Isotropic Gaussian random fields on the sphere

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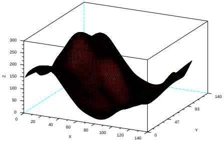
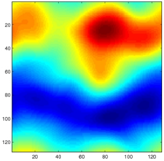
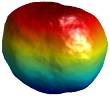
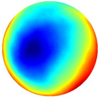
Examples of random fields

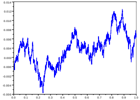
- collection of random variables
- stochastic processes, e.g., Brownian motion
- solutions of stochastic (partial) differential equations
- solutions of random partial differential equations

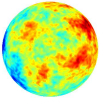


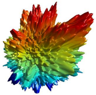






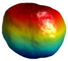


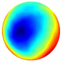
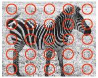
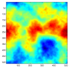
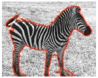
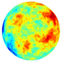
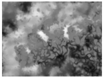
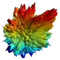




Outline

- isotropic Gaussian random fields
- *approximation* of random fields
- sample *regularity* of random fields
- stochastic processes & stochastic partial differential equations

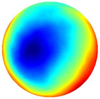


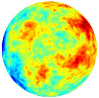








Random fields on spheres

- $(\Omega, \mathcal{A}, P)$ probability space
- $(\mathbb{S}^2, d)$ compact, metric space
  - $\mathbb{S}^2 = \{x \in \mathbb{R}^3, \|x\| = 1\}$ unit sphere
  - $d(x,y) = \arccos\langle x, y\rangle_{\mathbb{R}^3}$
- $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$, $\mathcal{A} \otimes \mathcal{B}(\mathbb{S}^2)$-measurable: *real-valued random field* on $\mathbb{S}^2$
- T *Gaussian random field*: $\forall k \in \mathbb{N}$, $x_1, \dots, x_k \in \mathbb{S}^2$, $a_1, \dots, a_k \in \mathbb{R}$: $\sum_{i=1}^k a_i T(x_i)$ Gaussian
- T *isotropic*, Gaussian: $\forall k \in \mathbb{N}$, $x_1, \dots, x_k \in \mathbb{S}^2$, $g \in \mathrm{SO}(3)$:

$$(T(x_1), \dots, T(x_k)) \sim (T(gx_1), \dots, T(gx_k))$$





Spherical harmonic functions

- *Legendre polynomials* $(P_\ell, \ell \in \mathbb{N}_0)$:

$$P_\ell(\mu) := 2^{-\ell} \frac{1}{\ell!} \frac{\partial^\ell}{\partial \mu^\ell} (\mu^2 - 1)^\ell, \quad \mu \in [-1, 1]$$

- *associated Legendre polynomials* $(P_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$:

$$P_{\ell m}(\mu) := (-1)^m (1 - \mu^2)^{m/2} \frac{\partial^m}{\partial \mu^m} P_\ell(\mu), \quad \mu \in [-1, 1]$$

- *spherical harmonic functions* $(Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$:

$$Y_{\ell m}(\vartheta, \varphi) := \begin{cases} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos\vartheta) e^{im\varphi} & m \geq 0 \\ (-1)^m \overline{Y_{\ell -m}(\vartheta, \varphi)}, & m < 0 \end{cases}$$

$(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$

Spherical Laplacian — Laplace–Beltrami operator

- $y = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \in \mathbb{S}^2$

$$\Delta_{\mathbb{S}^2} = (\sin\vartheta)^{-1}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + (\sin\vartheta)^{-2}\frac{\partial^2}{\partial\varphi^2}.$$

- eigenvalues & eigenfunctions

$$\Delta_{\mathbb{S}^2} Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}, \quad \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell$$

- spaces of eigenfunctions

$$\mathcal{H}_\ell(\mathbb{S}^2) = \operatorname{span}\{Y_{\ell m}, m = -\ell, \dots, \ell\}$$

- eigenbasis

$$H := L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell(\mathbb{S}^2)$$

Theorem ([Marinucci, Pecatti 11])

- *T isotropic Gaussian random field*

Then:

