Invariant measures of the stochastic Allen-Cahn Equation: The regime of small noise and large system size

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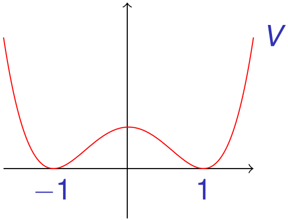
Stochastic Allen-Cahn equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x) - V'(u(t,x)) + \sqrt{2\varepsilon}\,\dot{W}(t,x)$$

$x \in [-L, L]$ one-dimensional.

V symmetric double-well potential.



$\dot{W}$ space-time white noise.

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Parameters:

- $\varepsilon \ll 1$ noise strength.
- $O(1)$ typical lengths of an interface.
- system size: $L \gg 1$.

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Questions:

Depending on L, ε

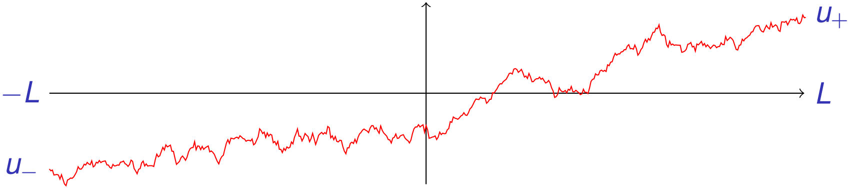
- Do we see nucleation (noise induced creation of new interfaces)?
- What is the influence of the boundary conditions $u(\pm L) = u_\pm$?

Invariant measures for (SAC)

Dirichlet boundary conditions: $u(\pm L) = u_\pm$

Auxiliary measure:

$\mathcal{W}^{u_-,u_+}_{\varepsilon,(-L,L)}$ Brownian bridge from $(-L, u_-)$ to (L, u_+). Variance ε.

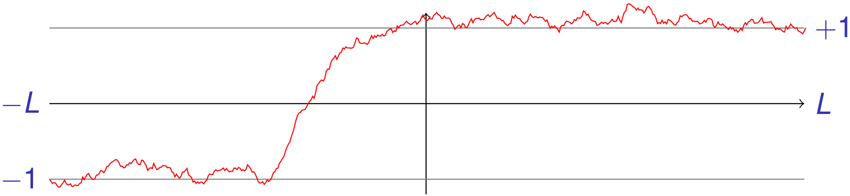


Invariant measures for (SAC)

Dirichlet boundary conditions: $u(\pm L) = u_\pm$

Invariant measure:

$$\mu(du) = \frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{\varepsilon} \int_{-L}^{L} V(u(x))\, dx \right) \mathcal{W}(du)$$



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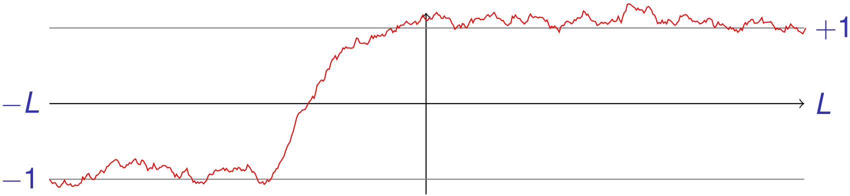

Question: How does the behaviour depend on ε and L (and $u_\pm$)?

Gibbs measure

Energy functional:

$$E(u) := \int_{-L}^{L} \frac{1}{2}(\partial_x u(x))^2 + V(u(x))\, dx$$

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Invariant measure of stochastic Allen-Cahn equation

$$\begin{aligned}\dot{u}(t,x) &= \partial_x^2 u(t,x) - V'(u(t,x)) + \sqrt{2\varepsilon}\,\dot{W}(t,x)\\ &= -\nabla_{L^2} E(u) + \sqrt{2\varepsilon}\,\dot{W}(t,x).\end{aligned}$$

Energy functional on the full line

$$E(u) := \int_{-\infty}^{\infty} \frac{1}{2}(\partial_x u(x))^2 + V(u(x))\, dx$$

