Quantitative estimates in stochastic homogenization

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Motivation: Effective **large scale behavior of random media**

- description by **statistics**
- effective large scale behavior
  $\Rightarrow$ **stochastic homogenization**
- qualitative theory
  $\Rightarrow$ well-established
  $\Rightarrow$ **formula** for effective properties

**In practice:** Evaluation of **formula** requires **approximation**

- only few results; non-optimal estimates for approximation error
- lack of understanding on very basic level
Motivation: Effective large scale behavior of random media

- description by statistics
- effective large scale behavior
  \[ \leadsto \] stochastic homogenization
- qualitative theory
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- lack of understanding on very basic level

Our motivation: Quantitative methods leading to optimal estimates
...model problem: linear, elliptic, scalar, on \( \mathbb{Z}^d \)
Summary

- Framework: discrete elliptic equation with random coefficients
- Qualitative homogenization
- Homogenization formula and corrector – periodic case
- Corrector equation in probability space
- Main results
- A decay estimate for a diffusion semigroup
Discrete elliptic equation with random coefficients
Discrete elliptic equation with random coefficients

\[ \nabla^* a(x) \nabla u(x) = f(x) \]

**Coefficient field**

\[ a : \mathbb{Z}^d \to \mathbb{R}^{d \times d}_{\text{diag, } \lambda} \]

\[ 0 < \lambda \leq a(x) \leq 1 \]

(uniform ellipticity)
Discrete elliptic equation with random coefficients

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(uniform ellipticity)

**Lattice** \( \mathbb{Z}^d \)

sites \( x, y \), coord. directions \( e_1, \ldots, e_d \)

**Gradient** \( \nabla \)

\[ \nabla u = (\nabla_1 u, \ldots, \nabla_d u), \quad \nabla_i u(x) = u(x + e_i) - u(x) \]

(negative) **Divergence** \( \nabla^* \) \( (= \ell^2\text{-adjoint of } \nabla) \)

\[ \nabla^* g = \nabla_1^* g_1 + \ldots + \nabla_d^* g_d, \quad \nabla_i^* g_i(x) = g(x - e_i) - g(x). \]
Discrete elliptic equation with random coefficients

\[ \nabla^* a(x) \nabla u(x) = f(x) \]

**Coefficient field**

\[ a : \mathbb{Z}^d \to \mathbb{R}^{d \times d}_{\text{diag}, \lambda} \]

\[ 0 < \lambda \leq a(x) \leq 1 \]

(uniform ellipticity)

**Random coefficients**

\[ \Omega := (\mathbb{R}^{d \times d}_{\text{diag}, \lambda})^{\mathbb{Z}^d} \]

= space of coefficient fields

\[ \langle \cdot \rangle = \text{probability measure on } \Omega \]

= ”the ensemble”

Behavior in the large \( \rightsquigarrow \) stochastic homogenization
Simplest setting: \( \{a(x)\}_{x \in \mathbb{Z}^d} \) are independent and identically distributed according to a random variable \( A \)

Most general setting: \( \langle \cdot \rangle \) is stationary and ergodic

Stationarity: \( \forall z \in \mathbb{Z}^d : a(\cdot) \) and \( a(\cdot + z) \) have same distribution

Ergodicity: If \( \forall z \in \mathbb{Z}^d \) \( F(a(\cdot + z)) = F(a) \) then \( F = \langle F \rangle \) a. s.
Qualitative homogenization
Numerical simulation - 1d, Dirichlet problem

\[ \nabla^* a(x) \nabla u(x) = 1, \quad x \in (0, L) \cap \mathbb{Z}, \quad L \gg 1 \]

\[ u(0) = u(L) = 0 \]

statistics of \( a \) independent, identically, distributed uniformly in \((0.2, 1)\)
Numerical simulation - 1d, Dirichlet problem

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\[ L = 50 \]
Numerical simulation - 1d, Dirichlet problem

\[ \nabla^* \alpha(x) \nabla u(x) = 1, \quad x \in (0, L) \cap \mathbb{Z}, \quad L \gg 1 \]

\[ u(0) = u(L) = 0 \]

statistics of \( \alpha \) independent, identically, distributed uniformly in \((0.2, 1)\)

