Selection of an invariant measure of a dynamical system by noise. An example.

Etienne Pardoux

joint with J. Mattingly

Introduction 0

Very roughly, the question we want to address in this talk can be formulated as follows.

Take a dynamical system (an ODE) whose large time behavior depends dramatically upon the initial condition, e.g. because of some conserved quantities.

Could it be that when adding a very small noise (possibly together with some small damping term), the system forgets its initial condition, and becomes ergodic, in such a way that this remains true in the small noise limit (i.e. those invariant measures would converge to a uniquely selected invariant measure of the dynamical system).

Introduction 1

- Our work is motivated by the following open problem. Consider a $2D$ Navier–Stokes equation with additive white noise of the form

$$\dot{u} - \varepsilon\Delta u + (u \cdot \nabla)u + \nabla p = \sqrt{\varepsilon}\dot{W}, \quad \mathrm{div}(u) = 0,$$

  where W is an $L^2(\mathbb{R}^2)$–valued BM such that $\forall\varepsilon > 0$, the above has a unique invariant measure μ_ε (see Hairer, Mattingly (06)). Kuksin (06) shows that $\{\mu_\varepsilon, \, \varepsilon > 0\}$ is tight, and that any limit of a converging subsequence is an invariant measure of the Euler equation. But does the whole sequence converge, and if yes, towards which particular invariant measure of the Euler equation ?
- We do not claim to solve this difficult *true problem*. Rather, we consider a much simpler problem, namely a $3D$ SDE with damping of the order of ε and additive white noise multiplied by $\sqrt{\varepsilon}$. Our very simple *toy problem* has however in common with the *true problem* the property that the limiting deterministic undamped ODE possesses conserved quantities and infinitely many invariant measures.

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Introduction 2

- Consider the following three dimensional ordinary differential equation:

$$\dot{X}_t = Y_t Z_t$$
$$\dot{Y}_t = X_t Z_t$$
$$\dot{Z}_t = -2X_t Y_t,$$

- This equation has two conserved quantities : $2X_t^2 + Z_t^2$ and $2Y_t^2 + Z_t^2$.
- We consider, for $\varepsilon > 0$, the following damped/noisy version of the above ODE

$$\dot{X}_t^\varepsilon = Y_t^\varepsilon Z_t^\varepsilon - \varepsilon X_t^\varepsilon + \sigma_x \sqrt{\varepsilon} \dot{B}_t$$
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Ergodicity for each $\varepsilon > 0$

- The respective scalings of the damping factor and of the noise are chosen in such a way that

$$\sup_{0<\varepsilon\le 1}\sup_{t\ge 0}\mathbb{E}\left[\|(X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)\|^2\right] < \infty.$$

- Provided at least two of the three parameters σ_x, σ_y and σ_z are non zero, which we assume from now on, then the solution of the three–dimensional SDE has a unique invariant measure μ_ε for each $\varepsilon > 0$.
- Our aim is to study the limit of μ_ε, as $\varepsilon \to 0$.

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Large time behavior of the solution of the ODE

- The existence of the two conserved quantities implies that all of the orbits of the ODE are bounded and most are closed orbits, topologically equivalent to a circle. All orbits live on the surface of a sphere whose radius is dictated by the values of the conserved quantities.
- To any initial point (X_0, Y_0, Z_0) on one of the closed orbits, we can associate a measure defined by the following limit

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \delta_{(X_s,Y_s,Z_s)} ds \, .$$

- Any such defined measure is an invariant measure for the ODE. Hence we see that the ODE has infinitely many invariant measures.
- Our result is that under appropriate conditions, there exists a unique probability measure μ such that μ_ε converges weakly as μ as $\varepsilon \to 0$.

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Convergence on $[0, T]$

- We first note that as $\varepsilon \to 0$, the process $(X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)$ converges to the solution of the ODE on any finite time interval.
- A simple calculation yields

$$\mathbb{E}\left(\|(X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)\|^2\right) = e^{-2\varepsilon t}\|(X_0, Y_0, Z_0)\|^2 + \|\sigma\|^2\left(1 - e^{-2\varepsilon t}\right)/2.$$

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which is consistent with the convergence towards the solution of the ODE, and the conservation of the norm along solutions of the ODE.
- However

$$\mathbb{E}\left(\|(X_{t/\varepsilon}^\varepsilon, Y_{t/\varepsilon}^\varepsilon, Z_{t/\varepsilon}^\varepsilon)\|^2\right)$$

