Amplitude Equation for the generalized Swift Hohenberg Equation with Noise

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Introduction

The generalized Swift Hohenberg equation

$$\partial_t u = ru - (1 + \partial_x{}^2)^2 u + \alpha u^2 - u^3 \qquad \text{(SH)}$$

is a qualitative model for Rayleigh Benard convection.

Introduction

It is well known (Cross/Hohenberg 93, Hilali 95, Burke/Knobloch 06) that

$$u(t,x) \approx \sqrt{|r|} \cdot A(|r|t) \cdot e^{ix} + \sqrt{|r|} \cdot \overline{A(|r|t)} \cdot e^{-ix}.$$

where the complex amplitude $A(T)$ of the dominant mode e^{ix} is the solution of

$$\partial_T A = \text{sign}(r)A + 3(\tfrac{38}{27}\alpha^2 - 1)|A|^2 A,$$

Setting

We consider the following stochastic version of (SH)

$$\partial_t u = \nu \varepsilon^2 u - (1 + \partial_x{}^2)^2 u + \alpha u^2 - u^3 + \varepsilon \sigma \partial_t \beta, \qquad \text{(SSH)}$$

where

- $\beta(t)$ is a real valued standard Brownian motion,
- α, σ and ν are real-valued constants,
- the small parameter $\varepsilon > 0$ relates the distance from bifurcation to the noise strength.

We estimate (SSH) by a similar amplitude equation as in the deterministic case:

$$dA = (\nu A + 3(\tfrac{38}{27}\alpha^2 - 1)A|A|^2 + 3(\alpha^2 - \tfrac{1}{2})\sigma^2 A)dT + 2\alpha\sigma A d\tilde{\beta}\ . \qquad \text{(AE)}$$

Result on bounded domains

Theorem

Let $T_0 > 0$, $\alpha \in \mathbb{R}$ *with* $\alpha^2 < \frac{27}{38}$ *and* $0 < \kappa$. *Let* u *be a mild solution of (SSH) with* $\|u(0)\|_\infty = \mathcal{O}(\varepsilon^{1-\kappa})$. *Let* $A(T) \in \mathbb{C}$, $T \in [0, T_0]$ *solve (AE), then* $\forall p \in \mathbb{N} : \exists C_p$ *such that*

$$\mathbb{P}\left(\sup_{t\in[0,T_0]} \|u(t) - u_A(t) - e^{-t(1+\partial_x^2)^2} u_s(0)\|_\infty > \varepsilon^{2-19\kappa} \right) \leq C_p \varepsilon^p$$

with the approximation

$$u_A(t,x) = \varepsilon A(\varepsilon^2 t) e^{ix} + \varepsilon \bar{A}(\varepsilon^2 t) e^{-ix} + \varepsilon Z_\varepsilon(\varepsilon^2 t)$$

where Z_ε *is the Ornstein-Uhlenbeck process defined by*

$$Z_\varepsilon(T) := \varepsilon^{-1} \sigma \int_0^T e^{-\varepsilon^{-2}(T-s)} d\tilde{\beta}(s).$$

Comparing the two Amplitude equations

Deterministic:

$$\partial_T A = rA + 3(\tfrac{38}{27}\alpha^2 - 1)|A|^2 A$$

With added noise:

$$dA = (\nu A + 3(\tfrac{38}{27}\alpha^2 - 1)A|A|^2)dT \\ + 3(\alpha^2 - \tfrac{1}{2})\sigma^2 A)dT + 2\alpha\sigma A d\tilde{\beta}\ .$$

Where does the difference come from?

- Noise - Nonlinearity interaction,
- Averaging ($\int aZ_\varepsilon^2 dt \approx \int a\sigma dt$).

Idea of Proof / Rescaling

We rescale the the solutions of (SSH) to their natural timescale:

$$v(T) := \varepsilon^{-1} u(\varepsilon^2 T).$$

Since we are on bounded domains we can write v as

$$v = ae^{ix} + \varepsilon\Phi e^{i2x} + c.c. + \varepsilon\Psi + Z_\varepsilon + \sum_{|k|\geq 3} v_k e^{ikx} + e^{-T\varepsilon^{-2}(1+\partial_x^2)^2} v_s(0)$$

The mild solution of v_k looks as follows

$$v_k(T) = \int_0^T e^{-\varepsilon^{-2}(1-k^2)^2(T-s)} \left[\nu v_k(s) + \varepsilon^{-1}\alpha(\widehat{v^2})_k(s) - (\widehat{v^3})_k(s)\right] ds,$$

Idea of Proof / Reduction

With this we can show that until a stopping time τ_* ($\|v(T)\|_\infty = \mathcal{O}(\varepsilon^{0-})$ for $T \in [0, \tau_*]$) we have

$$\|v - a - Z - e^{-T\varepsilon^{-2}(1+\partial_x^2)^2} v_s(0)\|_\infty = \mathcal{O}(\varepsilon^{1-})$$

and by calculating $\widehat{(v^2)}_i$ and $\widehat{(v^3)}_i$ ($i \in 1, 2, 3$) we get

$$da = (\nu a + 2\alpha\bar{a}\Phi + 2\alpha a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + \varepsilon^{-1}2\alpha a Z_\varepsilon + R_1)dT$$
$$d\Phi = (-9\varepsilon^{-2}\Phi + \varepsilon^{-2}\alpha a^2 + R_2)dT$$
$$d\Psi = (-\varepsilon^{-2}\Psi + \varepsilon^{-2}\alpha|a|^2 + \varepsilon^{-2}\alpha Z_\varepsilon^2 + R_3)dT$$

We exchange the $a\phi$, $a\psi$ and $eps^{-1}2\alpha a Z_\varepsilon$ terms by applying Itô differentiation on these terms.

The rest of the proof

The rest of the proof consists of three parts:

- Show that $\int aZ_\varepsilon^2 dt \approx \int a\sigma dt$ (Averaging Lemma)
- Show that a is approximately A
- Show that the stopping time τ_* is long enough (i.e. bigger then a fixed time independent of ε)

The SSH equation on unbounded domains

The solution to the amplitude equation $A(T)$ has values in the Sobolev space $\mathcal{H}^\alpha$, $\alpha > 1/2$ defined by

$$\mathcal{H}^\alpha := \{u \in L^2(\mathbb{R};\mathbb{C}) : \mathcal{F}^{-1}((1+k^2)^{\alpha/2}\mathcal{F}u) \in L^2(\mathbb{R};\mathbb{C})\}.$$

The solution to (SSH) is approximated by

$$u(t,x) \approx \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} + \varepsilon \bar{A}(\varepsilon^2 t, \varepsilon x)e^{-ix} + Z_\varepsilon$$

Problems on unbounded domains

- No Fourier series, but Fourier transform with bands of Eigenvalues.
- SDEs for the modes feature a full linear operator instead of a scalar, which makes the exchanging of mixed products $(a\Phi, a\Psi, ..)$ much more difficult.
- Bounds still depend a lot on A being in $\mathcal{H}^{1/2+}$ which prohibits more general noise (which is at most $\mathcal{H}^{1/2-}$).

Thank you.

Thank you for your attention!