Quantitative estimates in stochastic homogenization

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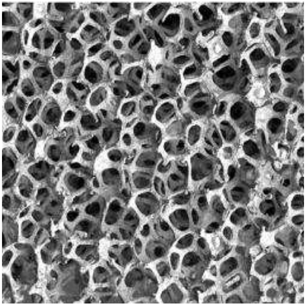
joint work with Antoine Gloria and Felix Otto

RDS 2012 – Bielefeld

Motivation: Effective large scale behavior of random media



- description by **statistics**
- effective large scale behavior

  ⇝ **stochastic homogenization**
- qualitative theory
  - ⇝ well-established
  - ⇝ **formula** for effective properties

In practice: Evaluation of **formula** requires **approximation**

- only few results; non-optimal estimates for approximation error
- lack of understanding on very basic level

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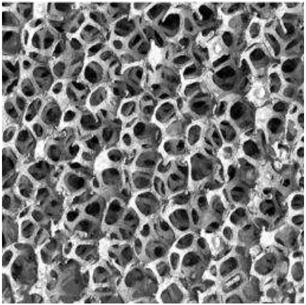

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Our motivation:
Quantitative methods leading to **optimal estimates**
...**model problem:** linear, elliptic, scalar, on $\mathbb{Z}^d$

Summary

- Framework: discrete elliptic equation with random coefficients
- Qualitative homogenization
- Homogenization formula and corrector – periodic case
- Corrector equation in probality space
- Main results
- A decay estimate for a diffusion semigroup

Discrete elliptic equation with random coefficients

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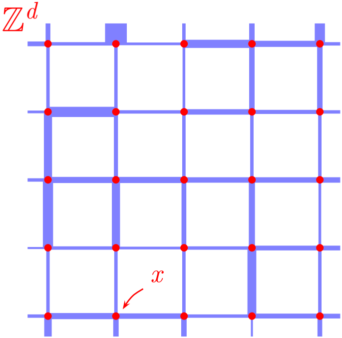


$$\nabla^* \boldsymbol{a}(x) \nabla u(x) = f(x)$$

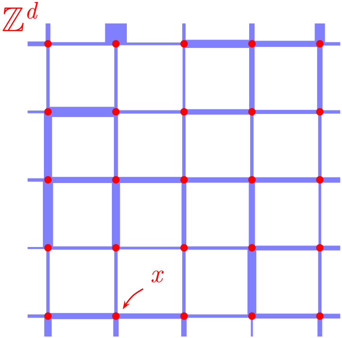
Coefficient field

$$\boldsymbol{a} : \mathbb{Z}^d \to \mathbb{R}^{d \times d}_{\mathrm{diag}, \lambda}$$

$$0 < \lambda \leqslant \boldsymbol{a}(x) \leqslant 1$$

(uniform ellipticity)

Discrete elliptic equation with random coefficients



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Lattice $\mathbb{Z}^d$

sites x, y, coord. directions $e_1, \ldots, e_d$

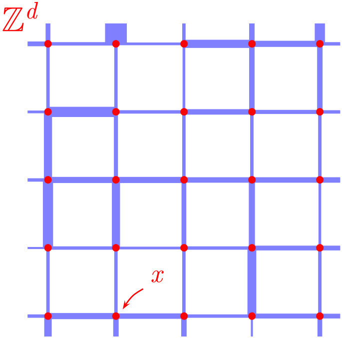
Gradient ∇

$$\nabla u = (\nabla_1 u, \ldots, \nabla_d u), \ \ \nabla_i u(x) = u(x+e_i) - u(x)$$

(negative) **Divergence ∇^*** (= ℓ^2-adjoint of ∇)

$$\nabla^* g = \nabla_1^* g_1 + \ldots + \nabla_d^* g_d, \ \ \nabla_i^* g_i(x) = g(x-e_i) - g(x).$$

Discrete elliptic equation with random coefficients



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Random coefficients

$$\Omega := \left(\mathbb{R}^{d\times d}_{\mathrm{diag},\lambda}\right)^{(\mathbb{Z}^d)}$$

= space of coefficient fields

$\langle\cdot\rangle$ = probability measure on Ω

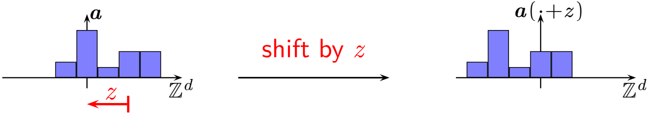
= "the ensemble"

