Completeness and semi-flows for stochastic differential equations with monotone drift

Michael Scheutzow

Technische Universität Berlin

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Stochastic Differential Equations

An SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sum_{j=1}^m \sigma_j(X_t)\,\mathrm{d}W_j(t),\;\; X_0 = x \in \mathbb{R}^d.$$

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Existence and uniqueness of solutions, continuous dependence on initial condition and existence of solution flow of homeomorphisms if b, σ globally Lipschitz.

Are these properties still true (at least locally) in case

- infinite number of driving BM (or Kunita-type sdes)
- local one-sided Lipschitz condition?

Existence and uniqueness of local solutions

Consider the sde

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + M(\mathrm{d}t, X_t),\ X_0 = x \in \mathbb{R}^d,$$

where b continuous, M cont. martingale field s.t.
$a(x,y) := \frac{\mathrm{d}}{\mathrm{d}t}[M(t,x), M(t,y)]$ is cont. and determ.,
$\mathcal{A}(x,y) := a(x,x) - a(x,y) - a(y,x) + a(y,y) (= \frac{\mathrm{d}}{\mathrm{d}t}[M(t,x) - M(t,y)])$
and

One-sided local Lipschitz condition

$$2\langle b(x) - b(y), x - y\rangle + \mathrm{Tr}\mathcal{A}(x,y) \le K_N |x-y|^2,\ |x|, |y| \le N.$$

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The sde has a unique local solution.

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In the "classical" case

$M(t,x) = \sum_{j=1}^m \sigma_j(x) W_j(t)$
$\mathcal{A}(x,y) = (\sigma(x) - \sigma(y))(\sigma(x) - \sigma(y))^t,\ \mathrm{Tr}\mathcal{A}(x,y) = \|\sigma(x) - \sigma(y)\|^2$

Idea of proof (Krylov, Prévôt/Röckner): Euler approx.

For $n \in \mathbb{N}$ let $\phi_0^n := x$ and for $t \in (\frac{k}{n}, \frac{k+1}{n}]$:

$$\phi_t^n := \phi_{k/n}^n + \int_{k/n}^t b(\phi_{k/n}^n)\,\mathrm{d}s + \int_{k/n}^t M(\mathrm{d}s, \phi_{k/n}^n).$$

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So,

Up to appropriate stopping time:

$$|\phi_t^n - \phi_t^m|^2 \leq ... \leq 2K_R \int_0^t |\phi_s^n - \phi_s^m|^2\,\mathrm{d}s + \int_0^t \text{ sth. small}\,\mathrm{d}s + N_t$$

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Now use

Stochastic Gronwall Lemma (v. Renesse, S., 2010)

Let $Z \geq 0$, H be adapted cts., N cts. local mart, $N_0 = 0$ s.t.

$$Z_t \leq K \int_0^t Z_u^*\,\mathrm{d}u + N_t + H_t,\ t \geq 0.$$

Then, for each $0 < p < 1$ and $\alpha > \frac{1+p}{1-p}$ $\exists c_1, c_2$:

$$\mathbb{E}(Z_T^*)^p \leq c_1 \exp\{c_2 KT\}(\mathbb{E}H_T^{*\alpha})^{p/\alpha}.$$

Existence of global solutions

Theorem

If, in addition, there exists a nondecr. $\rho : [0, \infty) \to (0, \infty)$ s.t. $\int_0^\infty 1/\rho(u)\,\mathrm{d}u = \infty$ and

$$2\langle b(x), x\rangle + \mathrm{Tr}(a(x,x)) \leq \rho(|x|^2),\ \ x \in \mathbb{R}^d,$$

then the local solution of the sde is global ((*weakly*) *complete*).