- *T has Karhunen–Loève expansion*

$$T = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m},$$

- $(a_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ *random variables,* $\perp\!\!\!\perp$
  - $\operatorname{Re} a_{\ell m} \perp\!\!\!\perp \operatorname{Im} a_{\ell m} \sim \mathcal{N}(0, A_\ell/2)$
  - $\operatorname{Re} a_{\ell 0} \sim \mathcal{N}(0, A_\ell)$, $\operatorname{Im} a_{\ell 0} = 0$
  - $\operatorname{Re} a_{00} \sim \mathcal{N}(\mathbb{E}(T) 2\sqrt{\pi}, A_0)$
- $(a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, -1)$ *given by*
  - $\operatorname{Re} a_{\ell m} = (-1)^m \operatorname{Re} a_{\ell -m}$
  - $\operatorname{Im} a_{\ell m} = (-1)^{m+1} \operatorname{Im} a_{\ell -m}$

Lemma

- T *centered, isotropic Gaussian random field*
- $\ell \in \mathbb{N}$, $m = 1, \ldots, \ell$, $\vartheta \in [0, \pi]$:

$$L_{\ell m}(\vartheta) := \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos\vartheta)$$

- $((X^1_{\ell m}, X^2_{\ell m}), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$, $\perp\!\!\!\perp$
  - $X^i_{\ell m} \sim \mathcal{N}(0,1)$, $i = 1, 2$, $m \neq 0$
  - $X^2_{\ell 0} = 0$
- $y = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$

$$\Longrightarrow T(y) \sim \sum_{\ell=0}^{\infty} \Big(\sqrt{A_\ell} X^1_{\ell 0} L_{\ell 0}(\vartheta) + \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(X^1_{\ell m}\cos(m\varphi) + X^2_{\ell m}\sin(m\varphi)) \Big)$$

$$T^\kappa(y) := \sum_{\ell=0}^{\kappa} \Big(\sqrt{A_\ell} X_{\ell 0}^1 L_{\ell 0}(\vartheta) + \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(X_{\ell m}^1 \cos(m\varphi) + X_{\ell m}^2 \sin(m\varphi)) \Big)$$

Theorem ([L., Schwab 13])

- *T centered, isotropic Gaussian random field*
- $\exists C > 0, \alpha > 2, \ell_0 \in \mathbb{N} : \forall \ell > \ell_0 : A_\ell \leq C \cdot \ell^{-\alpha}$

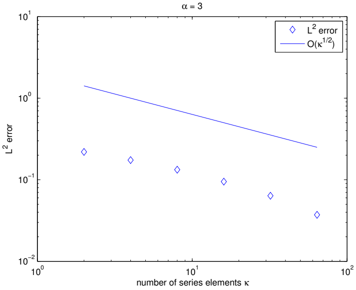
Then:

1. $\forall 0 < p < +\infty : \exists \hat{C}_p > 0 : \forall \kappa \in \mathbb{N} :$

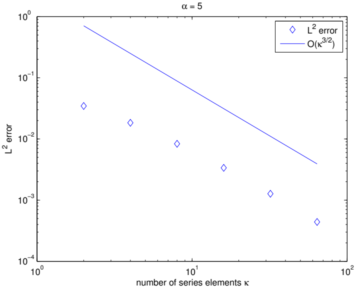
$$\|T - T^\kappa\|_{L^p(\Omega;H)} \leq \hat{C}_p \cdot \kappa^{-(\alpha-2)/2}$$

2. *asymptotically:* $\forall \beta < (\alpha - 2)/2 : \|T - T^\kappa\|_H \leq \kappa^{-\beta}$, P*-a.s.*

L^2 error, 1000 Monte Carlo samples



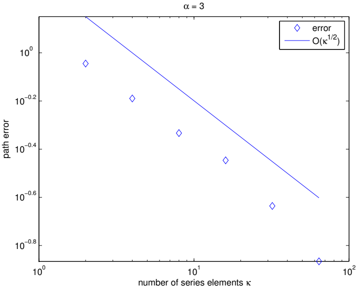
(a) $\alpha = 3$



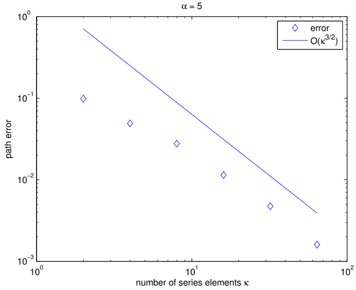
(b) $\alpha = 5$

Figure: error depending on series truncation

Sample error



(a) $\alpha = 3$



(b) $\alpha = 5$

Figure: error depending on series truncation

Second moments — covariance kernels

- *mixed second moments* $k_T : \mathbb{S}^2 \to \mathbb{R}$

$$k_T(x,y) := \mathbb{E}(T(x)T(y)) = \sum_{\ell=0}^{\infty} A_\ell \sum_{m=-\ell}^{\ell} Y_{\ell m}(x)\overline{Y_{\ell m}}(y)$$
$$= \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{4\pi} P_\ell(\langle x,y\rangle_{\mathbb{R}^3})$$