Behavior in the large $\rightsquigarrow$ stochastic homogenization

Simplest setting: $\{\boldsymbol{a}(x)\}_{x\in\mathbb{Z}^d}$ are **i**ndependent and **i**dentically **d**istributed according to a random variable A

Most general setting: $\langle\cdot\rangle$ is stationary and ergodic

Stationarity: $\forall z \in \mathbb{Z}^d$: $\boldsymbol{a}(\cdot)$ and $\boldsymbol{a}(\cdot + z)$ have same distribution



Ergodicity: If $\forall z \in \mathbb{Z}^d$ $F(\boldsymbol{a}(\cdot + z)) = F(\boldsymbol{a})$ then $F = \langle F\rangle$ a. s.

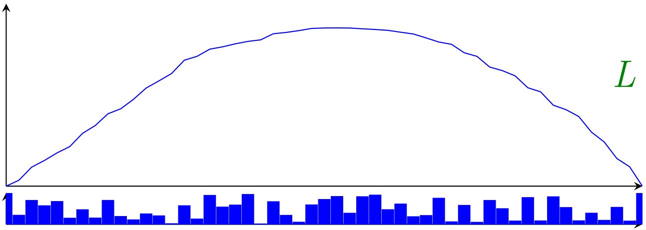
Qualitative homogenization

Numerical simulation - 1d, Dirichlet problem

$$\nabla^* \boldsymbol{a}(x) \nabla u(x) = 1, \qquad x \in (0, L) \cap \mathbb{Z}, \qquad L \gg 1$$

$$u(0) = u(L) = 0$$

statistics of $\boldsymbol{a}$ **i**ndependent, **i**dentically, **d**istributed uniformly in $(0.2, 1)$



$L = 50$

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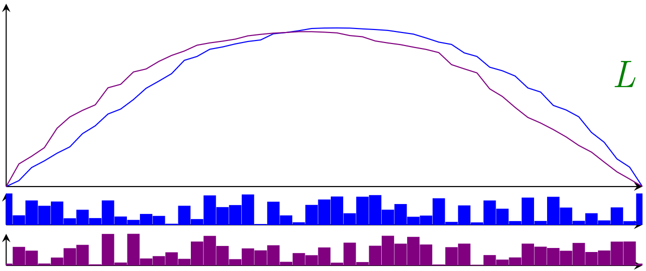

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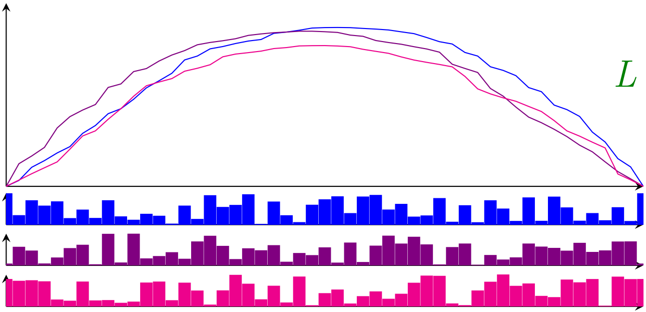

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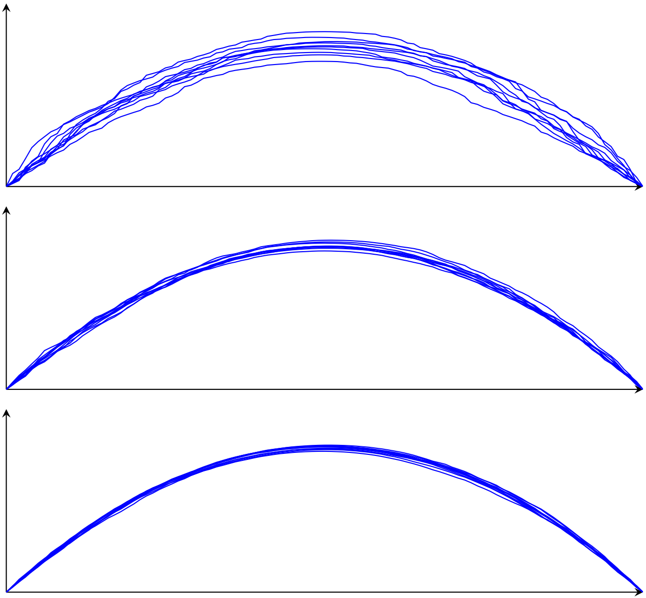

