Accuracy and Stability of the Continuous-Time 3DVAR Filter for 2D Navier-Stokes Equation

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joint work with:

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Filtering

Methods used for high dimensional data assimilation problems (for example in weather forecasting)

Widely applied, but not that well studied
in the nonlinear, stochastic & infinite dimensional setting.

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Basic Idea of Filtering

- estimate time-evolution of a trajectory based on partial observation & knowledge of the model
- use model to predict next step
- use data to correct prediction

Problem:
only partial and noisy observations/data

Our Approach:

- As starting point simple 3DVAR-filter
  (in this talk: not many details about filter)
- Example for the underlying dynamical system:
  deterministic 2D-Navier-Stokes equation
- Limit of high frequency noisy observations yields stochastic PDE (continuous time filters/noisy observer)
- Study Accuracy & Stability $\Longrightarrow$ Stochastic Dyn. Syst.

Accuracy & Stability

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Accuracy

Trajectories from observer get close to the true trajectory (on the order of observational noise)

$\Rightarrow$ Filter gives the true answer (recover unknown solution from partial noisy observations)

The Model

For simplicity:

In this talk only an ODE instead of 2D Navier Stokes.

Thus $\mathcal{H} = \mathbb{R}^n$, $n \gg 1$ very large.

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- $\mathcal{A}$ diagonal operator, $\mathcal{A} \geq 1$, $\delta > 0$,
- $\mathcal{B} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ – symmetric bilinear map
- f deterministic forcing (could be time dependent)

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trajectory u is the unknown we want to observe

Existence & Uniqueness (well known)

For 2D-Navier-Stokes see [Temam 95, 97], [Robinson 01].

Theorem

Suppose $\langle \mathcal{B}(u,u), u \rangle \leq 0$ and f is bounded.

Then for all initial conditions $u(0)$ there exists a global solution in $C^1([0,\infty), \mathcal{H})$.

Furthermore there is a global attractor in $B_R(0) \subset \mathcal{H}$ containing all bounded solutions.

Very brief description of 3DVAR

[Harvey 91, ...]

Consider

- S_h – one step in the model (of time $h > 0$)
- $u_j = u(jh) = S_h(u_{j-1})$ – unknown true trajectory
- $y_j = Pu_j + \mathcal{N}(0, \Gamma)$ – observation (noisy & partial)
- P – projection
- $\hat{m}_j$ – estimation

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Assume Gaussianity:
$$u_j | y_1 \dots y_j \sim \mathcal{N}(\hat{m}_j, C)$$
$$u_{j+1} | y_1 \dots y_j \sim \mathcal{N}(m_{j+1}, C)$$

Kalman mean update (Bayes' rule + some work)

$$\hat{m}_{j+1} = m_{j+1} + CP(\Gamma + PCP)^{-1}(y_{j+1} - Pm_{j+1})$$

High Frequency Observation Limit

Limit ($h \downarrow 0$) of high frequency noisy observations for 3DVAR yields for sufficiently large observational noise a stochastic equation (noisy observer/continuous time filter)

Key Point:

The discrete time filter can be written as
an Euler-Maruyama discretization of the observer

The formal limit is true in a much more general setting and for several filter (no rigorous result yet)

Discrete time case for 2D-Navier Stokes: [Law, Stuart, et. al. 11]

Noisy observer

$$\partial_t \hat{m} = -\delta \mathcal{A}\hat{m} + \mathcal{B}(\hat{m},\hat{m}) + f + \omega \mathcal{A}^{-2\alpha} P_\lambda [u - \hat{m} + \sigma \mathcal{A}^{-\beta} \partial_t W]$$

- P_λ – proj. onto the observed low modes
  (for 2D Navier-Stokes approximately λ^2 many)
- **Assume**: $\Gamma = \frac{1}{h}\sigma^2 \mathcal{A}^{-2\beta} P_\lambda$ covariance of the (given) observational noise (think of $\beta = 0$)
- $C = \omega\sigma^2 \mathcal{A}^{-2(\alpha+\beta)}$ how to weight data or the model
- W – standard cylindrical Wiener process
  (space-time white noise)

λ and ω are free parameters of the filter (also C and α)

Generation of SDS

Stochastic Dynamical System: (not all details)

$S(t,s,W)\hat{m}_0$ solution of observer

- at time $t > 0$
- given path $\{W(t)\}_{t\geq s}$
- given initial condition $\hat{m}(s) = \hat{m}_0$

Flow Property: $S(t,r,W)S(r,s,W) = S(t,s,W)$

Theorem: (by standard methods, details later)

The noisy observer generates a SDS in $\mathcal{H}$.

Remark: No Random Dynamical System is generated as the observer is non-autonomous due to u and possibly f.

