Two Perspectives on Travelling Waves and Stochasticity

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Overview

Topic 1: **Travelling Waves and Anomalous Diffusion**

- Review of reaction-diffusion models
- Nagumo travelling waves
- Perturbations and Riesz-Feller operators
- Anomalous diffusion

joint work with Franz Achleitner (TU Vienna)

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Topic 2: **Travelling Waves for the FKPP SPDE**

- Critical transitions for SDEs
- Stochastic warning signs
- Numerics of FKPP waves

Reaction-Diffusion Models

Simplest case $u = u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \qquad (x, t) \in \mathbb{R} \times [0, \infty).$$

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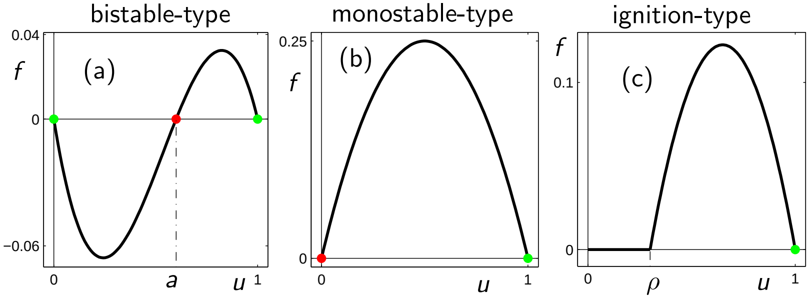
Classical nonlinearities:

(a) Nagumo/Allen-Cahn/RGL $f(u) = u(1-u)(u-a)$,

(b) Fisher-Kolmogorov-Petrovskii-Piscounov $f(u) = u(1-u)$,

(c) combustion nonlinearity $f|_{[0,\rho]} \equiv 0$, $f|_{(\rho,1)} > 0$, f(1)=0.



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Look for travelling front with

- bistable nonlinearity $f(U) = U(1-U)(U-a)$,
- boundary conditions

$$\lim_{\xi \to -\infty} U(\xi) = 0 \qquad \text{and} \qquad \lim_{\xi \to \infty} U(\xi) = 1.$$

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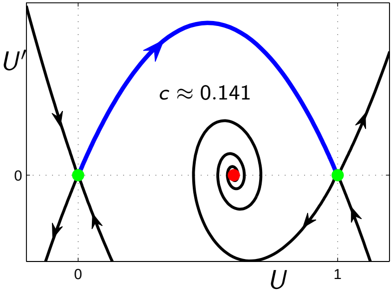
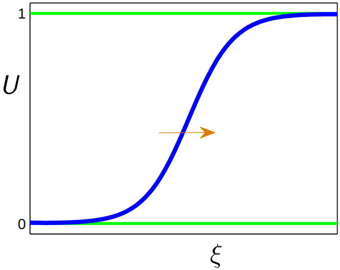


heteroclinic orbit ↔ front



Theorem (Aronson, Fife, McLeod, Nagumo, Weinberger, ...)

For $a \in (0,1)$, there exists an exponentially stable travelling front $u(x,t) = U(x-ct) = U(\xi) \in C^1(\mathbb{R})$ to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a),$$

which is unique up to translation and satisfies

$$\lim_{\xi \to -\infty} U(\xi) = 0, \ \lim_{\xi \to \infty} U(\xi) = 1, \ \lim_{|\xi| \to \infty} U'(\xi) = 0, \ U'(\xi) > 0.$$

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- existence: $\exists c \in \mathbb{R}$ s.t. the associated ODE has a heteroclinic.
- stability: $\exists \kappa > 0$ s.t. for $u(\cdot, 0) = u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$

$$\|u(\cdot, t) - U(\cdot - ct + \gamma)\|_{L^\infty(\mathbb{R})} \le K e^{-\kappa t}, \qquad \text{for all } t \ge 0.$$

  for some constants γ and K depending upon u_0.
- uniqueness: any other pair $(\tilde{U}, \tilde{c})$ satisfies

$$c = \tilde{c}, \qquad \tilde{U}(\cdot) = U(\cdot + \xi_0), \text{ for some } \xi_0 \in \mathbb{R}.$$

A Possible Generalization...

