Ginzburg-Landau energy for stochastically perturbed equation

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Classical Ginzburg-Landau equation

Evolution equations related to the *Ginzburg-Landau energy* (introduced by Bethuel, Brezis and Hélein in '94):

$$E_\varepsilon(u_\varepsilon) = \int_D \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2$$

with $\varepsilon > 0$, $D \subset \mathbb{R}^2$ and $u_\varepsilon \in H^1(D, \mathbb{C})$

Example: heat flow

(E'94; Lin'96; Jerrard, Soner'98; Sandier, Serfaty'07)

$$\begin{cases} \frac{1}{\log \frac{1}{\varepsilon}} \partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}\left(1 - |u_\varepsilon|^2\right) u_\varepsilon \\ u_\varepsilon|\partial D = g, \quad g : \partial D \to \mathbb{S}^1 \end{cases}$$

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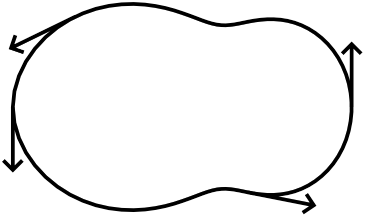
Vortices

If $\deg g \neq 0$, then u_ε has at least $n := |\deg g|$ vortices:



- Point singularities of u_ε
- Energy concentration points:

$$E_\varepsilon(u_\varepsilon) \geqslant \pi n \log \frac{1}{\varepsilon} - C$$

Main goal:

PDE for $u_\varepsilon \Rightarrow$ ODE for vortex paths $a_k(t)$:

$$\dot{a} + \nabla W(a) = 0$$

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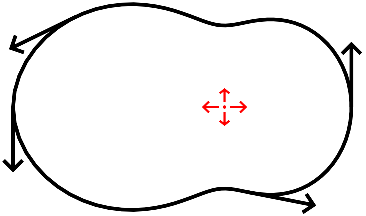

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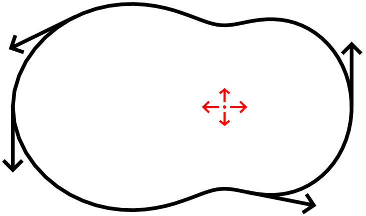

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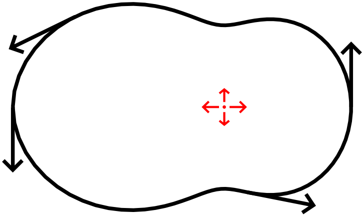

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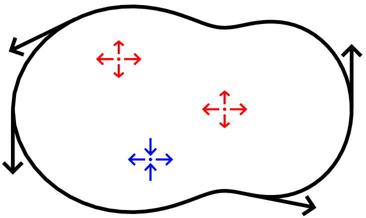

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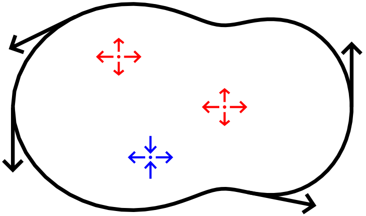

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Localization of vortex positions

Main tool:

the Jacobian $J(u_\varepsilon) := \det \nabla u_\varepsilon = \frac{1}{2}\mathrm{curl}\,(u_\varepsilon \wedge \nabla u_\varepsilon)$

$$\int_D J(u_\varepsilon)dx = \pi \deg(u_\varepsilon, \partial D)$$

Theorem (Jerrard-Soner'02)

If $\sup_{\varepsilon>0} \frac{1}{\log \frac{1}{\varepsilon}} E_\varepsilon(u_\varepsilon) < \infty$, *then* $J(u_\varepsilon)$ *is relatively compact in* $\dot{W}^{-1,1} := \left(C_0^{0,1}\right)^*$ *and*

$$J(u_\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \pi \sum_{k=1}^{N} d_k \delta_{a_k}$$

Note: $\|\delta_a - \delta_b\|_{\dot{W}^{-1,1}} = |a - b|$

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Ginzburg-Landau equation with noise

$$du_\varepsilon = \log\frac{1}{\varepsilon}\underbrace{\left(\Delta u_\varepsilon + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon\right)}_{f(u_\varepsilon)} dt + \frac{1}{\sqrt{\log\frac{1}{\varepsilon}}}\nabla u_\varepsilon \cdot F(x)\circ dB_t$$

Why this form of noise?

- typical model for interaction with external current, e.g. in Landau-Lifshitz-Gilbert model (Kurzke, Melcher, Moser, Spirn'11)
- Stochastic Allen-Cahn equation (Funaki'99, Röger, Weber'10)
- GL equation with non-random convection term (O.C.'10):

$$\dot{a} + \nabla W(a) = -F; \quad a(t) \in D$$

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Ginzburg-Landau energy in stochastic case

Itô's equation for E_ε

$$dE_\varepsilon(t) = -\log\frac{1}{\varepsilon}\int_D |f(u_\varepsilon)|^2 dxdt -$$

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Theorem (O.C.)

There exists a constant C depending on F, such that the process $\xi_\varepsilon(t) := E_\varepsilon(u_\varepsilon(t)) \cdot \exp\left(-\frac{Ct}{\log\frac{1}{\varepsilon}}\right)$ is a supermartingale.

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Towards tightness of the Jacobian

Energy bounds

Doob's inequality:

$$\mathbb{P}\{\sup_{0\leqslant t\leqslant T} \frac{1}{\log\frac{1}{\varepsilon}} E_\varepsilon(u_\varepsilon(t)) \geqslant \lambda\} \leqslant \frac{1}{\lambda}\frac{E_\varepsilon(u_\varepsilon(0))}{\log\frac{1}{\varepsilon}} \exp\left(\frac{CT}{\log\frac{1}{\varepsilon}}\right)$$

+ relative compactness of $J(u_\varepsilon)$ by Jerrard and Soner

$=$ Pointwise relative compactness of the Jacobian:

Theorem

Fix $T > 0$ and $0 < \varepsilon_0 < 1$, then $\forall \delta > 0\ \exists K(\delta) \Subset \dot{W}^{-1,1}$ such that $\forall \varepsilon < \varepsilon_0$

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- Tightness of $J(u_\varepsilon(t))$ in $C(0,T;E)$:
  pointwise relative compactness + equicontinuity

- Description of vortex motion with an SODE of the form

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