\( L = 50 \)
\[ L = 100 \]
\[ L = 500 \]
\[ L = 2000 \]

\[
- \nabla \cdot \mathbf{a}_{\text{hom}} \nabla u = 1 \\
u(0) = u(1) = 0
\]

\[ \mathbf{a}_{\text{hom}} = \langle \mathbf{a}^{-1} \rangle^{-1} \]
Qualitative homogenization result

Kozlov ['79], Papanicolaou & Varadhan ['79]

Suppose \( \langle \cdot \rangle \) is stationary & ergodic. Then:

\[ \exists \text{ unique } a_{\text{hom}} \in \mathbb{R}^{d \times d}_{\text{sym}} \text{ such that:} \]

Given \( f_0(\hat{x}) \) consider right-hand side \( f_L(x) = L^{-2} f_0(\frac{x}{L}) \), \( x \in \mathbb{Z}^d \)

Solve discrete Dirichlet problem:

\[
\begin{align*}
\n \nabla^* a(x) \nabla u_L & = f_L \quad \text{in } x \in ([-0, L) \cap \mathbb{Z})^d \\
 u_L & = 0 \quad \text{outside } ([0, L) \cap \mathbb{Z})^d
\end{align*}
\]

Solve continuum Dirichlet problem:

\[
\begin{align*}
\n -\nabla \cdot a_{\text{hom}} \nabla u_0 & = f_0 \quad \text{in } x \in [0, 1)^d \\
 u_0 & = 0 \quad \text{outside } [0, 1)^d
\end{align*}
\]

Then \( \lim_{L \uparrow \infty} u_L(L\hat{x}) = u_0(\hat{x}) \) almost surely.
Motivation of this talk: *approximation of* \( a_{\text{hom}} \)

...requires *quantitative estimates for* corrector problem
Motivation of this talk: \textbf{approximation of }\alpha_{\text{hom}} \\
...requires \textbf{quantitative estimates for corrector problem}

Related, but different:

- \textbf{homogenization error}, i.e. for $|u_L(L\cdot) - u_0(\cdot)|$
  (Naddaf et al., Conlon et al., ...)

- \textbf{correlation function in Euclidean field theory}
  (Naddaf/Spencer, Giacomin/Olla/Spohn,...)
Formula for $a_{hom}$
Formula for \( a_{hom} \)
— the periodic case —

Let \( \langle \cdot \rangle_L \) be stationary and concentrated on \( L \)-periodic coefficients:
\[
\forall z \in \mathbb{Z}^d \quad a(\cdot + Lz) = a(\cdot) \text{ a. s.}
\]
Formula for $a_{hom}$

— the periodic case —

Let $\langle \cdot \rangle_L$ be stationary and concentrated on $L$-periodic coefficients:

$$\forall z \in \mathbb{Z}^d \quad a(\cdot + Lz) = a(\cdot) \text{ a. s.}$$

We may think about the $L$-periodic ensemble $\langle \cdot \rangle_L$ as a periodic approximation of the stationary and ergodic ensemble $\langle \cdot \rangle$. 
Definition of $a_{\text{hom}, L} = a_{\text{hom}, L}(a)$

$$\forall e \in \mathbb{R}^d : \quad a_{\text{hom}, L} e := L^{-d} \sum_{x \in [0, L)^d} a(x)(e + \nabla \varphi(x))$$

where $\varphi(\cdot) = \varphi(a, \cdot)$ is the \textit{L-periodic} (mean-free) solution to

$$\nabla^* a(x)(e + \nabla \varphi(x)) = 0 \quad x \in [0, L)^d$$
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$\varphi$ is called the \textit{corrector} associated with $a$ and $e$
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$\varphi$ is called the **corrector** associated with $a$ and $e$

- existence and uniqueness by Poincaré’s inequality:
  $$\sum_{x \in [0,L)^d} |\varphi(x)|^2 \lesssim L^2 \sum_{x \in [0,L)^d} |\nabla \varphi(x)|^2$$

- stationarity: $\varphi(a(\cdot + z), \cdot) = \varphi(a, \cdot + z)$ for all $z \in \mathbb{Z}^d$ a.s.
Intuition of $a_{\text{hom},L}$:

Given $\mathbf{e} \in \mathbb{R}^d$ and associated $\varphi$, consider $u_L(x) := \mathbf{e} \cdot x + \varphi(x)$. Then

$$\nabla^* a \nabla u_L = 0$$

**average gradient** $= L^{-d} \sum_{x \in [0,L)^d} \nabla u_L(x) = \mathbf{e}$

**average flux** $= L^{-d} \sum_{x \in [0,L)^d} a(x) \nabla u_L(x) = a_{\text{hom},L} \mathbf{e}$
Formal passage $L \uparrow \infty$ yields:

**Def.** for stationary corrector $\varphi = \varphi(a, x)$ for $\langle \cdot \rangle$ defined by

(i) **corrector equation**

$$\nabla^* a(x)(e + \nabla \varphi(a, x)) = 0 \quad \text{for all } x \in \mathbb{Z}^d \text{ a.e. } a \in \Omega$$

(ii) **sublinear growth on average**

$$\lim_{L \uparrow \infty} L^{-d} \sum_{[0,L)^d} |L^{-1} \varphi(a, x)|^2 = 0.$$  

(iii) **stationarity**
Formal passage \( L \uparrow \infty \) yields:

**Def.** for **stationary corrector** \( \varphi = \varphi(a, x) \) for \( \langle \cdot \rangle \) defined by

1. **corrector equation**

   \[
   \nabla^* a(x)(e + \nabla \varphi(a, x)) = 0 \quad \text{for all } x \in \mathbb{Z}^d \text{ a.e. } a \in \Omega
   \]

2. **sublinear growth on average**

   \[
   \lim_{L \uparrow \infty} L^{-d} \sum_{[0,L)^d} \left| L^{-1} \varphi(a, x) \right|^2 = 0.
   \]

3. **stationarity**

**Def.** for **homogenized coefficient matrix**

\[
\mathbf{a}_{\text{hom}} e = \lim_{L \uparrow \infty} L^{-d} \sum_{[0,L)^d} a(e + \nabla \varphi)^{\text{ergodicity}} = \langle a(e + \nabla \varphi) \rangle
\]
Can we directly get existence of **stationary corrector for** $\langle \cdot \rangle$
from existence of **periodic corrector** by limit $L \uparrow \infty$ ?
Can we directly get existence of stationary corrector for $\langle \cdot \rangle$ from existence of periodic corrector by limit $L \uparrow \infty$?

No, since Poincaré’s inequality degenerates for $L \uparrow \infty$:

$$\sum_{x \in [0,L)^d} |\varphi(x)|^2 \lesssim L^2 \sum_{x \in [0,L)^d} |\nabla \varphi(x)|^2$$
Can we directly get existence of **stationary corrector for** $\langle \cdot \rangle$
from existence of **periodic corrector** by limit $L \uparrow \infty$?

**No**, since **Poincaré’s inequality degenerates** for $L \uparrow \infty$:

$$
\sum_{x \in [0,L)^d} |\varphi(x)|^2 \lesssim L^2 \sum_{x \in [0,L)^d} |\nabla \varphi(x)|^2
$$

In fact, for $d \leq 2$ **stationary correctors** in general do not exist!
The corrector equation in $L^2_{\langle . \rangle}$

$$D^* a(0)(e + D\phi) = 0.$$
From $\mathbb{Z}^d$ to $\Omega$ by stationarity

**Def.:** A random field $f(a, x)$ is called **stationary**, if

$$\forall x, z, a \quad f(a(\cdot + z), x) = f(a, x + z).$$

**Def.:** The **stationary extension** of a random variable $F(a)$ is defined by

$$\overline{F}(a, x) := F(a(\cdot + x)).$$
From $\mathbb{Z}^d$ to $\Omega$ by stationarity

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random variables $\overarrow{\overrightarrow{\bigcdot}}$ stationary random fields
From $\mathbb{Z}^d$ to $\Omega$ by stationarity

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$$\overline{F}(a, x) := F(a \cdot + x).$$

Random variables $\rightarrow$ stationary random fields

Physical space $(\nabla_i, \mathbb{Z}^d)$ $\overset{\sim}{\rightarrow}$ Stationarity $\overset{\sim}{\rightarrow}$ Probability space $(D_i, \Omega)$
The horizontal derivative

\[ \nabla_i \overline{F}(a, x) = \overline{F}(a, x + e_i) - \overline{F}(a, x) \]
\[ = \overline{F}(a(e_i), x) - \overline{F}(a, x) =: D_i \overline{F}(a, x), \]
The horizontal derivative

\[ \nabla_i \overline{F}(a, x) = \overline{F}(a, x + e_i) - \overline{F}(a, x) \]
\[ = \overline{F}(a(\cdot + e_i), x) - \overline{F}(a, x) =: D_i F(a, x), \]