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Modica-Mortola trick: Cost of transitions $u(\pm\infty) = \pm 1$

$$E(u) = \int_{-\infty}^{\infty} \frac{1}{2}\Big(\partial_x u(x) \pm \sqrt{2V(u(x))}\Big)^2 \mp \partial_x u(x)\sqrt{2V(u(x))}$$

$$\geq \int_{-1}^{1} \sqrt{2V(\tilde{u})}\, d\tilde{u} =: c_0.$$

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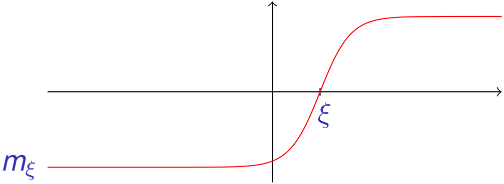
Optimal profiles Translation invariant $\mathcal{M} = \{m_\xi : \xi \in \mathbb{R}\}$.



Order one systems

System $L \approx 1$ fixed, noise strength $\varepsilon \ll 1$: Large deviation estimates!

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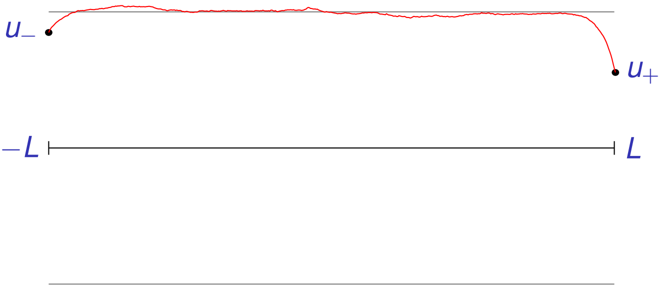
Concentration around E minimiser with given boundary conditions.



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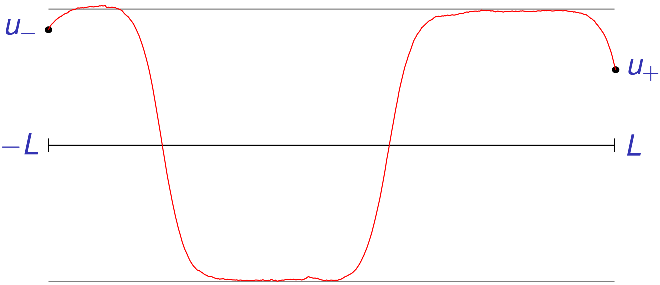

Extra transitions are exponentially unlikely

$$\mu_{\varepsilon,(L,L)}^{-1,1}(2 \text{ transitions }) \sim \exp\Big(-\frac{1}{\varepsilon}(2c_0 \pm \gamma)\Big).$$

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What happens when $\varepsilon \ll 1$ and $L \gg 1$?

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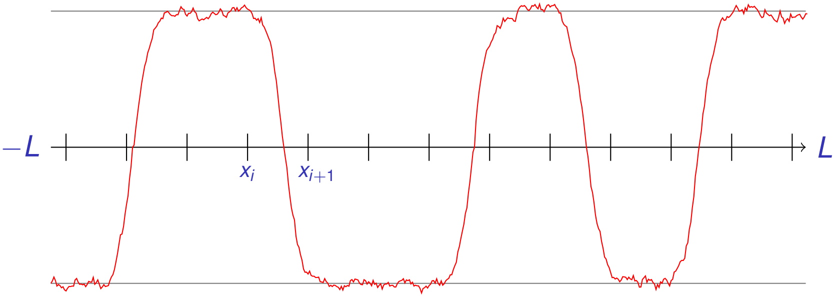

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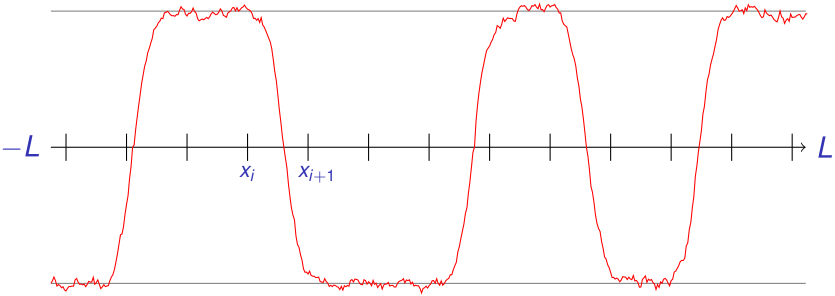