Numerical Results

$$\partial_t \hat{m} = -\delta \mathcal{A}\hat{m} + \mathcal{B}(\hat{m},\hat{m}) + f + \omega \mathcal{A}^{-2\alpha} P_\lambda [u - \hat{m} + \sigma \mathcal{A}^{-\beta}\partial_t W]$$

Parameter: λ large, $\omega = 100$, $\alpha = 1/2$, $\beta = 0$, $\sigma = 0.005$

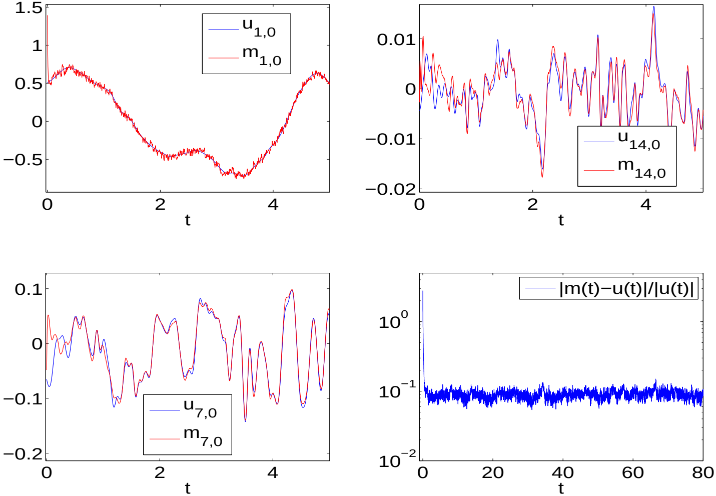
Spit step method

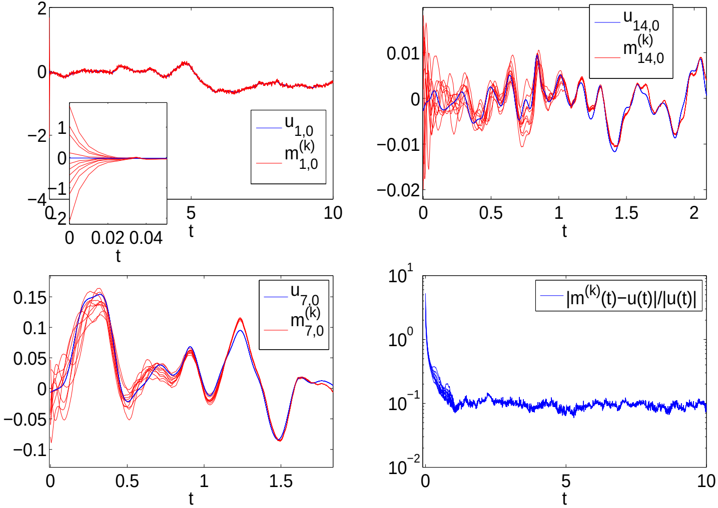
Pseudospectral method for Navier Stokes equation, using higher order method in time

Euler-Maruyama discretization for the OU-process (linear equation with noise)

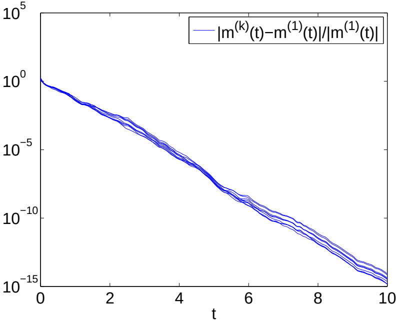
Attractivity of the observer

3 Fourier modes & relative error





u 1,0
m(k) 1,0
t
u 14,0
m(k) 14,0
u 7,0
m(k) 7,0
|m(k)(t)−u(t)|/|u(t)|



Relative error of an ensemble of trajectories.

Results forward in time

mean square & in probability

Forward Accuracy

Only in mean square – no almost sure results expected

$$\mathbb{E}|u(t) - \hat{m}(t)|^2 = \mathcal{O}(\text{noise-strength}^2) \qquad \text{for } t \to \infty$$

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Conjectures:

$$\liminf_{t\to\infty} \mathbb{P}\big(\hat{m}(t) \in B\big) > 0$$

and

$$\mathbb{P}\big(\exists\, t > 0 : \hat{m}(t) \in B\big) = 1$$

for all open $B \subset \mathcal{H}$

Well known for stochastic Navier-Stokes [Hairer, Mattingly 06].

