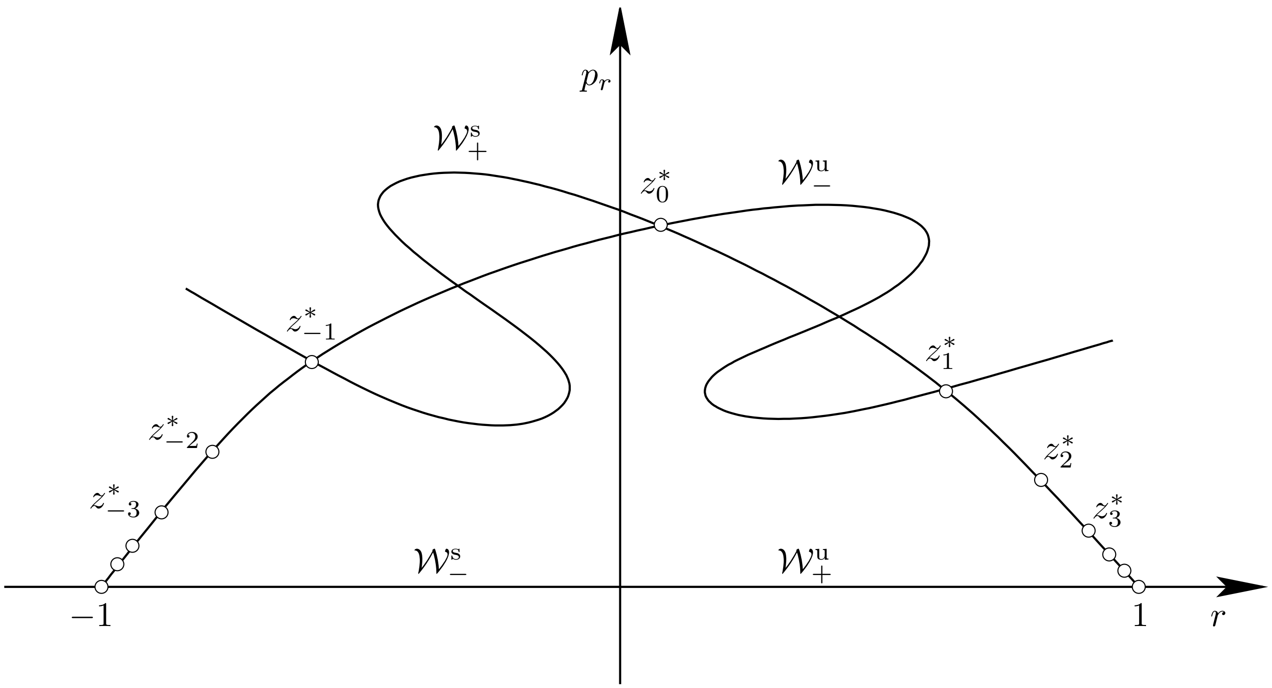
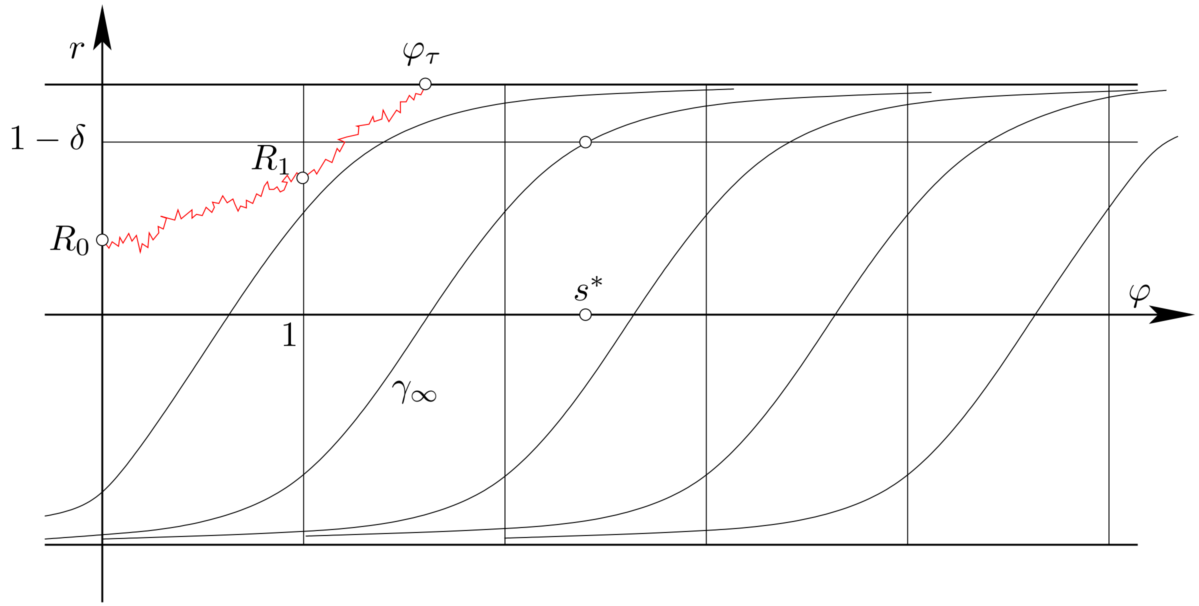