Entropic term:

$$\mu^{-1,1}_{\varepsilon,(-L,L)}(2n+1 \text{ transitions }) \sim L^{2n} \exp\Big(-\frac{1}{\varepsilon} 2nc_0\Big).$$

Probability of transitions

Transition Layer: u has a transition layer on (x_-, x_+) if

$u(x_\pm) = \pm 1$ or ∓ 1 and $|u(x)| < 1$ for all $x \in (x_-, x_+)$.

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Theorem

Boundary conditions: $u_\pm = \pm 1$.

System size: $1 \ll L \ll \exp\left(\frac{c_0'}{\varepsilon}\right)$ *for a* $c_0' < c_0$,

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Similar result for different boundary conditions (e.g. periodic, homogeneuous,...).

Location of jump

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Consider intervals of the type

$$J_x := [x - d, x + d],$$

for $d \gg |\log \varepsilon|$.

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Then

$$\sup_x \left| \mu^{-1,1}_{\varepsilon,(-L,L)}(\textit{transition in } I_x) \frac{L}{d} - 1 \right| \ll 1.$$

Related results

Bertini, Brassesco, Buttà '08: Same system $L = \frac{1}{4}|\log(\varepsilon)|$:

$\rightarrow$ Concentration around $\mathcal{M}$.

$\rightarrow$ Due to influence of the boundary the interface stays localized. In the limit interface location

$$\xi \sim \exp\Big(-A(cosh(\alpha z)-1)\Big)\,dz.$$

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W. '10: Same system for $L = \varepsilon^{-\gamma}$, $\gamma < \frac{2}{3}$:

$\rightarrow$ Concentration near energy minimisers.

Strategies

[BBB'08] use approach: u can be realized as

$$du(x) = a_\varepsilon(u(x))\,dx + \sqrt{\varepsilon}dw(x)$$

$$u(-L) = -1 \quad \text{conditioned on } u(L) = 1.$$

Difficulty:

→ a_ε is not known explicitly.

→ Conditioning on final condition.

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[W'10] use approach: Discretized measure

$$\mu^{N,\varepsilon} = \frac{1}{\mathcal{Z}^{N,\varepsilon}} \exp\left(- \frac{1}{\varepsilon} E(u)\right) d\mathcal{L}^N.$$

Use explicit bounds on the energy landscape of E.

Difficulty:

→ Error terms to large for $L > \varepsilon^{-\gamma}$.

Ingredients of proof

Two sided strong Markov property:

→ Left/right stopping points $x_- \leq \chi_- < \chi_+ \leq x_+$.

→ Φ nice test function

$$\mathbb{E}^{\mu_\varepsilon}\left(\Phi \middle| \mathcal{F}_{[x_-,\chi_-]} \vee \mathcal{F}_{[\chi_+,x_+]}\right) = \mathbb{E}^{\mu_\varepsilon,\mathbf{u}}_{(\chi_-,\chi_+)}(\Phi)\,.$$

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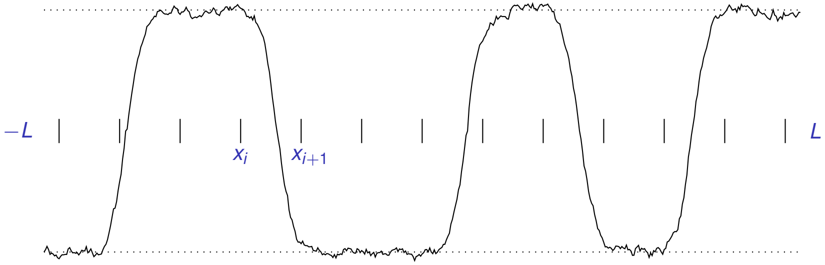
(Uniform) Large deviation bounds:

→ $\mathcal{A}$ ("nice") set of functions.