Forward Accuracy

Assumption on γ:

$$\gamma|h|^2 \leq 2\omega|\mathcal{A}^{-\alpha}P_\lambda h|^2 + \delta|\mathcal{A}^{1/2}h|^2 \qquad \forall\ h.$$

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Theorem [BLSZ12]

Suppose $\gamma = KR\delta^{-1} + \gamma_0$ for some $\gamma_0 > 0$,
where K is a constant defined by $\mathcal{B}$ and the bound on u.
Then

$$\mathbb{E}|\hat{m}(t)-u(t)|^2 \leq \mathrm{e}^{-\gamma_0 t}|\hat{m}(0)-u(0)|^2 + \omega^2\sigma^2\frac{1}{\gamma_0}\cdot\mathrm{tr}\big(\mathcal{A}^{-4\alpha-2\beta}P_\lambda\big).$$

Consequence: $\limsup_{t\to\infty}\mathbb{E}|\hat{m}(t)-u(t)|^2 = \mathcal{O}(\sigma^2).$

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Consequence: $\limsup_{t\to\infty}\mathbb{E}|\hat{m}(t)-u(t)|^2 = \mathcal{O}(\sigma^2)$.

Proof based on Itô-formula – SPDE for the error $\mathrm{e} = u - \hat{m}$

Forward Stability

Assume $R' = \sup_{t\in\mathbb{R}} |f + \omega\mathcal{A}^{-2\alpha} P_\lambda u|^2_{\mathcal{H}^{-1}} < \infty$

and define $R'' = \frac{K}{\delta^2} R' + \frac{K}{\delta}\omega^2\sigma^2 \mathrm{tr}\big(\mathcal{A}^{-4\alpha-2\beta} P_\lambda\big) < \infty$

Theorem [BLSZ12]

$\hat{m}_i$ trajectories of observer; initial condition $\hat{m}(0) = \hat{m}_i(0)$.

Suppose $\gamma = R'' + \gamma_0$ for some $\gamma_0 > 0$.

Then for all $\eta \in (0, \gamma_0)$

$$|\hat{m}_1(t) - \hat{m}_2(t)| e^{\eta t} \to 0 \qquad \text{in probability as } t \to \infty.$$

Key: $\mathbb{P}(\frac{1}{t}\int_0^t \|\hat{m}_i(s)\|^2 ds \le R'') \to 1$ for $t \to \infty$.

sending initial time to $-\infty$

almost sure and pathwise results

Transformation to a random PDE

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Define for $\Phi > 0$ and $\mathcal{W} = \omega\sigma\mathcal{A}^{-2\alpha-\beta}P_\lambda W$

$$Z_\Phi(W) = \int_{-\infty}^{0} e^{s(-\delta\mathcal{A}+\Phi)} d\mathcal{W}(s) \ .$$

The stationary OU-process

$$Z(t) = Z_\Phi(\vartheta_t W) = \int_{-\infty}^{t} e^{(t-s)(-\delta\mathcal{A}+\Phi)} d\mathcal{W}(s)$$

with measure preserving ergodic shift

$$\vartheta_t W = W(t+\cdot) - W(t)$$

solving $dZ = (-\delta\mathcal{A} + \Phi)Z\,dt + d\mathcal{W}$

Transformation to a random PDE

e.g. [Crauel, Debussche, Flandoli, 97]

Define $v(t) = S(t,s,W)\hat{m}_0 - Z(t)$, which solves

$$\begin{aligned}\partial_t v &= -\delta \mathcal{A} v + \mathcal{B}(v,v) + f \\ &\quad +2\mathcal{B}(v,Z) + \mathcal{B}(Z,Z) + \omega \mathcal{A}^{-2\alpha}(u - v - Z) - \phi Z\end{aligned}$$

Existence & Uniqueness of solutions is standard using pathwise PDE-results (needed for generation of SDS)

Bounds on v using

standard energy type methods & Gronwalls inequality

Birkhoff's ergodic theorem

For bounds on v we need bounds on integrals over $Z(t) = Z_\Phi(\vartheta_t W)$ inside exponentials.

Theorem

$$\frac{1}{t-s}\int_s^t \|Z_\Phi(\vartheta_\tau W)\|^2 d\tau \to \mathbb{E}\|Z_\Phi\|^2 \qquad \text{as} \quad s \to -\infty$$

where $\mathbb{E}\|Z_\Phi\|^2 \to 0$ for $\Phi \to \infty$

Accuracy

Assume $K\delta^{-1}(17\mathbb{E}\|Z_\phi\|^2 + 16R) < \gamma$. (not optimal)

Theorem (Pull-Back Accuracy) [BLSZ12]

There is a random radius $r(W) > 0$ such that for all $\hat{m}_0$

$$\limsup_{s\to-\infty} |S(t,s,W)\hat{m}_0 - u(t) - Z_\phi(\vartheta_t W)|^2 \leq r(\vartheta_t W) \; .$$

with an almost surely finite constant

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with an almost surely finite constant (due to Birkhoff)

$$r(W) = \frac{4}{\delta}\int_{-\infty}^0 \exp\Big(\int_\tau^0 (16K\delta^{-1}(\|Z\|^2 + R) - \gamma)d\eta\Big)\mathcal{T}^2 d\tau ,$$

where $\mathcal{T} := K\|Z\|(\|Z\| + 2R^{1/2}) + \phi|Z| + \omega|\mathcal{A}^{-2\alpha}P_\lambda Z|$.