**Def:** Horizontal derivative for \( F(a) \)

\[ D_i F(a) := F(a(\cdot + e_i)) - F(a), \]
\[ D_i^* F(a) := F(a(\cdot - e_i)) - F(a) \]
Corrector problem in probability

\[ D^* \alpha(0)(e + D\phi) = 0 \]
Corrector problem in probability

\[ D^* a(0)(e + D\phi) = 0 \]

Homogenization formula in probability

\[ a_{\text{hom}} e = \langle a(0)(e + D\phi) \rangle \]

\[ e \cdot a_{\text{hom}} e = \inf_{F \in L^2(\Omega)} \langle (e + DF) \cdot a(0)(e + DF) \rangle. \]
Does there exist $\phi$ s.t. $D^* a(0) (e + D\phi) = 0$?

**Yes** if

$$\exists \rho > 0 \ \forall F \quad \langle (F - \langle F \rangle)^2 \rangle \leq \frac{1}{\rho} \langle |DF|^2 \rangle$$

$\text{SG}(\rho)$ for $D^* D$
Does there exist \( \phi \) s.t. \( D^* a(0)(e + D\phi) = 0 \) ?

Yes if

\[
\exists \rho > 0 \ \forall F \quad \langle (F - \langle F \rangle)^2 \rangle \leq \frac{1}{\rho} \langle |DF|^2 \rangle
\]

SG(\( \rho \)) for \( D^* D \)

This is the case for the periodic ensemble \( \langle \cdot \rangle_L \).
However, SG(\( \rho_L \)) for \( D^* D \) in \( L^2_L \) degenerates for \( L \uparrow \infty \):

\[
\rho_L \sim \frac{1}{L^2}
\]
Does there exist $\phi$ s.t. $D^* a(0)(e + D\phi) = 0$?

**Yes** if

$$\exists \rho > 0 \forall F \left\langle (F - \langle F \rangle)^2 \right\rangle \leq \frac{1}{\rho} \left\langle |DF|^2 \right\rangle$$

SG($\rho$) for $D^* D$

This is the case for the **periodic ensemble** $\langle \cdot \rangle_L$. However, SG($\rho_L$) for $D^* D$ in $L^2 \langle \cdot \rangle_L$ degenerates for $L \uparrow \infty$:

$$\rho_L \sim \frac{1}{L^2}$$

Too many variables $\{a(x)\}_{x \in \mathbb{Z}^d}$ — too few derivatives $D_1, \ldots, D_d$. 
Our main assumption (inspired by Naddaf & Spencer ['97]):

Instead of (SG) for $D^* D$

$$\langle (F - \langle F \rangle)^2 \rangle \lesssim \sum_{i=1}^{d} \langle (D_i F)^2 \rangle$$

assume (SG) for $\sum_{x \in \mathbb{Z}^d} \left( \frac{\partial}{\partial x} \right)^2$

$$\langle (F - \langle F \rangle)^2 \rangle \leq \frac{1}{\rho} \sum_{x \in \mathbb{Z}^d} \langle \left( \frac{\partial F}{\partial x} \right)^2 \rangle$$
Def. **vertical derivative**

\[
\frac{\partial F}{\partial x} := F - \langle F | \{a(y)\}_{y \neq x} \rangle \sim \frac{\partial F}{\partial a(x)}
\]

...measure how sensitively \( F \) depends on \( a(x) \).

**Basic example**

\[
\{a(x)\}_{x \in \mathbb{Z}^d} \text{ i. i. d.} \Rightarrow SG(\rho) \text{ for } \sum_x \left( \frac{\partial}{\partial x} \right)^2
\]
Statement of main result
Theorem A [GNO, GO]

i) Let $d > 2$, suppose $SG(\rho)$ for $\sum x (\frac{\partial}{\partial x})^2$. Then

$$\forall q < \infty \quad \langle \phi^{2q} \rangle^{\frac{1}{2q}} \leq C(d, \lambda, \rho, q)$$

ii) Let $d = 2$, consider $\langle \cdot \rangle_L$. Suppose $SG(\rho)$ for $\sum x (\frac{\partial}{\partial x})^2$. Then