→ $\Delta E(\mathcal{A}) := \inf_{u\in\mathcal{A}} E(u) - \inf_{b.c.} E(u)$

$$\mu^{u_-,u_+}_{\varepsilon,(x_-,x_+)}(\mathcal{A}) \sim \exp\left(-\frac{1}{\varepsilon}(\Delta E(\mathcal{A}) \pm \gamma)\right).$$

Freidlin-Wentzel argument does not work directly!



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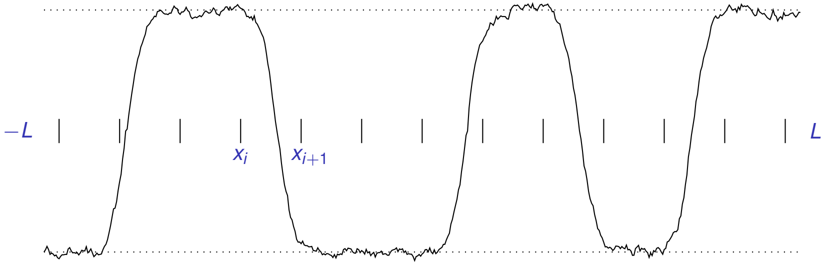

$$\mu_\varepsilon\Big(\text{transition in } [x_i, x_{i+1}]\Big)$$

$$= \int \nu_{x_{i-1},x_{i+2}}(du_{i-1}, du_{i+2})\, \mu_\varepsilon^{u_{i-1},u_{i+2}}\Big(\text{transition in } [x_i, x_{i+1}]\Big).$$

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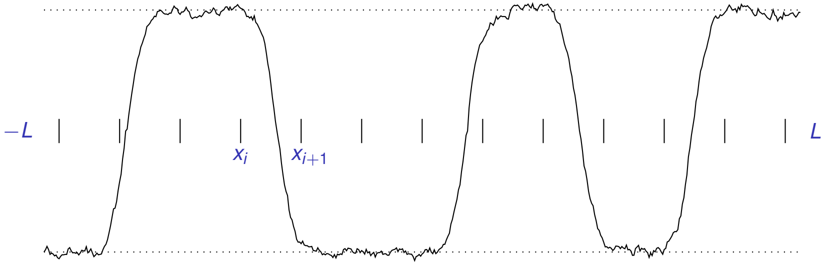

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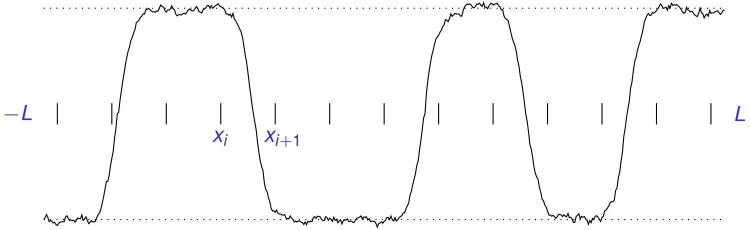
Large deviation estimate gives information on $\mu_\varepsilon^{u_{i-1},u_{i+2}}$.

But information about transition is contained in $\nu_{x_{i-1},x_{i+2}}$.

Symmetry helps

Idea: Transform the event into something we can estimate!



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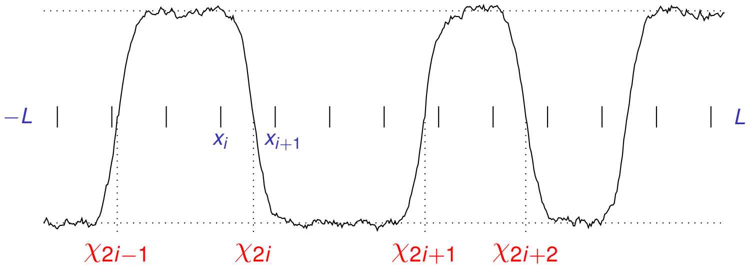