- $k : [0,\pi] \to \mathbb{R}$ as *function of distance* $r = d(x,y)$

$$k(r) := \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos r)$$

- $k_I : [-1,1] \to \mathbb{R}$ as *function of inner product* $\mu = \langle x,y\rangle_{\mathbb{R}^3}$

$$k_I(\mu) := k(\arccos \mu)$$

$$\Longrightarrow \quad k_T(x,y) = k(d(x,y)) = k_I(\langle x,y\rangle_{\mathbb{R}^3}), \quad x,y \in \mathbb{S}^2$$

Decay power spectrum $\Longleftrightarrow$ regularity kernel

Proposition ([L., Schwab 13])

- $n \in \mathbb{N}_0$

$$(\ell^{n+1/2} A_\ell, \ell \geq n) \in \ell^2(\mathbb{N}_0) \iff (1-\mu^2)^{n/2} \frac{\partial^n}{\partial \mu^n} k_I(\mu) \in L^2(-1,1)$$

i.e.,

$$\frac{1}{(4\pi)^2} \sum_{\ell \geq n} A_\ell^2 \frac{2\ell+1}{2} \ell^{2n} < +\infty \iff \int_{-1}^{1} \left| \frac{\partial^n}{\partial \mu^n} k_I(\mu) \right|^2 (1-\mu^2)^n \, d\mu < +\infty$$

extension to *non-integers* and *fractional weighted Sobolev spaces*

Sample regularity

Definition

X, Y random fields on $\mathbb{S}^2$

- Y *modification* of X: $\forall x \in \mathbb{S}^2 : P(X(x) = Y(x)) = 1$

Theorem ([L., Schwab 13])

- T *isotropic Gaussian random field with*

$$\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} < +\infty$$

1. $\beta \in (0,2]$: $\forall \gamma < \beta/2$
   $\exists$ *continuous modification with Hölder exponent* γ
2. $\beta > 0$: $\forall k < \beta/2 - 1$
   $\exists$ k*-times continuously differentiable modification*

Idea of proof

Lemma

$$\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} < +\infty, \ \beta \in [0,2] \quad \Longrightarrow \quad \forall r \in [0,\pi] : |k(0) - k(r)| \leq C_\beta r^\beta$$

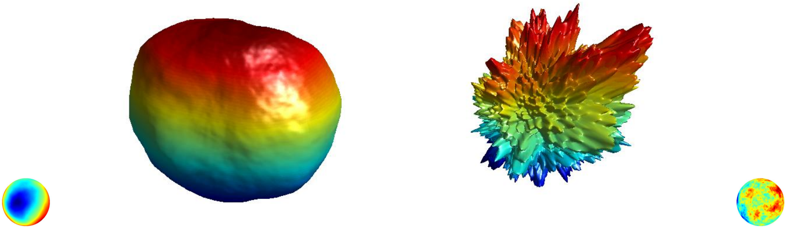
$$\Longrightarrow \quad \mathbb{E}(|T(x) - T(y)|^{2p}) \leq C_{\beta,p}\, d(x,y)^{\beta p}$$

Theorem (Kolmogorov–Chentsov theorem `[L., Schwab 13]`)

- T *random field on* $\mathbb{S}^2$
- $\exists p > 0, \epsilon \in (0,1], C > 0 : \mathbb{E}(|T(x) - T(y)|^p) \leq C d(x,y)^{2+\epsilon p}$

$\Longrightarrow \exists$ *continuous modification*
which is locally Hölder continuous with exponent $\gamma \in (0,\epsilon)$

Ice crystals & Sahara dust particles



- radius: *lognormal* random field $\exp(T)$
- same regularity as isotropic Gaussian random field T