$$\langle \phi^2 \rangle^{\frac{1}{2}} \leq C(d, \lambda, \rho) \ln L$$
Consider $\langle \cdot \rangle_L$ periodic ensemble and periodic proxy

$$a_{\text{hom},L}(a) := L^{-d} \sum_{x \in [0,L)^d} a(x)(e + D\varphi(a, x))$$

**Theorem B** [GNO]. Let $d \geq 2$, suppose $SG(\rho)$ for $\sum_{x \in [0,L)^d} \frac{\partial}{\partial x}$. Then

$$\text{Var}_{\langle \cdot \rangle_L} [a_{\text{hom},L}] \leq C(d, \lambda, \rho) L^{-d}$$
Consider $\langle \cdot \rangle_L$ periodic ensemble and periodic proxy

$$a_{\text{hom}, L}(a) := L^{-d} \sum_{x \in [0, L)^d} a(x)(e + D\varphi(a, x))$$

**Theorem B** [GNO].
Let $d \geq 2$, suppose $SG(\rho)$ for $\sum_{x \in [0, L)^d} \frac{\partial}{\partial x}$. Then

$$\text{Var}_{\langle \cdot \rangle_L}[a_{\text{hom}, L}] \leq C(d, \lambda, \rho) L^{-d}$$

**Remark:** $a_{\text{hom}, L}$ is spatial average of correlated r.v.
In fact, for $1 - \lambda \ll 1$ and $\{a(x)\}_{x \in [0, L)^d}$ i. i. d. have

$$\text{Cov}_{\langle \cdot \rangle_L} \left[ a(x)(e + \nabla \varphi(x)), a(z)(e + \nabla \varphi(z)) \right] \sim \nabla^2 G_L(x - z)$$

$$\text{Cov}_{\langle \cdot \rangle_L} \left[ \varphi(x), \varphi(z) \right] \sim G_L(x - z)$$

where $G_L$ is the $L$-periodic Green’s function for $\nabla^* \nabla$. 
Let $\langle \cdot \rangle_\infty$ be i.i.d. with base measure $\beta$, i.e.

$$\langle F \rangle_\infty = \int_{\Omega} F(a) \prod_{x \in \mathbb{Z}^d} \beta(da(x)).$$

Let $\langle \cdot \rangle_L$ be $L$-periodic and i.i.d. with base measure $\beta$, i.e.

$$\langle F \rangle_L = \int_{\Omega_L} F(a) \prod_{x \in [0,L)^d} \beta(da(x)).$$

**Theorem C [GNO]** Let $d \geq 2$. Then

$$|\langle a_{\text{hom},L} \rangle_L - a_{\text{hom}}|^2 \leq C(d, \lambda, \rho) L^{-2d}$$

(up to logarithmic corrections for $d = 2$)
Optimal estimate of systematic error

Let $\langle \cdot \rangle_\infty$ be i.i.d. with base measure $\beta$, i.e.

$$\langle F \rangle_\infty = \int_\Omega F(a) \prod_{x \in \mathbb{Z}^d} \beta(da(x)).$$

Let $\langle \cdot \rangle_L$ be $L$-periodic and i. i. d. with base measure $\beta$, i.e.

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**Theorem C [GNO]** Let $d \geq 2$. Then

$$|\langle a_{\text{hom},L} \rangle_L - a_{\text{hom}}|^2 \leq C(d, \lambda, \rho) L^{-2d}$$

(up to logarithmic corrections for $d = 2$)

combine with $\langle |a_{\text{hom},L} - \langle a_{\text{hom},L} \rangle|^2 \rangle \leq C(d, \lambda, \rho) L^{-d}$ to get total $L^2_{\langle \cdot \rangle_L}$-error.
Common analytic estimate of the proofs:

optimal decay estimate for the semigroup

\[ \exp(-D^* a(0)D) \]
Semigroup representation of $\phi$

$$u(t) := \exp(-tD^* a(0) D)f, \quad f = -D^* a(0)e.$$ 

then formally $\phi := \int_0^\infty u(t) \, dt$ solves

$$D^* a(0) D\phi = -D^* a(0)e \quad \text{in } L^q \langle \cdot \rangle.$$
Semigroup representation of $\phi$

$$u(t) := \exp(-tD^* a(0)D)f, \quad f = -D^* a(0)e.$$  

then formally $\phi := \int_0^\infty u(t) \, dt$ solves

$$D^* a(0)D\phi = -D^* a(0)e \quad \text{in } L^q_{\langle \cdot \rangle}$$

This is rigorous as soon as $\int_0^\infty \langle |u(t)|^q \rangle^{\frac{1}{q}} \, dt < \infty$!
Standard:

\[(SG)\text{ for } D^* D \Rightarrow \text{ exponential decay of } \exp(-D^* a(0) D)\]