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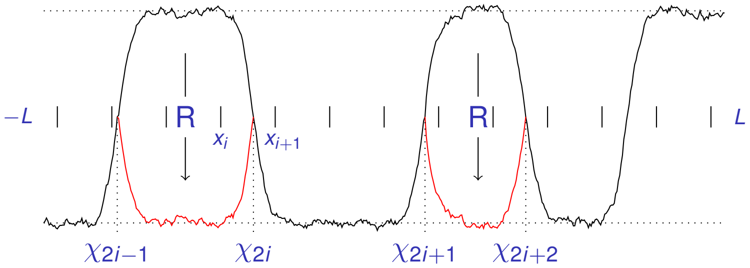

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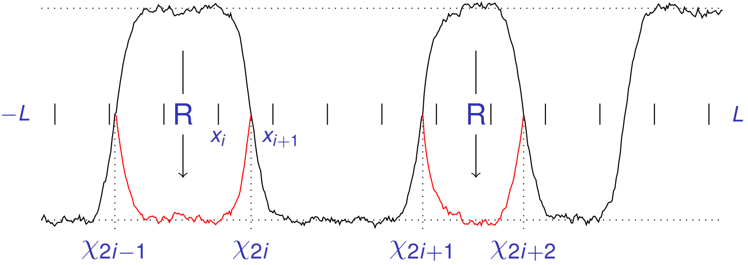


Reflection operator R preserves the measure!

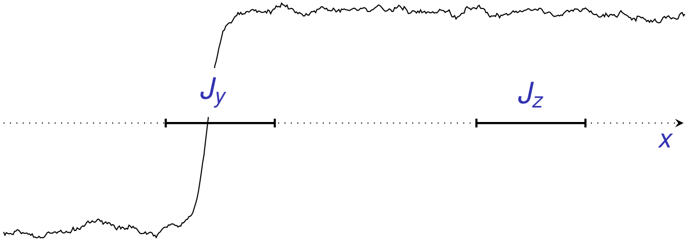
$$\mu_\varepsilon\Big(\text{transition in intervals } I_i\Big)$$

$$= \mu_\varepsilon\Big(\text{wasted excursions in intervals } I_i\Big)$$

$$= \int \nu_{x_{i-1},x_{i+2}}(du_{i-1}, du_{i+2})\, \mu_\varepsilon^{u_{i-1},u_{i+2}}\Big(\text{ wast. exc. in}[x_i, x_{i+1}]\Big).$$

Uniform distribution

Idea: Use symmetry again!



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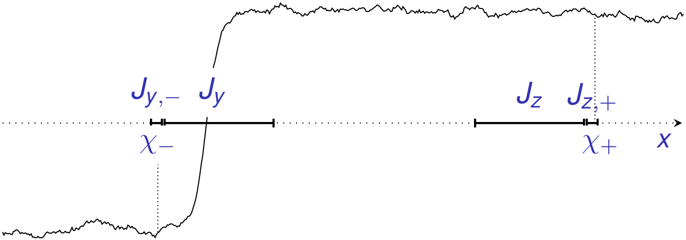

$\chi_\pm$ hitting points of ± 1 in auxilary intervals $J_{y,-}$ and $J_{z,+}$.

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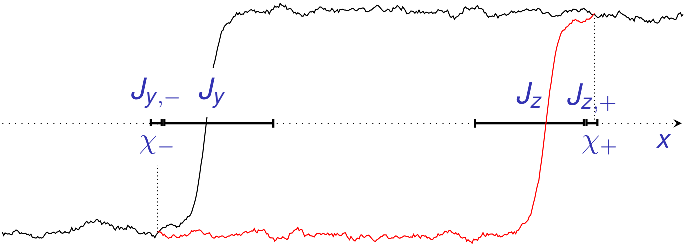

$\chi_\pm$ hitting points of ± 1 in auxilary intervals $J_{y,-}$ and $J_{z,+}$.