Generically optimal path (for infinite time) is isolated

Random Poincaré map

In polar-type coordinates (r, φ):



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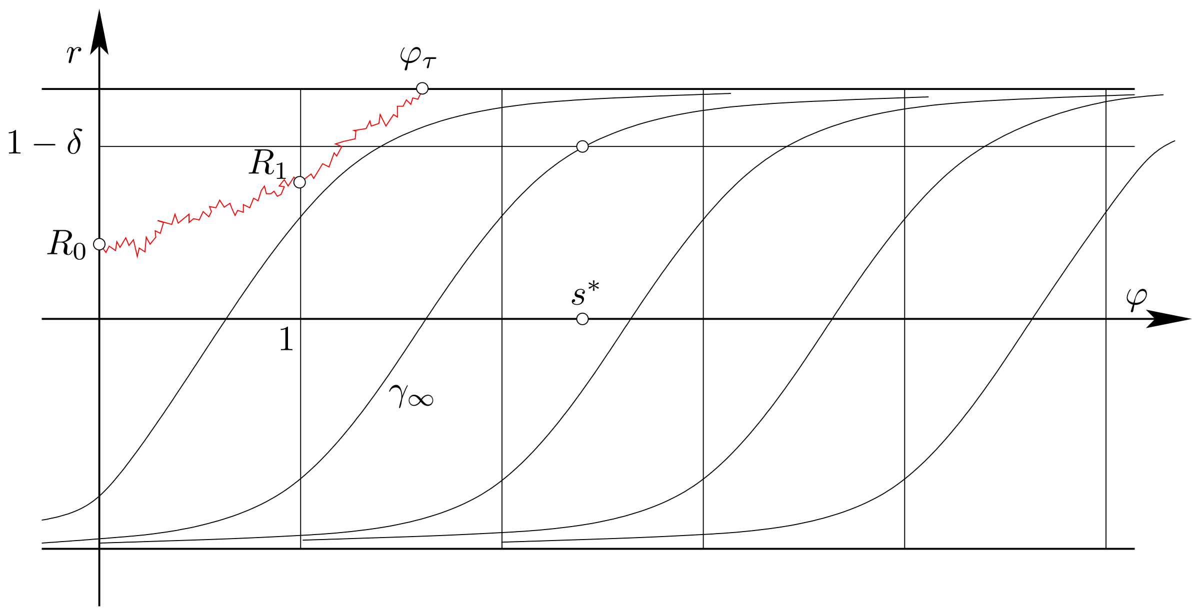


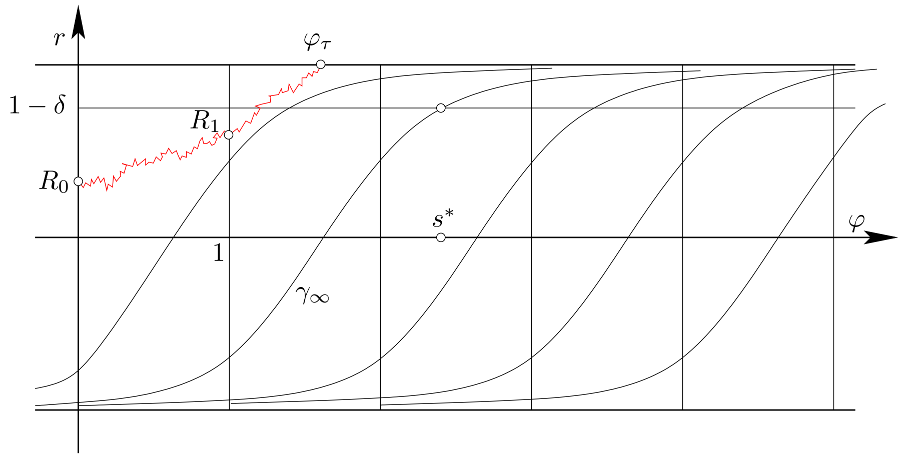
$$\mathbb{P}^{R_0}\{R_n \in A\} = \lambda_0^n h_0(R_0) \int_A h_0^*(y)\,\mathrm{d}y\Big[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\Big]$$