Our estimate:

\[(SG)\text{ for } \sum_x \left( \frac{\partial}{\partial x} \right)^2 \Rightarrow \text{ algebraic decay of } \exp(-D^* a(0) D)\]

(with optimal rate!)
**Theorem 1 [GNO]:** (optimal decay in $t$)

Let $d \geq 2$, suppose $SG(\rho)$ for $\sum_x (\frac{\partial}{\partial x})^2$. Then for $q < \infty$ have

$$\langle | \exp(-tD^*a(0)D)D^*g|^{2q} \rangle^{\frac{1}{2q}}$$

$$\leq C(d,\lambda,\rho,q)(t+1)^{-\left(\frac{d}{4} + \frac{1}{2}\right)} \left( \sum_{x \in \mathbb{Z}^d} \langle (\frac{\partial g}{\partial x})^{2q} \rangle^{\frac{1}{2q}} \right)$$
We explain a much simpler situation:
- constant coefficient semigroup $D^*D$ instead of $D^*a(0)D$
- initial data $f$ instead of $D^*g$
- linear exponent $p = 2$ instead $2q$
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- constant coefficient semigroup $D^* D$ instead of $D^* a(0) D$
- initial data $f$ instead of $D^* g$
- linear exponent $p = 2$ instead $2q$

**Theorem 2 [GNO]:** (optimal decay in $t$)

Let $d \geq 2$, suppose $SG(\rho)$ for $\sum_x (\frac{\partial}{\partial x})^2$. Then for $f$ with $\langle f \rangle = 0$ have

$$\langle | \exp (-tD^* D) f |^2 \rangle^{\frac{1}{2}} \leq \frac{1}{\sqrt{\rho}} \left( \sum_{x \in \mathbb{Z}^d} G^2(t, x) \right)^{\frac{1}{2}} \sum_{x \in \mathbb{Z}^d} \langle (\frac{\partial f}{\partial x})^2 \rangle^{\frac{1}{2}},$$

where $G(t, x)$ denotes the parabolic Green’s function for $(\partial_t + \nabla^* \nabla)$.

$$\left( \sum_{x \in \mathbb{Z}^d} G^2(t, x) \right)^{\frac{1}{2}} \sim (1+t)^{-\frac{d}{4}}, \quad \left( \sum_{x \in \mathbb{Z}^d} |\nabla G(t, x)|^2 \right)^{\frac{1}{2}} \sim (1+t)^{-\left(\frac{d}{4} + \frac{1}{2}\right)}.$$
Argument for Theorem 2:
Set $u(t) := \exp(-tD^* D)f$. 
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Set $u(t) := \exp(-tD^* D)f$.

Stationary extension $\overline{u}$ characterized by parabolic equation
\[(\partial_t + \nabla^* \nabla)\overline{u}(t, x) = 0, \quad \overline{u}(t = 0, x) = \overline{f}(x)\]
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Set \( u(t) := \exp(-tD^* D)f \).

Stationary extension \( \overline{u} \) characterized by parabolic equation
\[
(\partial_t + \nabla^* \nabla)\overline{u}(t, x) = 0, \quad \overline{u}(t = 0, x) = \overline{f}(x)
\]

Green's representation for \( u \) and \( \frac{\partial u}{\partial y} \)
\[
u(t) = \sum_{z \in \mathbb{Z}^d} G(t, z)\overline{f}(z), \quad \frac{\partial u}{\partial y}(t) = \sum_{z \in \mathbb{Z}^d} G(t, z)\frac{\partial \overline{f}}{\partial y}(z)
\]
Argument for Theorem 2:
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Green's representation for $u$ and $\frac{\partial u}{\partial y}$

$$u(t) = \sum_{z \in \mathbb{Z}^d} G(t, z) \overline{f}(z), \quad \frac{\partial u}{\partial y}(t) = \sum_{z \in \mathbb{Z}^d} G(t, z) \frac{\partial \overline{f}}{\partial y}(z)$$