Point reflection operator

$$Ru(x) := \begin{cases} u(x) & \text{for } x \leq \chi_-, \\ -u(\chi_- + \chi_+ - x) & \text{for } \chi_- < x < \chi_+, \\ u(x) & \text{for } x \geq \chi_+, \end{cases}$$

leaves μ_ε invariant and moves the transition in J_y close to J_z.

Choice of auxiliary intervals

$\mathcal{J}_y := \{u\colon\ u$ has a δ^-up layer in J_y (+ extra conditions)$\}$.

Lemma (“Hitting Lemma")

Auxiliary intervals $|J_y^-|, |J_z^+| \approx \bar{K}|\log(\varepsilon)|$.

- *Then*

$$\mu^{-1,1}_{\varepsilon,(-L,L)}\Big(u \in \mathcal{J}_y\colon \textit{no hitting of } -1 \textit{ in } J_{y,-}\Big) \leq E_1(\varepsilon)\,\mu^{-1,1}_{\varepsilon,(-L,L)}(\mathcal{J}_y).$$

- *Error term*

$$E_1(\varepsilon) \leq \lambda^{\bar{K}} + L\exp\left(-\frac{c_0-\gamma}{\varepsilon}\right) + 2\exp\left(-\frac{c_1}{2\varepsilon}\right).$$

- *Same for* $J_{z,+}$.

Crucial step for “Hitting Lemma”

Lemma (“Close to 1”)

- *For* $\varepsilon \leq \varepsilon_0$*, small.*
- $K_\varepsilon \sim \log\left(\sqrt{\frac{\varepsilon_0}{\varepsilon}}\right)$ *and* $\ell_\varepsilon := (2K_\varepsilon + 1)\ell_0$.

Crucial step for "Hitting Lemma"

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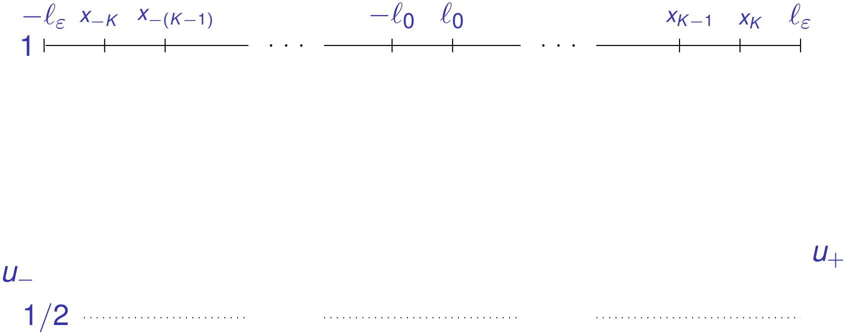
- *For* $\varepsilon \leq \varepsilon_0$*, small.*
- $K_\varepsilon \sim \log\left(\sqrt{\frac{\varepsilon_0}{\varepsilon}}\right)$ *and* $\ell_\varepsilon := (2K_\varepsilon + 1)\ell_0$*.*

Then and all $u_\pm \in [1/2, 3/2]$*, we have*

$$\mu^{u_-,u_+}_{\varepsilon,(-\ell_\varepsilon,\ell_\varepsilon)}\left(\sup_{x\in[-\ell_0,\ell_0]} |u(x) - 1| \geq \sqrt{\frac{\varepsilon}{\varepsilon_0}} \;\middle|\; |u(\pm(2k-1)\ell_0) - 1| \leq \frac{1}{2},\ k = 1, 2, \ldots, K_\varepsilon\right)$$