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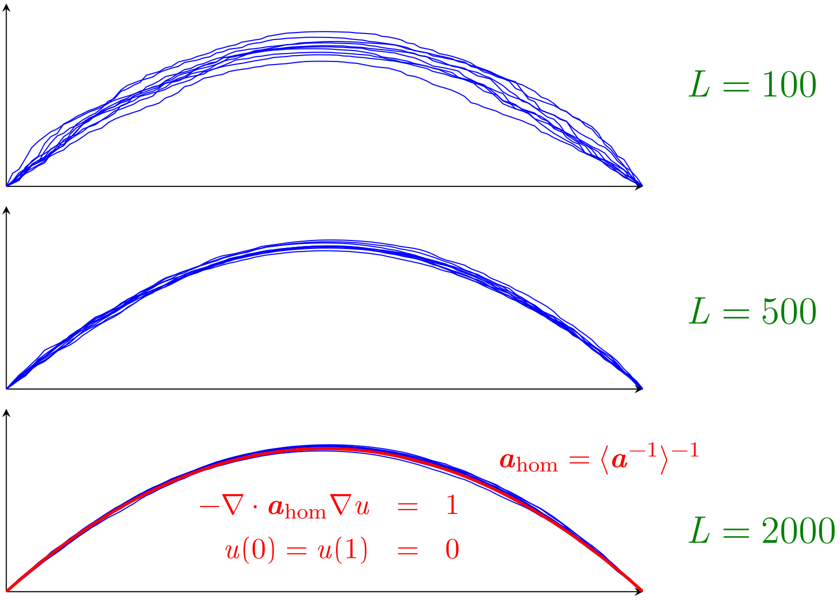
$L = 500$

$L = 2000$


$L = 100$
$L = 500$
$a_{\text{hom}} = \langle a^{-1} \rangle^{-1}$
$-\nabla \cdot a_{\text{hom}} \nabla u = 1$
$u(0) = u(1) = 0$
$L = 2000$

Qualitative homogenization result

Kozlov ['79], Papanicolaou & Varadhan ['79]

Suppose $\langle \cdot \rangle$ is **stationary & ergodic**. Then:

$\exists$ unique $\boldsymbol{a}_{\mathrm{hom}} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that:

Given $f_0(\hat{x})$ consider right-hand side $f_L(x) = L^{-2} f_0(\frac{x}{L})$, $x \in \mathbb{Z}^d$

Solve discrete Dirichlet problem :
$$\begin{cases} \nabla^* \boldsymbol{a}(x) \nabla u_L = f_L & \text{in } x \in ([-0, L) \cap \mathbb{Z})^d \\ u_L = 0 & \text{outside } ([0, L) \cap \mathbb{Z})^d \end{cases}$$

Solve continuum Dirichlet problem :
$$\begin{cases} -\nabla \cdot \boldsymbol{a}_{\mathrm{hom}} \nabla u_0 = f_0 & \text{in } x \in [0,1)^d \\ u_0 = 0 & \text{outside } [0,1)^d \end{cases}$$

Then $\lim_{L\uparrow\infty} u_L(L\hat{x}) = u_0(\hat{x})$ **almost surely.**

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...requires **quantitative estimates** for **corrector problem**

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Related, but different:

- **homogenization error**, i.e. for $|u_L(L\cdot) - u_0(\cdot)|$
  (Naddaf et al., Conlon et al., ...)
- **correlation function** in **Euclidean field theory**
  (Naddaf/Spencer, Giacomin/Olla/Spohn,...)

Formula for $\boldsymbol{a}_{hom}$

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— the **periodic** case —

Let $\langle\cdot\rangle_L$ be stationary and concentrated on L-**periodic coefficients**:

$$\forall z \in \mathbb{Z}^d \quad \boldsymbol{a}(\cdot + Lz) = \boldsymbol{a}(\cdot) \text{ a. s.}$$

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We may think about the L-periodic ensemble $\langle\cdot\rangle_L$ as a **periodic approximation** of the stationary and ergodic ensemble $\langle\cdot\rangle$.