`[L., Schwab 13]`

Random fields $\longrightarrow$ stochastic processes

- Brownian motion – Wiener process $W = (W(t), t \geq 0)$

$$W(t) = (W(t) - W(t_n)) + (W(t_n) - W(t_{n-1})) + \cdots + (W(t_1) - W(0))$$

- independent increments

$$(W(t_n) - W(t_{n-1})) \sim \mathcal{N}(0, t_n - t_{n-1}) = \mathcal{N}(0, \Delta t)$$

- generate $\mathcal{N}(0, \Delta t)$-distributed *random numbers*
- resp. $\mathcal{N}(0, \Delta t Q)$-distributed *random field*

Stochastic process

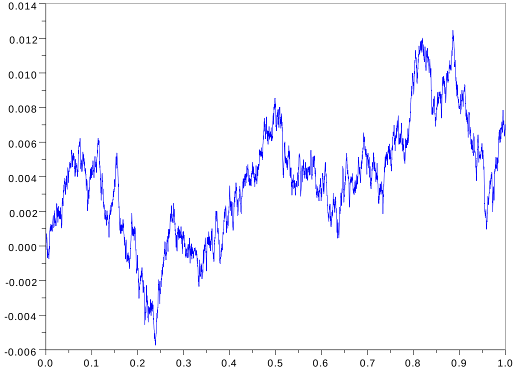
- probability space $(\Omega, \mathcal{A}, P)$
- filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ satisfies "'usual conditions"'
- separable Hilbert space U — e.g., $\mathbb{R}^d$, $L^2(D)$, $H^\alpha(D)$
- stochastic process $X = (X(t), t \geq 0)$ with values in U, e.g.,

$$U = L^2(\mathbb{S}^2): \; X = (X(t,x), t \geq 0, x \in \mathbb{S}^2) \text{ with } X(t,x,\omega) \in \mathbb{R}$$

- property: (often) *P-a.s. nowhere differentiable*

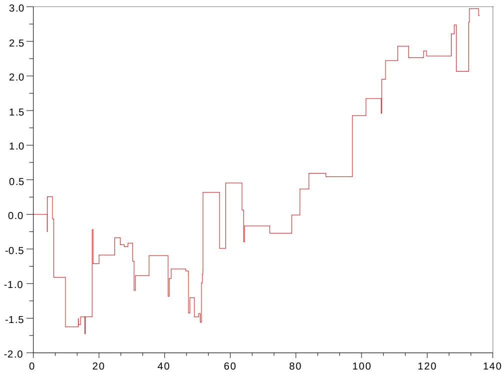
Examples

- Brownian motion — Q–Wiener process $W = (W(t), t \geq 0)$



Examples

- Lévy process $L = (L(t), t \geq 0)$



Isotropic Q-Wiener process

$$W(t,y) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(t) Y_{\ell m}(y)$$

$$= \sum_{\ell=0}^{\infty} \Big(\sqrt{A_\ell} \beta^1_{\ell 0}(t) L_{\ell 0}(\vartheta)$$

$$+ \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(\beta^1_{\ell m}(t)\cos(m\varphi) + \beta^2_{\ell m}(t)\sin(m\varphi))\Big)$$

- $y = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$
- Brownian motions $((\beta^1_{\ell m}, \beta^2_{\ell m}), \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$, $\perp\!\!\!\perp$, $\beta^2_{\ell 0} = 0$

Covariance operator Q

$$QY_{LM} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}((W(1), Y_{LM})_H \overline{(W(1), Y_{\ell m})_H}) Y_{\ell m}$$
$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}(a_{LM}(1) \overline{a_{\ell m}(1)}) Y_{\ell m}$$
$$= A_L Y_{LM}$$

$\Longrightarrow$ Q *characterized* by

- *eigenvalues*: $(A_\ell, \ell \in \mathbb{N}_0)$ angular power spectrum
- *eigenfunctions*: $(Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ spherical harmonics