$$\leq 4\exp\left(-\frac{1}{C\varepsilon_0}\right).$$

Proof of “Close to 1" Lemma



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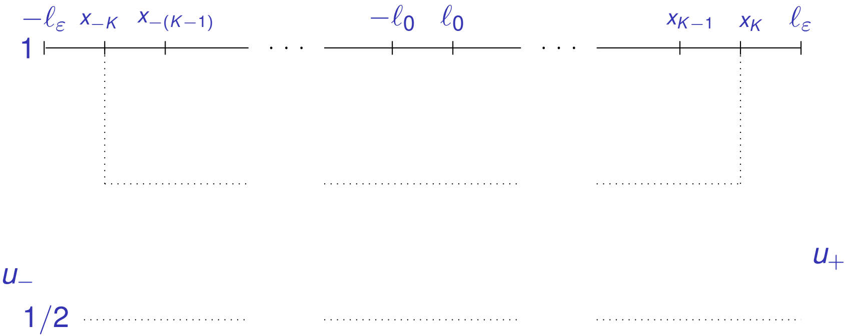

$$\mu^{u_-,u_+}_{\varepsilon,(x_{-(K+1)},x_{K+1})}\left(\sup_{x\in[x_{-K},x_K]} |u(x)-1| \geq \frac{1}{4} \,\middle|\, u \in \mathcal{A} \right)$$

$$\leq 2\,K \exp\left(-\frac{1}{C\varepsilon}\right).$$

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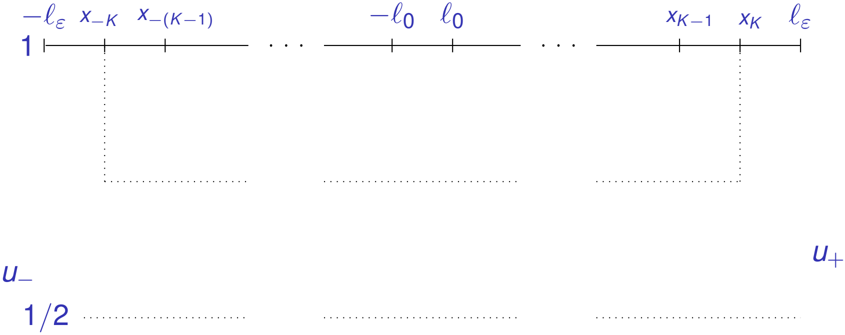

Rescaling: $\hat{u}(x) = 2(u(x) - 1) + 1$.

Rescaled energy

$$\frac{1}{\varepsilon}\hat{E}(\hat{u}) = \frac{1}{4\varepsilon}\int \frac{1}{2}|\partial_x \hat{u}|^2 + 4\, V\left(\frac{1}{2}(\hat{u} - 1) + 1\right)\, dx.$$

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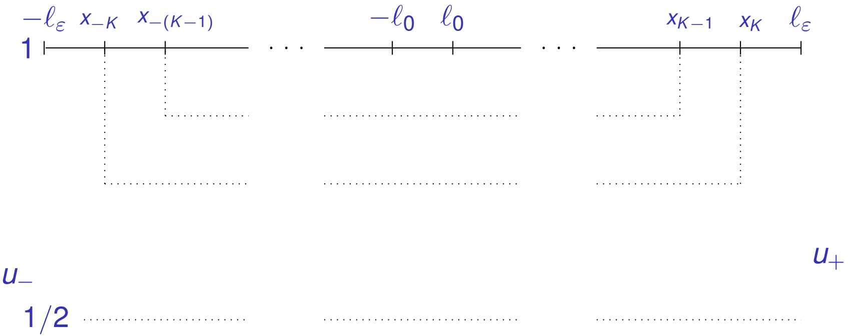

$$\mu^{u_-,u_+}_{\varepsilon,(x_{-(K+1)},x_{K+1})}\left(\sup_{x\in[x_{-K+1},x_{K-1}]}|u(x)-1|\geq\frac{1}{8}\,\middle|\,u\in\hat{\mathcal{A}}\right)$$
$$\leq 2\,(K-1)\exp\left(-\frac{1}{C4\varepsilon}\right).$$