Definition of $\boldsymbol{a}_{\mathrm{hom},L} = \boldsymbol{a}_{\mathrm{hom},L}(\boldsymbol{a})$

$$\forall e \in \mathbb{R}^d : \qquad \boldsymbol{a}_{\mathrm{hom},L}\, e := L^{-d} \sum_{x\in[0,L)^d} \boldsymbol{a}(x)(e + \nabla\varphi(x))$$

where $\varphi(\cdot) = \varphi(\boldsymbol{a},\cdot)$ is the L-**periodic** (mean-free) solution to

$$\nabla^* \boldsymbol{a}(x)(e + \nabla\varphi(x)) = 0 \qquad x \in [0,L)^d$$

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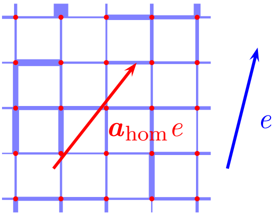
- **existence and uniqueness** by **Poincaré's inequality**: $\sum_{x\in[0,L)^d} |\varphi(x)|^2 \lesssim L^2 \sum_{x\in[0,L)^d} |\nabla\varphi(x)|^2$
- **stationarity**: $\varphi(\boldsymbol{a}(\cdot + z), \cdot) = \varphi(\boldsymbol{a}, \cdot + z)$ for all $z \in \mathbb{Z}^d$ a.s.

Intuition of $\boldsymbol{a}_{\mathrm{hom},L}$:
Given $e \in \mathbb{R}^d$ and associated φ,
consider $u_L(x) := e \cdot x + \varphi(x)$. Then



$$\nabla^* \boldsymbol{a} \nabla u_L = 0$$

$$\textbf{average gradient} = L^{-d} \sum_{x \in [0,L)^d} \nabla u_L(x) = \boldsymbol{e}$$

$$\textbf{average flux} = L^{-d} \sum_{x \in [0,L)^d} \boldsymbol{a}(x) \nabla u_L(x) = \boldsymbol{a}_{\mathrm{hom},L}\, \boldsymbol{e}$$

Formal passage $L \uparrow \infty$ yields:

Def. for **stationary corrector** $\varphi = \varphi(\boldsymbol{a}, x)$ for $\langle\cdot\rangle$ defined by

(i) **corrector equation**

$$\nabla^* \boldsymbol{a}(x)(e + \nabla\varphi(\boldsymbol{a}, x)) = 0 \qquad \text{for all } x \in \mathbb{Z}^d \text{ a.e. } \boldsymbol{a} \in \Omega$$

(ii) **sublinear growth on average**

$$\lim_{L\uparrow\infty} L^{-d} \sum_{[0,L)^d} \left|L^{-1}\varphi(\boldsymbol{a}, x)\right|^2 = 0.$$

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(iii) **stationarity**

Def. for **homogenized coefficient matrix**

$$\boldsymbol{a}_{\mathrm{hom}}\, e = \lim_{L\uparrow\infty} L^{-d} \sum\nolimits_{[0,L)^d} \boldsymbol{a}(e + \nabla\varphi) \overset{\text{ergodicity}}{=} \langle \boldsymbol{a}(e + \nabla\varphi)\rangle$$

Can we directly get existence of **stationary corrector for $\langle\cdot\rangle$**
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No, since **Poincaré's inequality degenerates** for $L \uparrow \infty$:

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In fact, for $d \leqslant 2$ **stationary correctors** in general **do not exist**!

The corrector equation in $L^2_{\langle\cdot\rangle}$

$$D^* \boldsymbol{a}(0)(e + D\phi) = 0 \ .$$

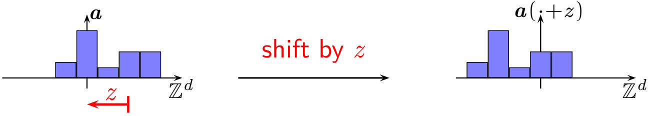
From $\mathbb{Z}^d$ to Ω by stationarity

Def.: A random field $f(\boldsymbol{a}, x)$ is called **stationary**, if

$$\forall x, z, \boldsymbol{a} \quad f(\boldsymbol{a}(\cdot + z), x) = f(\boldsymbol{a}, x + z).$$