Idea of Proof: Bounds on $v - u$ using the PDEs.

Conclusion

As $r(W) \approx \mathcal{O}(\|Z\|^2) \approx \mathcal{O}(\sigma^2)$:

$$
\begin{aligned}
S(t,s,W)\hat{m}_0 - u(t) &= v(t) - u(t) + Z_\phi(\vartheta_t W) \\
&= \mathcal{O}(\|Z\|^2) + Z_\phi(\vartheta_t W) \\
&= \mathcal{O}(\sigma)
\end{aligned}
$$

Thus we verified accuracy for the observer.

Birkhoff bounds

For stability we assume:

$\gamma > 0$ is sufficiently large that for some $\eta > 0$

$$\limsup_{s\to-\infty} \frac{1}{t-s}\int_s^t \|S(\tau,s,W)\hat{m}_0^{(1)}\|^2 d\tau < \frac{\gamma-2\eta}{4K}\delta \ .$$

Not proved. Should be possible, by varying $\Phi = \Phi(W)$
[Stannat, Es-Sahir 11], [Flandoli, Gatarek 95]
Also method in [Chueshov, Duan, Schmalfuss 03] might work.

Technical Problem:
$r(W)$ does not satisfy Birkhoff-theorem ($\mathbb{E}r = \infty$)

Theorem (Pull-Back Accuracy) [BLSZ12]

Assume one initial condition $\hat{m}_0^{(1)} \in \mathcal{H}$ satisfies Birkhoff-bounds.

Let $\hat{m}_0^{(2)} \in \mathcal{H}$ be any other initial condition.

Then

$$\lim_{s \to -\infty} |S(t,s,W)\hat{m}_0^{(1)} - S(t,s,W)\hat{m}_0^{(2)}| \cdot \mathrm{e}^{\eta(t-s)} = 0.$$

Stability

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Idea of proof: random PDE for $e = \hat{m}^{(1)} - \hat{m}^{(2)}$ energy type estimates – Gronwall's Lemma – Birkhoff bounds for integrals on Z and $\hat{m}^{(1)}$ in the exponential....

Generalized Observer I

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Other filter lead in the limit of high frequency observations to

$$\partial_t \hat{m} = -\delta \mathcal{A}\hat{m} + \mathcal{B}(\hat{m},\hat{m}) + CP\Gamma^{-1}P\,(u-\hat{m}) + CP\Gamma^{-\frac{1}{2}}\partial_t W$$
$$\partial_t C = LC + CL^* - CP\Gamma^{-1}PC$$

- Γ operator determined by observational noise
- linear operators L determined as part of the filter
- P projects onto the observed modes
- C is a covariance operator (symmetric, trace-class)

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- Γ operator determined by observational noise
- linear operators L determined as part of the filter
- P projects onto the observed modes
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Observation: If the Ricatti-type equation has an attracting stable steady state for C, then this algorithm simply converges to 3DVAR algorithm.

Generalized Observer II

$$\begin{aligned}\partial_t \hat{m} &= -\delta\mathcal{A}\hat{m} + \mathcal{B}(\hat{m},\hat{m}) + CP\Gamma^{-1}P\,(u-\hat{m}) + CP\Gamma^{-\frac{1}{2}}\partial_t W \\ \partial_t C &= LC + CL^* - CP\Gamma^{-1}PC\end{aligned}$$

Problem:
For filters like the extended Kalman Filter we have

$$L = -\delta\mathcal{A} + 2\mathcal{B}(\hat{m},\cdot)$$

and thus coupled equations.

Remarks

- Proof of the high frequency limit?
  Convergence of the Euler-Maruyama scheme!?
- General solutions u, non-autonomous f?
  We only needs boundedness of u for $t \to \infty$ (or $s \to -\infty$)
- Other types of equations/models?
  Proofs use only local and one-sided Lipschitz conditions.
- Birkhoff bounds ???
- stability & accuracy for generalized observer ???
  only partial ideas – work in progress

Summary

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- Stability & Accuracy via continuous time filter/observer
- 2D Navier-Stokes & 3DVAR as first example

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Thank you very much for you attention!