If $t = n + s$,

$$\mathbb{P}^{R_0}\{\varphi_\tau \in \mathrm{d}t\} = \lambda_0^n h_0(R_0) \int h_0^*(y)\mathbb{P}^y\{\varphi_\tau \in \mathrm{d}s\}\,\mathrm{d}y\Big[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\Big]$$

Periodically modulated exponential distribution

Random Poincaré map



Split into two Markov chains:

▷ Chain killed upon r reaching $1-\delta$ in $\varphi = \varphi_{\tau_-}$

$$\mathbb{P}^0\{\varphi_{\tau_-} \in [\varphi_1, \varphi_1 + \Delta]\} \simeq (\lambda_0^{\mathrm{s}})_1^{\varphi} \, \mathrm{e}^{-J(\varphi_1)/\sigma^2}$$

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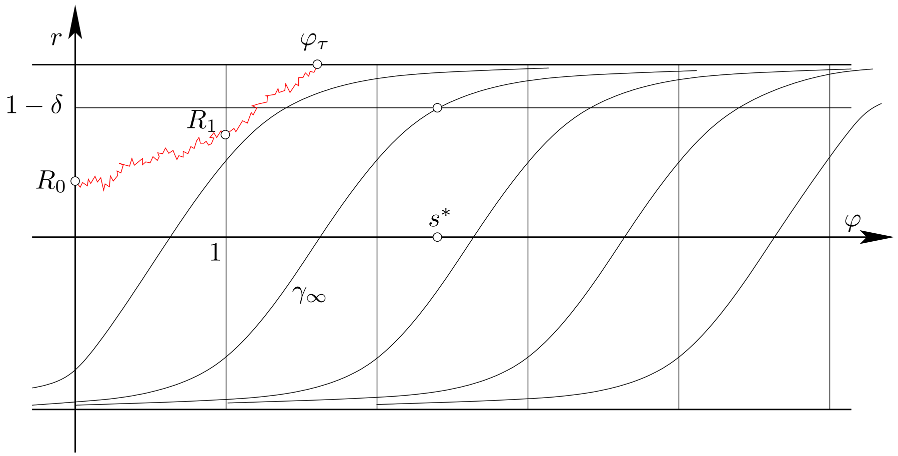

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▷ Chain killed at $r = 1 - 2\delta$ and on unstable orbit $r = 1$

– Principal eigenvalue: $\lambda_0^{\mathsf{u}} = \mathrm{e}^{-2\lambda_+ T_+}(1 + \mathcal{O}(\delta))$

$\lambda_+ =$ Lyapunov exponent, $T_+ =$ period of unstable orbit

– Using LDP:

$$\mathbb{P}^{\varphi_1}\{\varphi_\tau \in [\varphi, \varphi + \Delta]\} \simeq (\lambda_0^{\mathsf{u}})^{\varphi - \varphi_1} \, \mathrm{e}^{-[I_\infty + c(\mathrm{e}^{-2\lambda_+ T_+(\varphi-\varphi_1)})]/\sigma^2}$$

Main result: cycling

Theorem [B & Gentz, 2012]

$$\mathbb{P}^{r_0,0}\{\varphi_\tau \in [\varphi, \varphi + \Delta]\} = C(\sigma)(\lambda_0)^\varphi \chi_\Delta(\varphi) Q_{\lambda_+ T_+}\left(\frac{|\log\sigma| - \theta(\varphi) + \mathcal{O}(\delta)}{\lambda_+ T_+}\right)$$
$$\times \left[1 + \mathcal{O}(\mathrm{e}^{-c\varphi/|\log\sigma|}) + \mathcal{O}(\delta|\log\delta|)\right]$$