Def.: The **stationary extension** of a random variable $F(\boldsymbol{a})$ is defined by

$$\overline{F}(\boldsymbol{a}, x) := F(\boldsymbol{a}(\cdot + x)).$$



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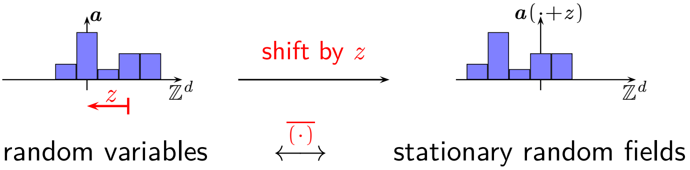

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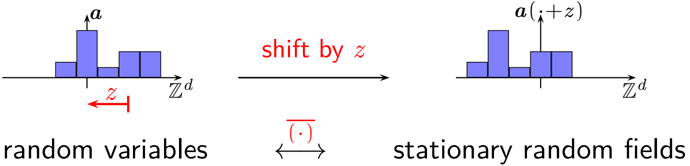

physical space | stationarity ⇝ | probability space
$(\nabla_i,\ \mathbb{Z}^d)$ | | $(D_i,\ \Omega)$

The horizontal derivative

$$\begin{aligned} \nabla_i \overline{F}(\boldsymbol{a}, x) &= \overline{F}(\boldsymbol{a}, x+e_i) - \overline{F}(\boldsymbol{a}, x) \\ &= \overline{F}(\boldsymbol{a}(\cdot+e_i), x) - \overline{F}(\boldsymbol{a}, x) =: \overline{D_i F}(\boldsymbol{a}, x), \end{aligned}$$

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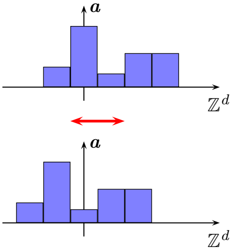
\end{aligned}
$$

Def: Horizontal derivative for $F(\boldsymbol{a})$

$$
\begin{aligned}
D_i F(\boldsymbol{a}) &:= F(\boldsymbol{a}(\cdot+e_i)) - F(\boldsymbol{a}), \\
D_i^* F(\boldsymbol{a}) &:= F(\boldsymbol{a}(\cdot-e_i)) - F(\boldsymbol{a})
\end{aligned}
$$



Corrector problem in probability

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Homogenization formula in probability

$$\begin{aligned} \boldsymbol{a}_{\mathrm{hom}}\, e &= \langle \boldsymbol{a}(0)(e + D\phi)\rangle \\ e \cdot \boldsymbol{a}_{\mathrm{hom}}\, e &= \inf_{F \in L^2(\Omega)} \langle (e + DF) \cdot \boldsymbol{a}(0)(e + DF)\rangle. \end{aligned}$$

Does there exists ϕ s.t. $D^* \boldsymbol{a}(0)(e + D\phi) = 0$?

Yes if $\boxed{\exists \rho > 0 \ \forall F \quad \langle (F - \langle F \rangle)^2 \rangle \leqslant \frac{1}{\rho} \langle |DF|^2 \rangle}$ SG(ρ) for $D^* D$

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This is the case for the **periodic ensemble** $\langle \cdot \rangle_L$.
However, SG(ρ_L) for $D^* D$ in $L^2_{\langle \cdot \rangle_L}$ **degenerates for** $L \uparrow \infty$:

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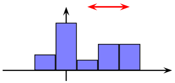
Too many variables $\{\boldsymbol{a}(x)\}_{x \in \mathbb{Z}^d}$ — too few derivatives $D_1, \ldots D_d$.

Our main assumption (inspired by Naddaf & Spencer ['97]):

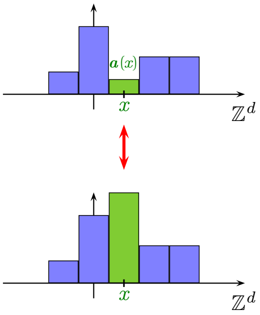
Instead of **(SG) for** $D^* D$

$$\langle (F - \langle F \rangle)^2 \rangle \lesssim \sum_{i=1}^{d} \langle (D_i F)^2 \rangle$$



assume **(SG) for** $\sum_{x \in \mathbb{Z}^d} \left(\frac{\partial}{\partial x}\right)^2$

$$\langle (F - \langle F \rangle)^2 \rangle \leqslant \frac{1}{\rho} \sum_{x \in \mathbb{Z}^d} \left\langle \left(\frac{\partial F}{\partial x} \right)^2 \right\rangle$$



Def. vertical derivative

$\frac{\partial F}{\partial x} := F - \langle F \mid \{\boldsymbol{a}(y)\}_{y \neq x} \rangle \ \sim \ \frac{\partial F}{\partial \boldsymbol{a}(x)}$

...measure how sensitively F depends on $\boldsymbol{a}(x)$.