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A different time scale

- This suggests to consider the "asymptotically constant quantities" in the time scale t/ε.
- We define

$$U_t^\varepsilon = 2(X_{t/\varepsilon}^\varepsilon)^2 + (Z_{t/\varepsilon}^\varepsilon)^2, \quad V_t^\varepsilon = 2(Y_{t/\varepsilon}^\varepsilon)^2 + (Z_{t/\varepsilon}^\varepsilon)^2.$$

- We have

$$dU_t^\varepsilon = [2\sigma_x^2 + \sigma_z^2 - 2U_t^\varepsilon]dt + 4\sigma_x X_{t/\varepsilon}^\varepsilon dB_t + 2\sigma_z Z_{t/\varepsilon}^\varepsilon dD_t,$$
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- The main step of our work consists in showing that the limit (U_t, V_t) as $\varepsilon \to 0$ of $(U_t^\varepsilon, V_t^\varepsilon)$ satisfies the following SDE.

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The (U, V) equation

- $$\begin{aligned} dU_t &= [2\sigma_x^2 + \sigma_z^2 - 2U_t]dt + \sigma_x\sqrt{8\big(U_t - \Gamma(U_t, V_t)\big)}dB_t \\ &\quad + 2\sigma_z\sqrt{\Gamma(U_t, V_t)}dD_t, \\ dV_t &= [2\sigma_y^2 + \sigma_z^2 - 2V_t]dt + \sigma_y\sqrt{8\big(V_t - \Gamma(U_t, V_t)\big)}dC_t \\ &\quad + 2\sigma_z\sqrt{\Gamma(U_t, V_t)}dD_t. \end{aligned}$$

- where
$$\Gamma(u, v) = \lim_{t\to\infty} \frac{1}{t}\int_0^t Z_s^2 ds,$$
if (X_t, Y_t, Z_t) follows the ODE, starting from any point $(x, y, z) \in \mathbb{R}^3$ such that $(2x^2 + z^2, 2y^2 + z^2) = (u, v)$.

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- More explicitly

$$\Gamma(u,v) = u \wedge v \; \Lambda\left(\frac{u \wedge v}{u \vee v}\right),$$

- where $\Lambda(r)$ is a continuous and strictly increasing function on $[0,1]$ with $\Lambda(0) = \frac{1}{2}$ and $\Lambda(1) = 1$. Furthermore as $\varepsilon \to 0^+$,

$$\Lambda(\varepsilon) = \frac{1}{2} + \frac{1}{16}\epsilon + \frac{1}{32}\epsilon^2 + o(\epsilon^2)$$

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- Suppose for a moment that

$$(H) \quad \text{the } (U, V) \text{ SDE has a unique weak solution.}$$

- Then we can show

Proposition

As $\varepsilon \to 0$,

$$\{(U_t^\varepsilon, V_t^\varepsilon),\ t \geq 0\} \to \{(U_t, V_t),\ t \geq 0\}$$

weakly in $C([0, +\infty); \mathbb{R}^2)$.

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The invariant measures of the ODE

- To each $(x, y, z) \in \mathbb{R}^3 \backslash \{0\}$, we attach $(u, v) = (2x^2 + z^2, 2y^2 + z^2) \in (0, +\infty)^2$.
- To each $(u, v) \in (0, +\infty)^2$, one can associate two orbits of the ODE starting from (x, y, z), which, in addition to (u, v) depend only upon the sign of

$$\sigma(x, y, z) = \text{sign}(\mathbf{1}_{\{|x| \geq |y|\}} x + \mathbf{1}_{\{|x| < |y|\}} y).$$

- We denote by $\mathcal{O}(u, v, +1)$ and $\mathcal{O}(u, v, -1)$ those two orbits, and by $\nu_{(u,v,+1)}(dx, dy, dz)$ (resp. $\nu_{(u,v,-1)}(dx, dy, dz)$) the probability measure which is the mean over $(x, y, z) \in \mathcal{O}(u, v, +1)$ (resp. over $(x, y, z) \in \mathcal{O}(u, v, -1)$) of the Dirac masses at (x, y, z).

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The limit of μ_ε as $\varepsilon \to 0$

- Define the probability measure μ on $\mathbb{R}^3$ by

$$\mu(dx, dy, dz) = \frac{1}{2} \int_{\mathbb{R}^2} \lambda(du, dv) \left[\nu_{(u,v,+1)}(dx, dy, dz) + \nu_{(u,v,-1)}(dx, dy, dz)\right].$$

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Does the (U, V) equation have a unique solution ?