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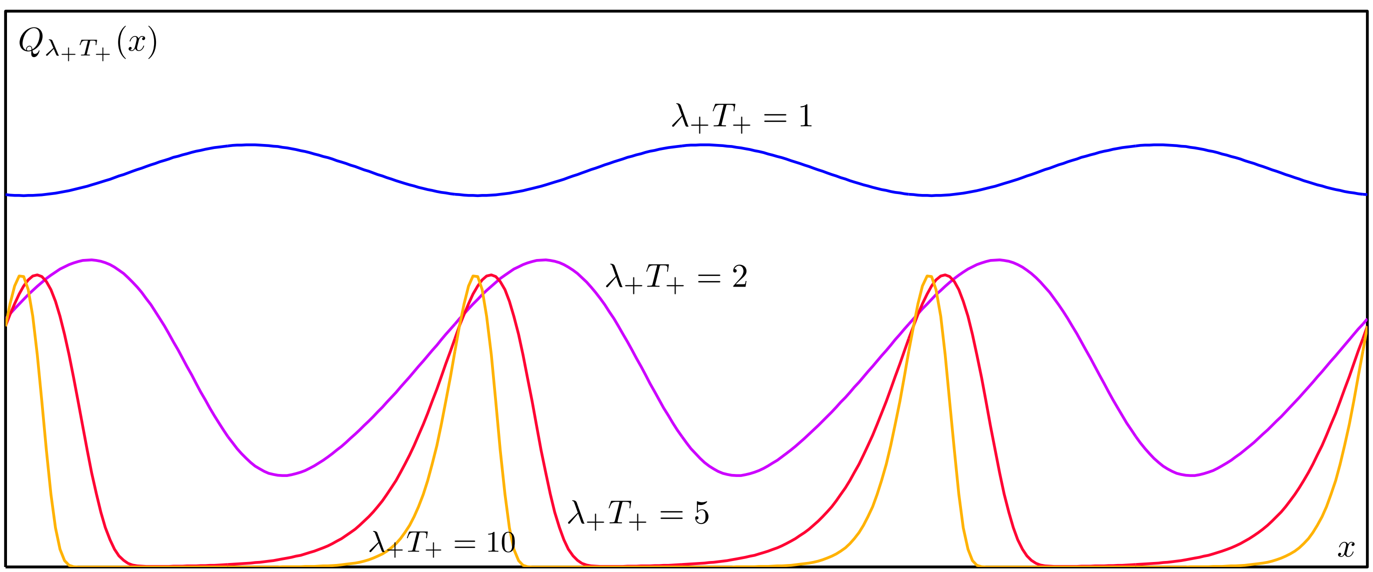
▷ $Q_{\lambda T}(x) = \sum_{n=-\infty}^{\infty} A(\lambda T(n - x))$ with $A(x) = \frac{1}{2} \exp\{-2x - \frac{1}{2} \mathrm{e}^{-2x}\}$

Cycling profile, periodicised Gumbel distribution



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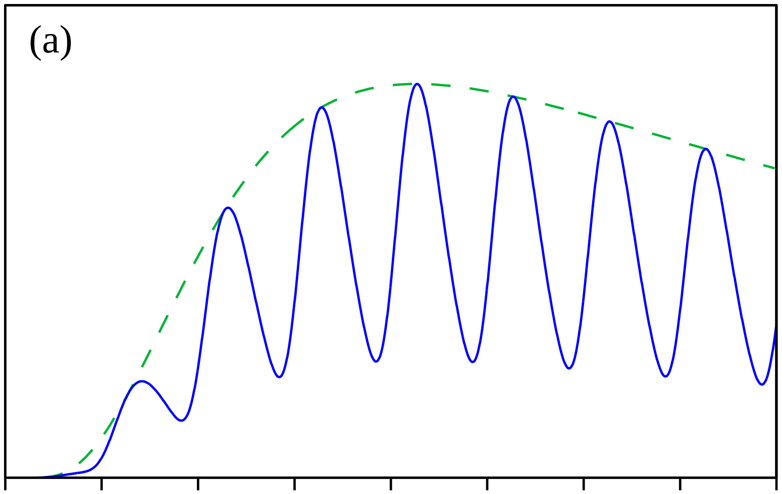
Cycling: periodic dependence on $|\log\sigma|$

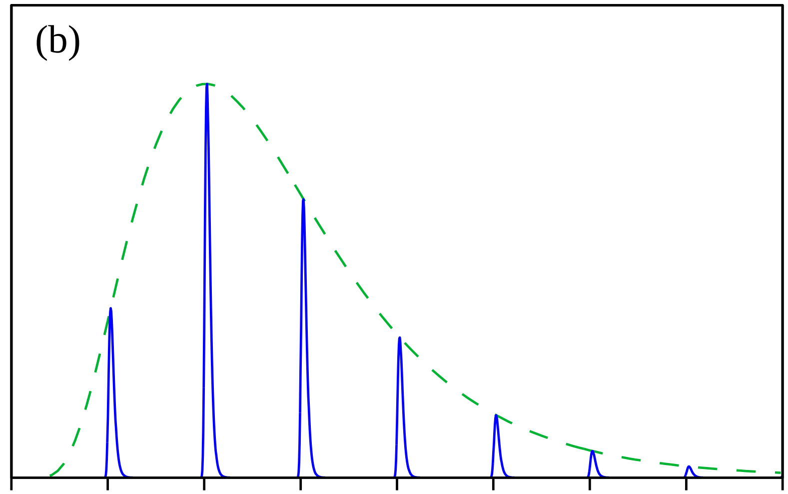
[Day'90, Maier & Stein '96, Getfert & Reimann '09]