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Itô's formula implies

$$X_\tau^2 - X_0^2 = \int_0^\tau 2\langle b(X_u), X_u\rangle + \mathrm{Tr}(a(X_u, X_u))\,\mathrm{d}u + N_\tau$$
$$\le \int_0^\tau \rho(|X_u|^2)\,\mathrm{d}u + N_\tau.$$

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Lemma (v. Renesse, S., 2010)

Let $Z \ge 0$ be adapted cts. defined on $[0,\sigma)$, N cts. local mart, $N_0 = 0$, $C \ge 0$ s.t.

$$Z_t \le \int_0^t \rho(Z_u^*)\,\mathrm{d}u + N_t + C,\ t \in [0,\sigma)$$

and $\lim_{t\uparrow\sigma} Z_t^* = \infty$ on $\{\sigma < \infty\}$. Then $\sigma = \infty$ almost surely.

Continuous dependence on initial condition

Question

Are our conditions sufficient for

- Continuous dependence on initial conditions (or even the semi-flow property)?
- In particular: Do conditions for global existence of solutions ensure existence of a continuous map $\varphi : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ which is a modification of the solution map (*strong completeness*)?

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Answer: No!

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Answer: No!

There exists a 2d sde without drift driven by a single BM with bounded and C^∞ coefficient which is not strongly complete.
Reference: Li, S.: Lack of strong completeness .. (Ann. Prob. 2011)

Continuous modification

Lemma

Assume that for some $\mu, K \geq 0$, and all $x, y \in \mathbb{R}^d$

$$2\langle b(x) - b(y), x - y\rangle + \mathrm{Tr}\mathcal{A}(x,y) + \mu\|\mathcal{A}(x,y)\| \leq K|x-y|^2.$$

Then the sde is weakly complete. Denote solutions by $\phi_t(x)$. For each $q \in (0, \mu + 2)$, there exist c_1, c_2 s.t.

$$\mathbb{E}\sup_{0\leq s\leq T}|\phi_s(x) - \phi_s(y)|^q \leq c_1|x-y|^q \exp\{c_2 KT\}$$

holds for all x, y, T.

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Proof

Show: above condition implies suff. cond. for global existence.

Define $Z_t := |\phi_t(x) - \phi_t(y)|^{\mu+2}$. Then $\mathrm{d}Z_t = \ldots$ and

$$Z_t \leq |x - y|^{\mu+2} + (\tfrac{\mu}{2} + 1)K \int_0^t Z_s \,\mathrm{d}s + N_t.$$

Applying Stochastic Gronwall Lemma yields assertion.

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Lemma

Assume that for some $\mu \geq 0$, and nondecr. $f : [0, \infty) \to (0, \infty)$
$2\langle b(x) - b(y), x - y\rangle + \mathrm{Tr}\mathcal{A}(x, y) + \mu \|\mathcal{A}(x, y)\| \leq f(|x| \vee |y|)|x - y|^2$.
Assume the sde is weakly complete. Then for $q \in (0, \mu + 2)$, $\varepsilon > 0$ and $B := \frac{(\mu+2)(1+\varepsilon)}{\mu+2-q}$:

$$\mathbb{E} \sup_{0 \leq s \leq T} |\phi_s(x) - \phi_s(y)|^q$$

$$\leq c_{q,\varepsilon,\mu} |x - y|^q \Big(\mathbb{E} \exp\{\frac{qB}{2} \int_0^T f(|\phi_s(x)| \vee |\phi_s(y)|)\, \mathrm{d}s\}\Big)^{1/B}$$

$$\leq c_{q,\varepsilon,\mu} |x - y|^q \max_{z \in \{x,y\}} \Big(\mathbb{E} \exp\{qB \int_0^T f(|\phi_s(z)|)\, \mathrm{d}s\}\Big)^{1/B} \leq \ldots$$

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Corollary

If the ass. holds for some $\mu > d - 2$ and the last expectation is locally bounded in z for some $q > d$ and $T > 0$, then the sde is strongly complete.

Criteria for strong completeness

$$2\langle b(x),x\rangle + \mathrm{Tr}(a(x,x)) \le c(1+|x|)^2, \qquad f(x) = \beta \log^+ x$$

$$2\langle b(x),x\rangle, \mathrm{Tr}(a(x,x)) \le c(1+|x|)^2, \qquad f(x) = \beta (\log^+ x)^2$$

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NB

If, for some $\mu > d-2$, we have
$2\langle b(x) - b(y), x - y\rangle + \mathrm{Tr}\mathcal{A}(x,y) + \mu\|\mathcal{A}(x,y)\| \le K_N|x-y|^2$,
then the sde has a *locally continuous modification* (up to explosion).