Basic example

$$\{\boldsymbol{a}(x)\}_{x \in \mathbb{Z}^d} \text{ i. i. d. } \Rightarrow \quad SG(\rho) \text{ for } \sum_x \left(\frac{\partial}{\partial x}\right)^2$$

Statement of main result

Existence and higher moment bounds

Theorem A [GNO, GO]

i) Let $d > 2$, suppose $SG(\rho)$ for $\sum_x (\frac{\partial}{\partial x})^2$. Then

$$\forall q < \infty \quad \langle \phi^{2q} \rangle^{\frac{1}{2q}} \leqslant C(d, \lambda, \rho, q)$$

ii) Let $d = 2$, consider $\langle \cdot \rangle_L$. Suppose $SG(\rho)$ for $\sum_x (\frac{\partial}{\partial x})^2$. Then

$$\langle \phi^2 \rangle_L^{\frac{1}{2}} \leqslant C(d, \lambda, \rho) \ln L$$

Optimal variance estimate for periodic ensemble

Consider $\langle\cdot\rangle_L$ periodic ensemble and periodic proxy

$$\boldsymbol{a}_{\mathrm{hom},L}(\boldsymbol{a}) := L^{-d} \sum_{x\in[0,L)^d} \boldsymbol{a}(x)(e + D\varphi(\boldsymbol{a},x))$$

Theorem B [GNO].
Let $d \geqslant 2$, suppose $SG(\rho)$ for $\sum_{x\in[0,L)^d} \frac{\partial}{\partial x}$. Then

$$\mathrm{Var}_{\langle\cdot\rangle_L}\left[\boldsymbol{a}_{\mathrm{hom},L}\right] \leqslant C(d,\lambda,\rho)L^{-d}$$

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$$\mathrm{Var}_{\langle\cdot\rangle_L}\left[\boldsymbol{a}_{\mathrm{hom},L}\right] \leqslant C(d,\lambda,\rho)L^{-d}$$

Remark: $\boldsymbol{a}_{\mathrm{hom},L}$ is **spatial average** of **correlated r.v.**
In fact, for $1-\lambda \ll 1$ and $\{\boldsymbol{a}(x)\}_{x\in[0,L)^d}$ i. i. d. have

$$\mathrm{Cov}_{\langle\cdot\rangle_L}\Big[\boldsymbol{a}(x)(e+\nabla\varphi(x)), \boldsymbol{a}(z)(e+\nabla\varphi(z))\Big] \sim \nabla^2 G_L(x-z)$$

$$\mathrm{Cov}_{\langle\cdot\rangle_L}\Big[\varphi(x),\ \varphi(z)\Big] \sim G_L(x-z)$$

where G_L is the L-periodic Green's function for $\nabla^*\nabla$.

Optimal estimate of systematic error

Let $\langle\cdot\rangle_\infty$ be i.i.d. with base measure β, i.e.

$$\langle F\rangle_\infty = \int_\Omega F(\boldsymbol{a}) \prod_{x\in\mathbb{Z}^d} \beta(d\boldsymbol{a}(x)).$$

Let $\langle\cdot\rangle_L$ be L-periodic and i. i. d. with base measure β, i.e.

$$\langle F\rangle_L = \int_{\Omega_L} F(\boldsymbol{a}) \prod_{x\in[0,L)^d} \beta(d\boldsymbol{a}(x)).$$

Theorem C [GNO] Let $d \geqslant 2$. Then

$$|\langle \boldsymbol{a}_{\mathrm{hom},L}\rangle_L - \boldsymbol{a}_{\mathrm{hom}}|^2 \leqslant C(d,\lambda,\rho) L^{-2d}$$

(up to logarithmic corrections for $d = 2$)

Optimal estimate of systematic error

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combine with $\langle|\boldsymbol{a}_{\mathrm{hom},L} - \langle\boldsymbol{a}_{\mathrm{hom},L}\rangle|^2\rangle \leqslant C(d,\lambda,\rho)L^{-d}$ to get total $L^2_{\langle\cdot\rangle_L}$-error.