Main result: cycling

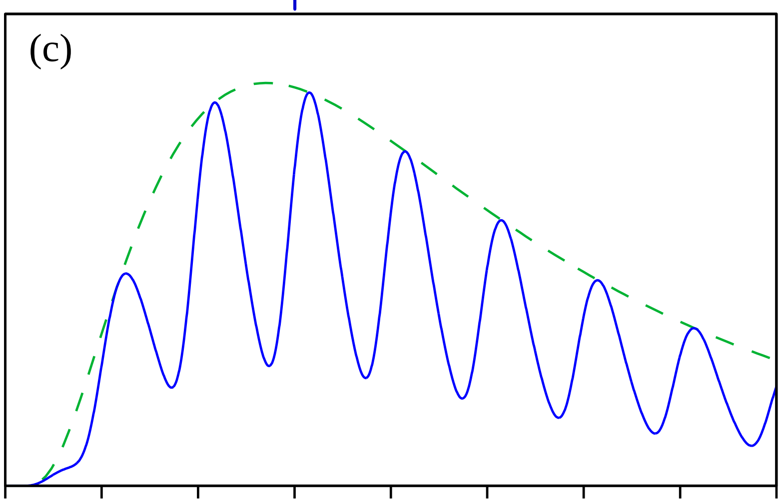
$V = 0.5$, $\lambda_+ = 1$



$\sigma = 0.4$, $T_+ = 2$



$\sigma = 0.4$, $T_+ = 20$



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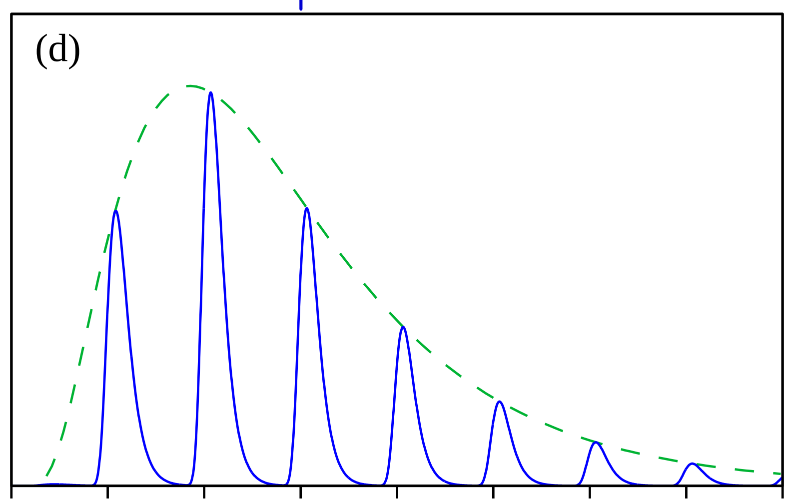

$\sigma = 0.5$, $T_+ = 5$

Application 2: Stochastic FitzHugh–Nagumo equations

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}$$
$$\mathrm{d}y_t = [a - x_t]\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^{(2)}$$

▷ $x \propto$ membrane potential of neuron

▷ $y \propto$ proportion of open ion channels (recovery variable)

▷ $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes

▷ $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

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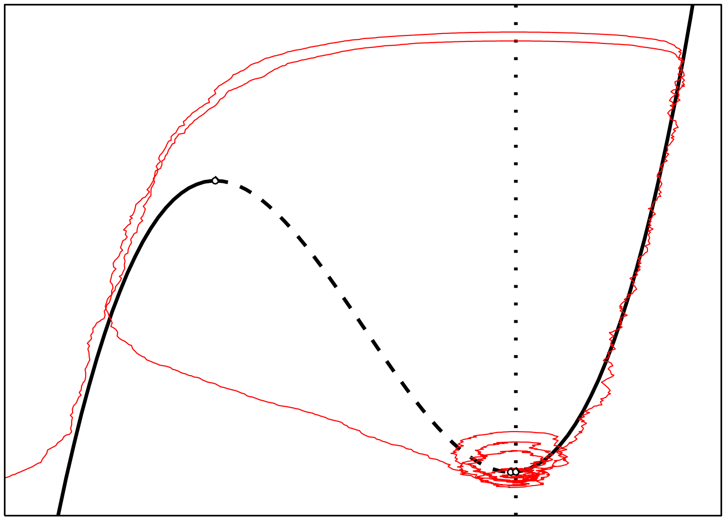
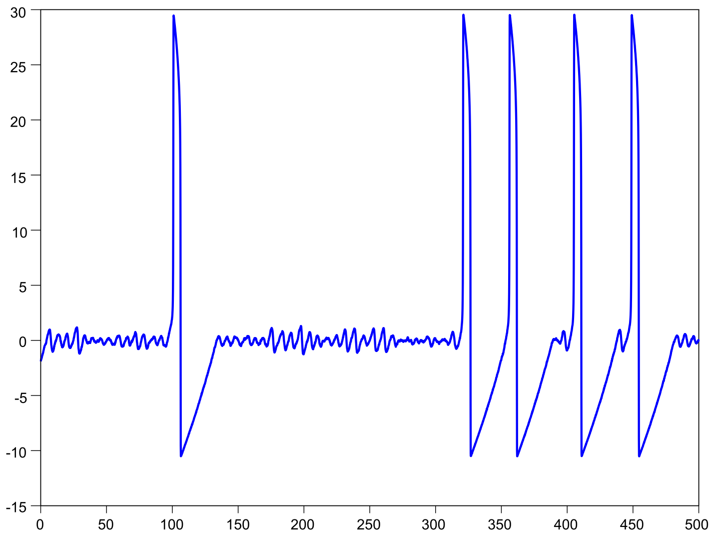
$\varepsilon = 0.1$
$\delta = \frac{3a^2-1}{2} = 0.02$
$\sigma_1 = \sigma_2 = 0.03$