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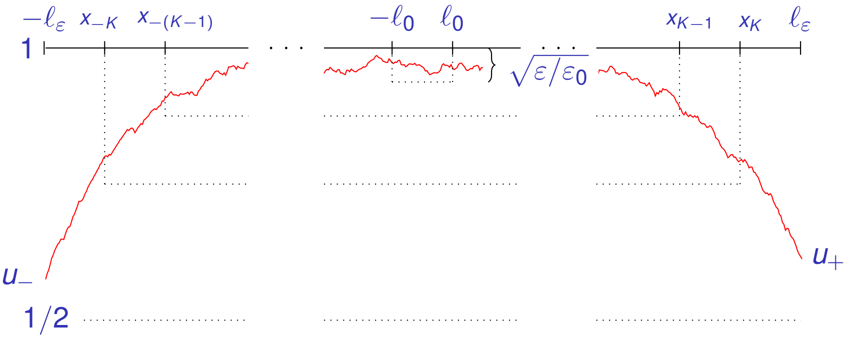

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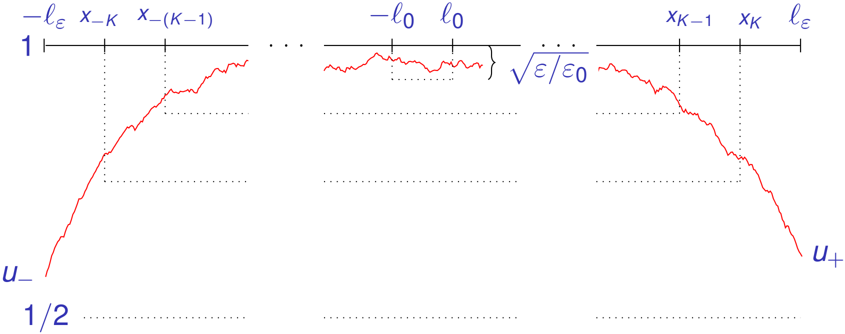

$$\mu^{u_-,u_+}_{\varepsilon,(x_{-(K+1)},x_{K+1})}\left(\sup_{x\in[-\ell_0,\ell_0]}|u(x)-1|\geq\frac{1}{2^{K_\varepsilon}}\,\Big|\,u\in\hat{\mathcal{A}}_k\right)$$

$$\leq 2\left(-\frac{1}{C4^{K_\varepsilon}\varepsilon}\right).$$

Along the way: Tails of the one point distribution

Lemma ("One point distribution")

- *M large, ε small (depending on M).*

$$\mu^{-1,1}_{\varepsilon,(-L,L)}\Big(|u(x_0)| \geq M\Big) \;\leq\; \exp\left(-\frac{M}{\varepsilon\, C}\right).$$

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- *M large, ε small (depending on M).*

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Comment:

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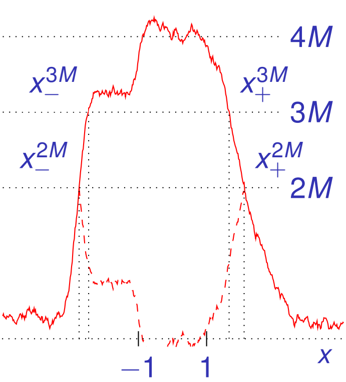
- Closely related to decay of the ground state of the Schrödinger operator

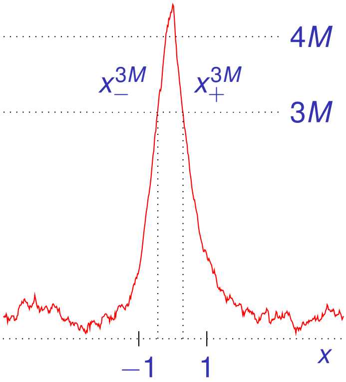
$$\varepsilon \partial_x^2 + V$$

in semiclassical limit.

Argument for "one point distribution" Lemma



(a) **Case 1**: Treated with another reflection argument.



(b) **Case 2:** Treated with Large deviation estimates.

Outlook

Alternative arguments for “close to 1” Lemma and “One point distribution” Lemma based on tricks from Statistical Mechanics (FKG inequality, Brascamp Lieb inequality).

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Relation to diffusion bridges (in higher dimensional asymmetric potentials)?

Conclusion

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Key ideas of proof: Local large deviation bounds, global symmetries, detailed properties of energy landscape.