Common analytic estimate of the proofs:

optimal decay estimate for the semigroup

$$\exp(-D^* \boldsymbol{a}(0) D)$$

Semigroup representation of ϕ

$$u(t) := \exp(-tD^* \boldsymbol{a}(0) D) f, \qquad f = -D^* \boldsymbol{a}(0) e.$$

then **formally** $\phi := \int_0^\infty u(t)\, dt$ solves

$$D^* \boldsymbol{a}(0) D \phi = -D^* \boldsymbol{a}(0) e \qquad \text{in } L^q_{\langle \cdot \rangle}$$

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This is rigorous as soon as $\int_0^\infty \langle |u(t)|^q \rangle^{\frac{1}{q}}\, dt < \infty$!

Standard:

(SG) for $D^* D$ $\quad\Rightarrow\quad$ exponential decay of $\exp(-D^* \boldsymbol{a}(0) D)$

Our estimate:

(SG) for $\sum_x (\frac{\partial}{\partial x})^2$ $\quad\Rightarrow\quad$ **algebraic decay of** $\exp(-D^* \boldsymbol{a}(0) D)$

(with optimal rate!)

Theorem 1 [GNO]: (optimal decay in t)

Let $d \geqslant 2$, suppose $SG(\rho)$ for $\sum_x (\frac{\partial}{\partial x})^2$. Then for $q < \infty$ have

$$\langle | \exp\big(-tD^* \boldsymbol{a}(0) D\big) D^* g|^{2q} \rangle^{\frac{1}{2q}}$$

$$\leqslant C_{(d,\lambda,\rho,q)} \, (t+1)^{-(\frac{d}{4}+\frac{1}{2})} \left(\sum_{x \in \mathbb{Z}^d} \langle (\frac{\partial g}{\partial x})^{2q} \rangle^{\frac{1}{2q}} \right)$$

We explain a much simpler situation:

- constant coefficient semigroup D^*D instead of $D^*\boldsymbol{a}(0)D$
- initial data f instead of D^*g
- linear exponent $p = 2$ instead $2q$

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Theorem 2 [GNO]: (optimal decay in t)
Let $d \geqslant 2$, suppose $SG(\rho)$ for $\sum_x(\frac{\partial}{\partial x})^2$. Then for f with $\langle f\rangle =$ have

$$\langle|\exp\big(-tD^*D\big)f|^2\rangle^{\frac{1}{2}} \leqslant \frac{1}{\sqrt{\rho}}\left(\sum_{x\in\mathbb{Z}^d} G^2(t,x)\right)^{\frac{1}{2}} \sum_{x\in\mathbb{Z}^d}\langle(\frac{\partial f}{\partial x})^2\rangle^{\frac{1}{2}},$$

where $G(t,x)$ denotes the parabolic Green's function for $(\partial_t+\nabla^*\nabla)$.

$$\left(\sum_{x\in\mathbb{Z}^d} G^2(t,x)\right)^{\frac{1}{2}} \sim (1+t)^{-\frac{d}{4}}, \quad \left(\sum_{x\in\mathbb{Z}^d} |\nabla G(t,x)|^2\right)^{\frac{1}{2}} \sim (1+t)^{-(\frac{d}{4}+\frac{1}{2})}$$

Argument for Theorem 2:
Set $u(t) := \exp(-tD^*D)f$.