Spectral gap estimate

$$\langle u^2(t) \rangle^{\frac{1}{2}} \leq \frac{1}{\sqrt{\rho}} \left( \sum_{y \in \mathbb{Z}^d} \langle \left( \frac{\partial u}{\partial y} (t) \right)^2 \rangle \right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\rho}} \left( \sum_{y \in \mathbb{Z}^d} \left( \left( \sum_{z \in \mathbb{Z}^d} G(t, z) \frac{\partial \overline{f}}{\partial y}(z) \right)^2 \right) \right)^{\frac{1}{2}}$$

$$\overset{\text{stat.}}{=} \frac{1}{\sqrt{\rho}} \left( \sum_{y \in \mathbb{Z}^d} \left( \left( \sum_{x \in \mathbb{Z}^d} G(t, y-x) \frac{\partial \overline{f}}{\partial x}(y-x) \right)^2 \right) \right)^{\frac{1}{2}}$$
\[
\left( \sum_{y \in \mathbb{Z}^d} \left\langle \left( \sum_{x \in \mathbb{Z}^d} G(t, y-x) \frac{\partial f}{\partial x}(y-x) \right)^2 \right\rangle \right)^{\frac{1}{2}}
\]

\[
\triangle\text{-inequality}
\]

\[
\leq \sum_{x \in \mathbb{Z}^d} \left( \sum_{y \in \mathbb{Z}^d} \left\langle G^2(t, y-x) \left| \left( \frac{\partial f}{\partial x} \right)(y-x) \right| \right\rangle \right)^{\frac{1}{2}}
\]

\[
G \text{ is deterministic, stationarity}
\]

\[
= \sum_{x \in \mathbb{Z}^d} \left( \sum_{y \in \mathbb{Z}^d} G^2(t, y-x) \left\langle \left| \frac{\partial f}{\partial x} \right| \right\rangle \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{y \in \mathbb{Z}^d} G^2(t, y-x) \right)^{\frac{1}{2}} \sum_{x \in \mathbb{Z}^d} \left\langle \left| \frac{\partial f}{\partial x} \right| \right\rangle^{\frac{1}{2}}.
\]
Source of difficulty for $\exp(-tD^*a(0)D)$ (Theorem 1)

Instead of representation

$$\frac{\partial u}{\partial y}(t) = \sum_{z \in \mathbb{Z}^d} G(t, z) \frac{\partial f}{\partial y}(z)$$

$\rightsquigarrow$ **Duhamel's formula** for divergence form initial data $D^*g$

$$\frac{\partial u(t)}{\partial y} = \sum_{z \in \mathbb{Z}^d} \nabla_z G(t, a, 0, z) \cdot \frac{\partial g}{\partial y}(z)$$

$$+ \int_0^t \sum_{z \in \mathbb{Z}^d} \nabla_z G(t - s, a, 0, z) \cdot \frac{\partial a(z)}{\partial y} \nabla_z \mathbf{u}(s, z) \, ds.$$
Quantitative analysis requires estimates on

\[ |\nabla_x G(t, \mathbf{a}, x, y)|^p \]

where \( G(t, \mathbf{a}, x, y) \) denotes parabolic, non-constant coefficient Green’s function on \( \mathbb{Z}^d \).

need...

- optimal decay in \( t \sim (t + 1)^{-(d/2 + 1/2)p} \)
- deterministic, i.e. uniform in \( \mathbf{a} \)
- exponent \( p > 2 \)

... can only expect

- averaged in space (with weight)

use: discrete elliptic & parabolic regularity theory
Caccioppoli estimate, Meyers’ estimate, Nash-Aronson, ...
Future directions

- from scalar to **systems** (elasticity)
  scalar case relies on testing with nonlinear functions $|u|^{p-2}u$

- from uniform ellipticity
to **supercritical percolation**
  random geometry of percolation cluster
  $\leadsto$ isoperimetric inequality
  $\leadsto$ Green’s function estimate

  have quantitative results for a toy problem

- application to homogenization error
**Ann. Probab.** 2011

**Ann. Appl. Probab.** 2012

- A. Gloria, S. N. & F. Otto. **work in progress**
  * Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics.
  * Approximation of effective coefficients by periodization in stochastic homogenization.