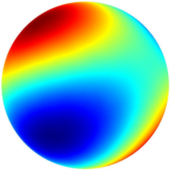
Heat equation with additive noise

- $H = L^2(\mathbb{S}^2)$ separable Hilbert space

$$dX(t) = \Delta_{\mathbb{S}^2} X(t)\, dt + dW(t), \qquad X(0) = X_0$$

- stochastic *integral* equation

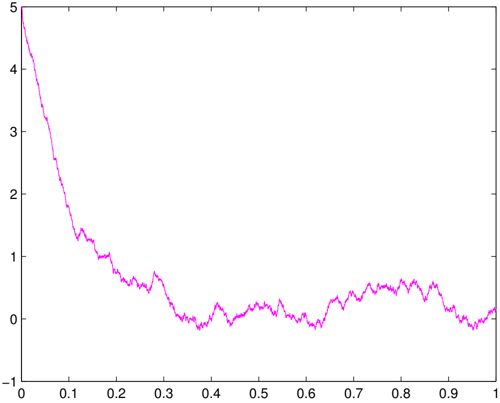
$$X(t) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + \int_0^t dW(s)$$
$$= X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + W(t)$$



Examples

- *Ornstein–Uhlenbeck* process

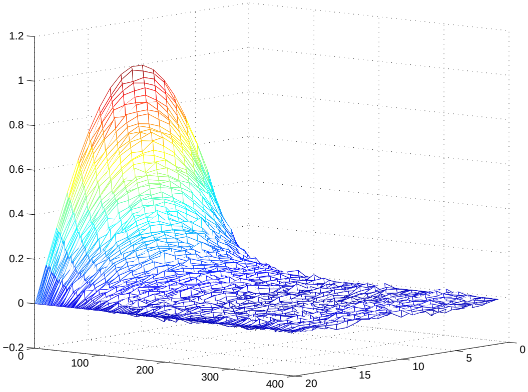
$$dX_t = \alpha X_t \, dt + dW_t$$



Examples

- *parabolic* differential equation (heat equation)

$$dX(t) = \frac{1}{2}\Delta X(t)\,dt + dL(t)$$



Spherical Laplacian — Laplace–Beltrami operator

- $y = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \in \mathbb{S}^2$

$$\Delta_{\mathbb{S}^2} = (\sin\vartheta)^{-1}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial}{\partial\vartheta}\right) + (\sin\vartheta)^{-2}\frac{\partial^2}{\partial\varphi^2}.$$

- eigenvalues & eigenfunctions

$$\Delta_{\mathbb{S}^2} Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}, \quad \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell$$

- spaces of eigenfunctions

$$\mathcal{H}_\ell(\mathbb{S}^2) = \operatorname{span}\{Y_{\ell m}, m = -\ell, \dots, \ell\}$$

- eigenbasis

$$H := L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell(\mathbb{S}^2)$$

Karhunen–Loève expansion

$$X(t) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + W(t)$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (X(t), Y_{\ell m})_H Y_{\ell m}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big((X_0, Y_{\ell m})_H + \int_0^t (X(s), Y_{\ell m})_H \, ds \Delta_{\mathbb{S}^2} + a_{\ell m}(t) \Big) Y_{\ell m}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big((X_0, Y_{\ell m})_H - \ell(\ell+1) \int_0^t (X(s), Y_{\ell m})_H \, ds + a_{\ell m}(t) \Big) Y_{\ell m}$$

stochastic *ordinary* differential equations

$$(X(t), Y_{\ell m})_H = (X_0, Y_{\ell m})_H - \ell(\ell+1) \int_0^t (X(s), Y_{\ell m})_H \, ds + a_{\ell m}(t)$$

Stochastic ordinary differential equations

$$(X(t), Y_{\ell m})_H = (X_0, Y_{\ell m})_H - \ell(\ell+1) \int_0^t (X(s), Y_{\ell m})_H \, ds + a_{\ell m}(t)$$

- *variation of constants* formula

$$(X(t), Y_{\ell m})_H = e^{-\ell(\ell+1)t} (X_0, Y_{\ell m})_H + \int_0^t e^{-\ell(\ell+1)(t-s)} \, da_{\ell m}(s)$$

- SPDE

$$X(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big(e^{-\ell(\ell+1)t} (X_0, Y_{\ell m})_H + \int_0^t e^{-\ell(\ell+1)(t-s)} \, da_{\ell m}(s) \Big) Y_{\ell m}$$

$$=: \sum_{\ell=0}^{\infty} X_\ell(t)$$

$$X(t) = \sum_{\ell=0}^{\infty} X_\ell(t)$$

- recursion formula

$$\begin{aligned} X_\ell(t+h) = e^{-\ell(\ell+1)h} X_\ell(t) & \\ + \sqrt{A_\ell} \Big[\int_t^{t+h} & e^{-\ell(\ell+1)(t+h-s)} \, d\beta^1_{\ell 0}(s) \, Y_{\ell 0} \\ & + \sqrt{2} \sum_{m=1}^{\ell} \Big(\int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta^1_{\ell m}(s) \, \mathrm{Re} \, Y_{\ell m} \\ & \qquad + \int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta^2_{\ell m}(s) \, \mathrm{Im} \, Y_{\ell m} \Big) \Big] \end{aligned}$$