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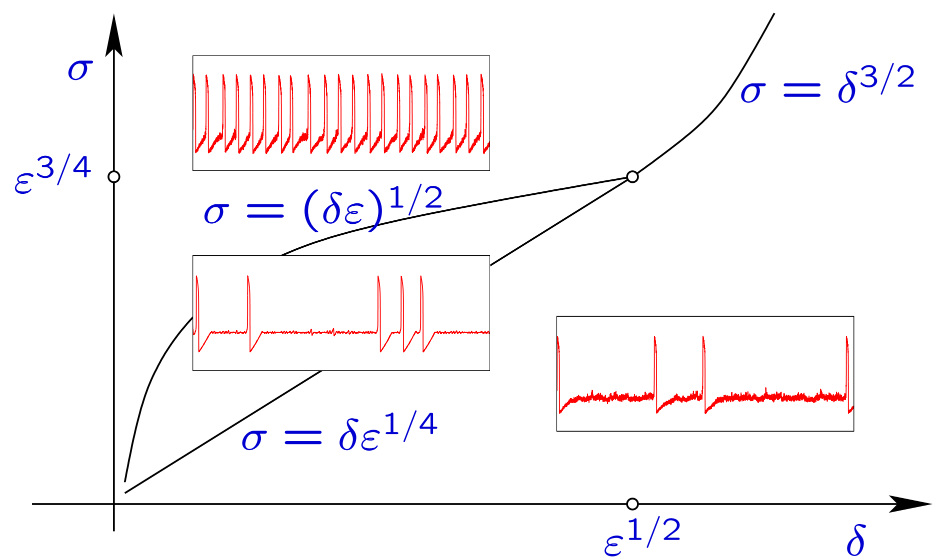
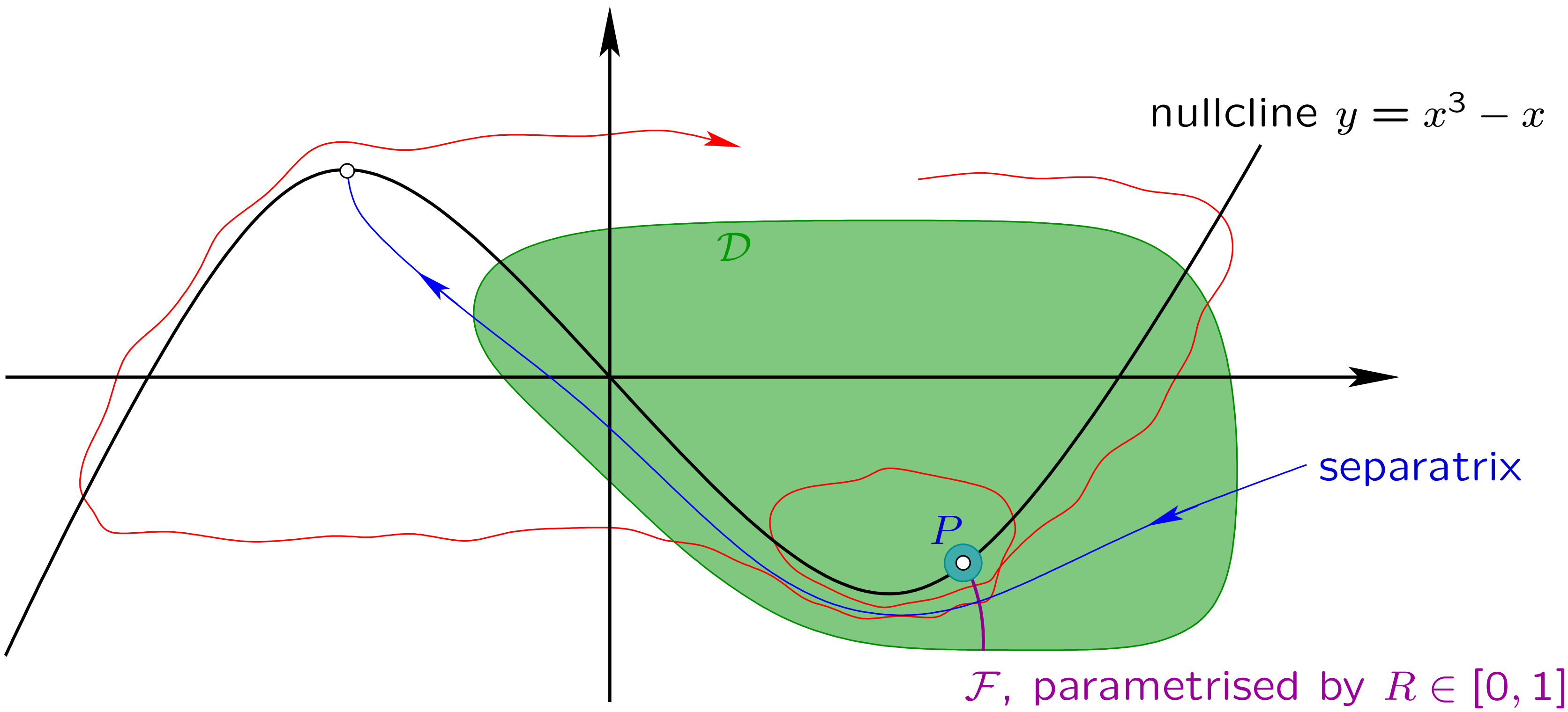
Different regimes