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Spectral gap estimate

$$\begin{aligned}
\langle u^2(t)\rangle^{\frac{1}{2}} &\leqslant \frac{1}{\sqrt{\rho}}\Big(\sum_{y\in\mathbb{Z}^d}\langle(\tfrac{\partial u}{\partial y}(t))^2\rangle\Big)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{\rho}}\left(\sum_{y\in\mathbb{Z}^d}\left\langle\left(\sum_{z\in\mathbb{Z}^d} G(t,z)\frac{\partial\overline{f}}{\partial y}(z)\right)^2\right\rangle\right)^{\frac{1}{2}} \\
&\overset{\substack{\text{stat.}\\ x=y-z}}{=} \frac{1}{\sqrt{\rho}}\left(\sum_{y\in\mathbb{Z}^d}\left\langle\left(\sum_{x\in\mathbb{Z}^d} G(t,y-x)\overline{\frac{\partial f}{\partial x}}(y-x)\right)^2\right\rangle\right)^{\frac{1}{2}}
\end{aligned}$$

$$\left(\sum_{y\in\mathbb{Z}^d}\left\langle\left(\sum_{x\in\mathbb{Z}^d}G(t,y-x)\overline{\frac{\partial f}{\partial x}}(y-x)\right)^2\right\rangle\right)^{\frac{1}{2}}$$

$$\overset{\substack{\triangle\text{-inequality}\\ \text{in } \left(\sum_{y\in\mathbb{Z}^d}\langle(\cdot)^2\rangle\right)^{\frac{1}{2}}}}{\leqslant} \sum_{x\in\mathbb{Z}^d}\left(\sum_{y\in\mathbb{Z}^d}\langle G^2(t,y-x)\,|\overline{\left(\frac{\partial f}{\partial x}\right)}(y-x)|^2\rangle\right)^{\frac{1}{2}}$$

$$\overset{\substack{G\text{ is deterministic,}\\ \text{stationarity}}}{=} \sum_{x\in\mathbb{Z}^d}\left(\sum_{y\in\mathbb{Z}^d}G^2(t,y-x)\langle|\frac{\partial f}{\partial x}|^2\rangle\right)^{\frac{1}{2}}$$

$$= \left(\sum_{y\in\mathbb{Z}^d}G^2(t,y-x)\right)^{\frac{1}{2}}\sum_{x\in\mathbb{Z}^d}\langle|\frac{\partial f}{\partial x}|^2\rangle^{\frac{1}{2}}.$$

□

Source of difficulty for $\exp(-tD^*\boldsymbol{a}(0)D)$ (Theorem 1)

Instead of representation $\frac{\partial u}{\partial y}(t) = \sum_{z\in\mathbb{Z}^d} G(t,z)\frac{\partial \overline{f}}{\partial y}(z)$

$\rightsquigarrow$ **Duhamel's formula** for divergence form initial data D^*g

$$\frac{\partial u(t)}{\partial y} = \sum_{z\in\mathbb{Z}^d} \nabla_z G(t,\boldsymbol{a},0,z)\cdot\frac{\partial \overline{g}}{\partial y}(z)$$
$$+\int_0^t \sum_{z\in\mathbb{Z}^d} \nabla_z G(t-s,\boldsymbol{a},0,z)\cdot\tfrac{\partial \boldsymbol{a}(z)}{\partial y}\nabla_z\overline{u}(s,z)\,ds.$$

Quantitative analysis requires estimates on

$$|\nabla_x G(t, \boldsymbol{a}, x, y)|^p$$

where $G(t, \boldsymbol{a}, x, y)$ denotes parabolic, non-constant coefficient Green's function on $\mathbb{Z}^d$.

need...

- optimal decay in $t \rightsquigarrow (t+1)^{-(\frac{d}{2}+\frac{1}{2})p}$
- deterministic, i. e. uniform in $\boldsymbol{a}$
- exponent $p > 2$

... can only expect

- averaged in space (with weight)

use: discrete elliptic & parabolic regularity theory
Caccioppoli estimate, Meyers' estimate, Nash-Aronson, ...

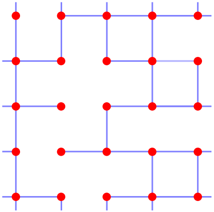
Future directions

- from scalar to **systems** (elasticity)

  scalar case relies on testing with nonlinear functions $|u|^{p-2}u$

- from uniform ellipticity
  to **supercritical percolation**

  random geometry of percolation cluster
  $\rightsquigarrow$ isoperimetric inequality
  $\rightsquigarrow$ Green's function estimate

  have quantitative results for a toy problem

- application to homogenization error

- A. Gloria & F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations.
  Ann. Probab. 2011
- A. Gloria & F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations.
  Ann. Appl. Probab. 2012
- A. Gloria, S. N. & F. Otto. **work in progress**
  * Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics.
  * Approximation of effective coefficients by periodization in stochastic homogenization.