- *Itô formula* (cf. [Jentzen, Kloeden 09])

$$\int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta^i_{\ell m}(s) \sim \mathcal{N}(0, (2\ell(\ell+1))^{-1}(1 - e^{-2\ell(\ell+1)h}))$$

Spectral approximation — $L^2(\Omega; H)$

$$X^\kappa(t) = \sum_{\ell=0}^{\kappa} X_\ell(t)$$

Lemma

- $\exists \ell_0 \in \mathbb{N}, \alpha > 0, C > 0 : \forall \ell > \ell_0 : A_\ell \leq C \cdot \ell^{-\alpha}$

$\Longrightarrow \forall t \in \mathbb{T} : \forall 0 = t_0 < \cdots < t_n = t :$

$$\|X(t) - X^\kappa(t)\|_{L^2(\Omega;H)} \leq \hat{C} \cdot \kappa^{-\alpha/2}, \qquad \kappa \geq \ell_0$$

where

$$\hat{C}^2 = \|X_0\|^2_{L^2(\Omega;H)} + C \cdot \left(\frac{2}{\alpha} + \frac{1}{\alpha+1}\right)$$

Spectral approximation — $L^p(\Omega; H)$ & P-a.s.

Lemma

- $\exists \ell_0 \in \mathbb{N}, \alpha > 0, C > 0 : \forall \ell > \ell_0 : A_\ell \leq C \cdot \ell^{-\alpha}$

$\Longrightarrow \forall t \in \mathbb{T} : \forall 0 = t_0 < \cdots < t_n = t : \forall p > 0 : \exists \hat{C}_p > 0 :$

$$\|X(t) - X^\kappa(t)\|_{L^p(\Omega;H)} \leq \hat{C}_p \cdot \kappa^{-\alpha/2}, \qquad \kappa \geq \ell_0$$

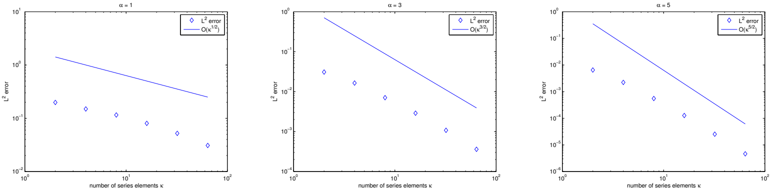
Corollary

- $\exists \ell_0 \in \mathbb{N}, \alpha > 0, C > 0 : \forall \ell > \ell_0 : A_\ell \leq C \cdot \ell^{-\alpha}$

$\Longrightarrow \forall t \in \mathbb{T} : \forall 0 = t_0 < \cdots < t_n = t : \forall \beta < \alpha/2 :$

$$\|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)} \leq \kappa^{-\beta}, \qquad P\text{-a.s.}$$

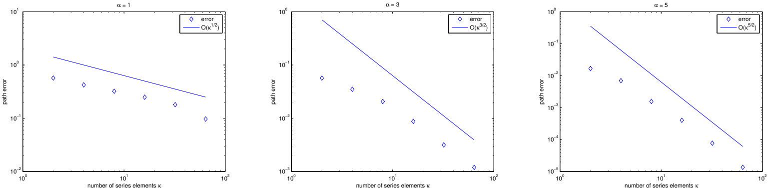
L^2 error, 100 Monte Carlo samples



(a) $\alpha = 1$ (b) $\alpha = 3$ (c) $\alpha = 5$

Figure: error depending on series truncation

Sample error



(a) $\alpha = 1$ (b) $\alpha = 3$ (c) $\alpha = 5$

Figure: error depending on series truncation

Conclusions & outlook

- isotropic Gaussian random fields on $\mathbb{S}^2$
- approximation & regularity
- stochastic partial differential equations
- regularity of random fields on manifolds [Andreev, L. 13]
- random partial differential equations

Thank you for your attention!