[Muratov & Vanden Eijnden '08]



Small-amplitude oscillations (SAOs)

Definition of random number of SAOs N:



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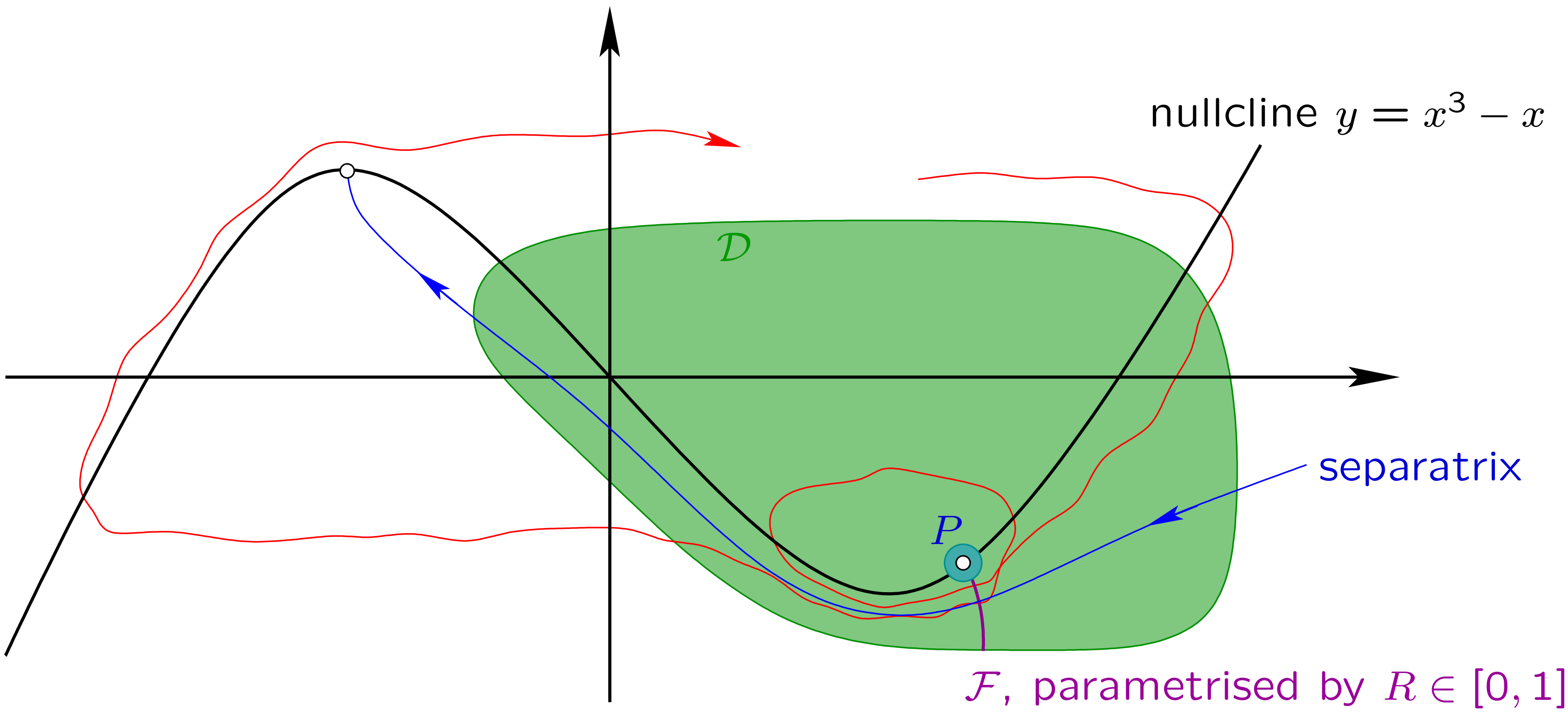