- It is easy to see that (U_t, V_t) stays in the quadrant $\{u \geq 0, v \geq 0\}$, and cannot reach any of the two axis $\{(0, v),\ v > 0\}$ and $\{(u, 0),\ u > 0\}$. Moreover, in each sector $\{u > v > 0\}$ and $\{v > u > 0\}$, the SDE for (U_t, V_t) has Lipschitz continuous coefficients, so we have strong uniqueness. However, the Lipschitz property of the diffusion coefficients is no longer true in the vicinity of the diagonal, and the SDE is degenerate on the diagonal.
- To understand the behavior near the diagonal $\{u = v\}$, it is better to write the equations for $H_t = 2^{-1}(U_t + V_t)$ and $K_t = 2^{-1}(U_t - V_t)$.

Does the (U, V) equation have a unique solution ?

- It is easy to see that (U_t, V_t) stays in the quadrant $\{u \geq 0, v \geq 0\}$, and cannot reach any of the two axis $\{(0, v),\ v > 0\}$ and $\{(u, 0),\ u > 0\}$. Moreover, in each sector $\{u > v > 0\}$ and $\{v > u > 0\}$, the SDE for (U_t, V_t) has Lipschitz continuous coefficients, so we have strong uniqueness. However, the Lipschitz property of the diffusion coefficients is no longer true in the vicinity of the diagonal, and the SDE is degenerate on the diagonal.
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We have (here $H_t \geq 0$, $K_t \in \mathbb{R}$)

$$\begin{aligned}
dH_t = & \left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2H_t\right] dt + \sqrt{2}\sigma_x \sqrt{H_t + K_t - \Gamma(H_t + K_t, H_t - K_t)}\, dB_t \\
& + \sqrt{2}\sigma_y \sqrt{H_t - K_t - \Gamma(H_t + K_t, H_t - K_t)}\, dC_t \\
& + 2\sigma_z \sqrt{\Gamma(H_t + K_t, H_t - K_t)}\, dD_t \\
dK_t = & \left[\sigma_x^2 - \sigma_y^2 - 2K_t\right] dt + \sqrt{2}\sigma_x \sqrt{H_t + K_t - \Gamma(H_t + K_t, H_t - K_t)}\, dB_t \\
& - \sqrt{2}\sigma_y \sqrt{H_t - K_t - \Gamma(H_t + K_t, H_t - K_t)}\, dC_t
\end{aligned}$$

The diagonal is hit

- Let us look at the equation for K_t. If the solution gets close to the diagonal (i. e. to $K_t = 0$), then, in e.g. the case $K_t > 0$,
- the argument of the square root in front of dB_t equals

$$\begin{aligned} H_t + K_t - \Gamma(H_t + K_t, H_t - K_t) &= H_t + K_t - (H_t - K_t)\Lambda\left(\frac{H_t - K_t}{H_t + K_t}\right) \\ &\simeq 2\left[K_t + \frac{H_t}{|\log(2K_t/H_t)|}\right] \\ &>> K_t \end{aligned}$$

- and similarly the argument of the square root in front of dC_t is

$$H_t - K_t - \Gamma(H_t + K_t, H_t - K_t) \simeq 2\frac{H_t}{|\log(2K_t/H_t)|} >> K_t.$$

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$\sigma_x = \sigma_y > 0$

- If $\sigma_x = \sigma_y > 0$, then the process stays on the diagonal once it is hit. In this case we have strong uniqueness of the SDE. Indeed, uniqueness holds until the diagonal is reached, and once on the diagonal, the process H_t solves the SDE

$$dH_t = (\|\sigma\|^2 - 2H_t)dt + 2\sigma_z\sqrt{H_t}dD_t,$$

which has a unique strong solution.

- If moreover $\sigma_z = 0$, then H_t solves a linear ODE, and $H_t \to \frac{\sigma_x^2+\sigma_y^2}{2} =: \overline{\sigma}$ as $t \to \infty$, and $\lambda = \delta_{(\overline{\sigma},\overline{\sigma})}$.
- If $\sigma_z > 0$, then H_t is the unique solution of a SDE on $\mathbb{R}_+$ (which can reach zero iff $\sigma_z^2 > \sigma_x^2 + \sigma_y^2$). H_t is ergodic, and the invariant measure λ is concentrated on the diagonal.

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- If however $\sigma_x \neq \sigma_y$, then when hitting for the first time the diagonal, the process (U_t, V_t) is instantaneously reflected into one of the open sectors $\{u > v > 0\}$ or $\{v > u > 0\}$, depending upon the sign of $\sigma_x^2 - \sigma_y^2$. Moreover the process hits again the diagonal infinitely often.
- So far, in that case, we have not been able to prove uniqueness of the law of the solution (U_t, V_t) of the above SDE. Hence in this case our results do not apply (yet ?).

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THANK YOU FOR YOUR ATTENTION !