Lemma

Assume that a, b are bounded and

$$(\star)\ 2\langle b(x)-b(y), x-y\rangle+\mathrm{Tr}\mathcal{A}(x,y)+\mu\|\mathcal{A}(x,y)\| \leq K|x-y|^2$$

for some $\mu \geq 0$. Then the sde is complete and for $q \in (0, \mu+2)$ and $T > 0$ there is $c > 0$ s.t.

$$\mathbb{E}|\phi_{st}(x) - \phi_{s't'}(y)|^q \leq c(|x-y|^q + |s-s'|^{q/2} + |t-t'|^{q/2})$$

for all $0 \leq s, s', t, t' \leq T$.

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- If $(\star)$ holds for some $\mu > d+2$, then the sde generates a global semi-flow.

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- If $(\star)$ holds for some $\mu > d+2$ locally, then the sde generates a local semi-flow.
- Local semi-flow + strong completeness $\Rightarrow$ global semi-flow.

Strong p-completeness

Possible Definitions

Let $p \in [0, d]$ and assume either

- nothing
- sde has locally continuous modif. φ
- sde generates local semi-flow φ.

Then the sde is called *strongly p-complete* if for every $A \subset \mathbb{R}^d$ of dimension at most p there exists a modif. φ of the local solution which restricted to A is continuous ($\mathbb{R}^d$-valued) in (t, x), where *dimension* may stand for either

- Hausdorff dimension
- Upper Minkowski dimension (= box dimension)
- something else

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NB

In local semi-flow case $d - 1$-completeness implies d-completeness.

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Then, the sde is strongly p-complete in the "nothing/upper Minkowski" sense.

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Proof

For $q \in (p, \mu + 2)$ we saw that

$$\mathbb{E}\sup_{0\leq s\leq T} |\phi_s(x) - \phi_s(y)|^q \leq c_1|x - y|^q \exp\{c_2KT\}.$$

Theorem 11.1/11.6 in Ledoux-Talagrand applied to a set $A \subset \mathbb{R}^d$ of upper Mink. dim. p implies the claim.

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In previous prop. the image of A is even bounded for each $t > 0$.

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Question

Is Proposition true with “local semi-flow/Hausdorff”?

Strong Completeness: a different approach

- $\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + M(\mathrm{d}t, X(t)),$

with b locally Lipschitz and $M \in B_{ub}^{1,\delta}$.

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$$\begin{aligned} F(t,z,x,\omega) &:= \{D\psi(t,z,\omega)\}^{-1} b(x) \\ \zeta(t,x,\omega) &:= \psi(t,.,\omega)^{-1}(x),\ t \geq 0,\ x,z \in \mathbb{R}^d \end{aligned}$$

Then

$$X(t,\omega) = \psi(t, x + \textstyle\int_0^t F(u, \zeta(u, X(u,\omega),\omega), X(u,\omega),\omega)\,\mathrm{d}u, \omega).$$

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Then

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If either

- there exists $\gamma \in (0,1)$ s.t. $|b(x)| \leq C(1+|x|^\gamma)$ or
- $|b(x)| \leq C(1+|x|)$, $\sup_{x,u} \|D\psi(u,x,\omega)^{-1}\| < \infty$,

then the SDE is strongly complete (Mohammed, S., JFA, '03).

Strong Completeness: Conjecture and Counterexample

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But:

b linear growth in radial dir., noise globally Lipschitz $\not\Rightarrow$ strong completeness.

Even:

$\langle b(x), x\rangle = 0$, noise globally Lipschitz and bounded $\not\Rightarrow$ strong completeness.

(Example on blackboard)

Proposition

If the SDE generates a global semi-flow, then it also generates a *random dynamical system*.

Proof

Follows as in Kager-S. (EJP, 1997).