$(R_0, R_1, \dots, R_{N-1})$ substochastic Markov chain with kernel

$$K(R_0, A) = \mathbb{P}^{R_0}\{R_\tau \in A\}$$

$R \in \mathcal{F}$, $A \subset \mathcal{F}$, τ = first-hitting time of $\mathcal{F}$ (after turning around P)
N = number of turns around P until leaving $\mathcal{D}$

Main results

Theorem 1: [B & Landon, 2012]

If $\sigma_1, \sigma_2 > 0$, then $\lambda_0 < 1$ and N is asymptotically geometric:

$$\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$$

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$$\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$$

Theorem 2: [B & Landon 2012]

Assume ε and $\delta/\sqrt{\varepsilon}$ sufficiently small

There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2/\log(\sqrt{\varepsilon}/\delta)$

- ▷ Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- ▷ Expected number of SAOs:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0)\exp\left\{\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

where $C(\mu_0)$ = probability of starting on $\mathcal{F}$ above separatrix

Transition from weak to strong noise

Linear approximation near separatrix:

$$\mathrm{d}z_t^0 = \Big(\frac{\delta - \sigma_1^2/\varepsilon}{\varepsilon^{1/2}} + t z_t^0\Big)\,\mathrm{d}t - \frac{\sigma_1}{\varepsilon^{3/4}} t\,\mathrm{d}W_t^{(1)} + \frac{\sigma_2}{\varepsilon^{3/4}}\,\mathrm{d}W_t^{(2)}$$

$$\Rightarrow \quad \mathbb{P}\{N=1\} \simeq \Phi\Big(-\pi^{1/4}\frac{\varepsilon^{1/4}(\delta-\sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2+\sigma_2^2}}\Big) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y$$

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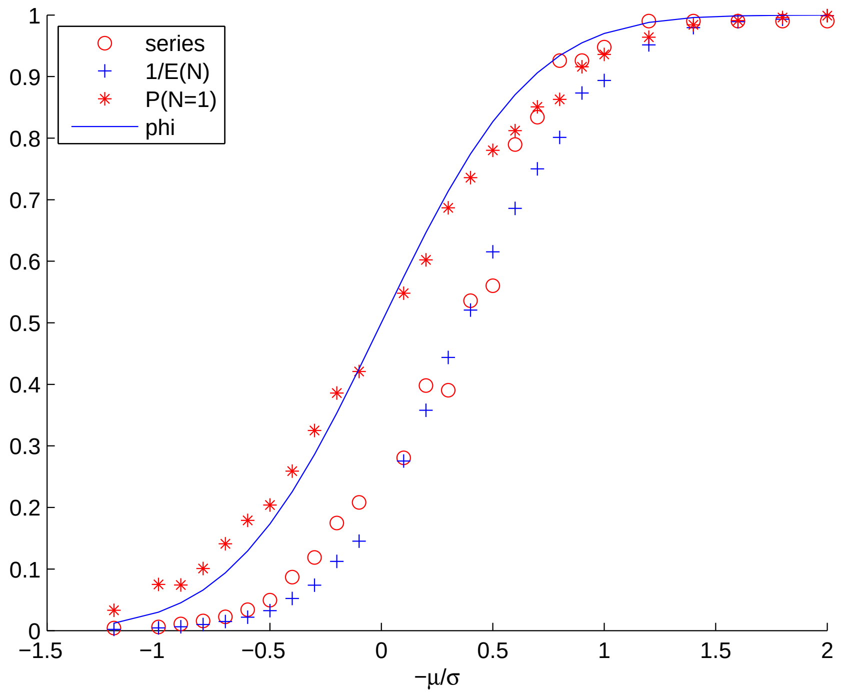

$*$: $\mathbb{P}\{\text{no SAO}\}$

$+$: $1/\mathbb{E}[N]$

$\circ$: $1 - \lambda_0$

curve: $x \mapsto \Phi(\pi^{1/4} x)$

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