Consider the abstract bistable nonlinearity

$$f \in C^1(\mathbb{R}), \quad f(0) = f(1) = f(a) = 0, \quad f|_{[0,a)} < 0, \quad f|_{(a,1]} > 0.$$

and define the convolution

$$J * S(u) := \int_{\mathbb{R}} J(x-y)S(u(y,t))\, dy.$$

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Theorem (Chen, 1997)

Let $f(u) := G(u, S^1(u), \cdots, S^n(u))$, assume (mild) conditions for

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + G(u, J_1 * S^1(u), \cdots, J_n * S^n(u)), \quad D \geq 0$$

$\Rightarrow$ *existence, uniqueness, exponential stability of a front hold.*

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The bistable nonlinearity $f(u)$

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$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) - v + I, \\ \frac{\partial v}{\partial t} = \epsilon(u - \gamma v), \end{cases} \qquad I, \gamma \in \mathbb{R},\ 0 < \epsilon \ll 1.$$

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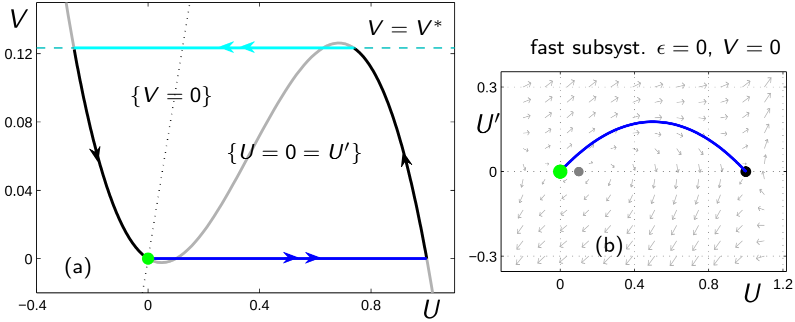

Return to 1-D Case

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Bistable case: front is robust under reaction-term perturbation.

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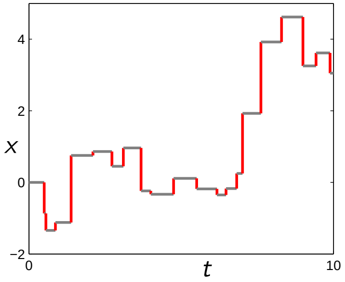
Answer: Go back to probabilistic fundamentals of diffusion.

Continuous-Time Random Walks and Diffusion



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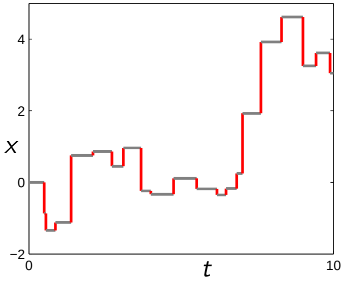

Choice of two distributions:

- waiting time in $(t, t + \Delta t)$ is

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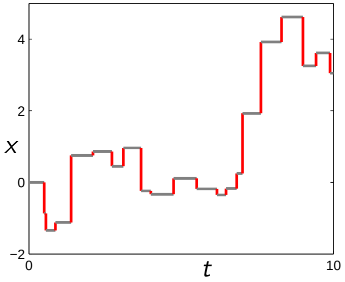

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- mean waiting time $T = \int_0^\infty w(t)t \; \mathrm{d}t$
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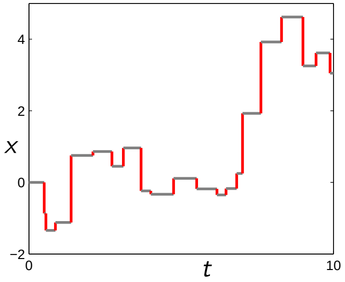

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Result: Assume $T, \Sigma^2 < \infty$, then central limit theorem implies $\mathbb{P}(\text{particle at } x \text{ at time } t) = u(x, t)$ obeys

$$\frac{\partial u}{\partial t} = K_1 \frac{\partial^2 u}{\partial x^2}, \qquad K_1 = \text{diffusion coefficient.}$$

Some Facts on Perturbed Models...

Case 1: $T = \infty$, $\Sigma^2 < \infty$, subdiffusive with long waiting time

- example: $w(t) \sim A_\beta \frac{1}{t^{1+\beta}}$ with $\beta \in (0,1)$,
- non-Markovian with "diffusion" equation

$$\frac{\partial u}{\partial t} = D^{1-\beta}_{\mathrm{RL},t} K_\alpha \frac{\partial^2 u}{\partial x^2}$$

involving the Riemann-Liouville fractional derivative

$$D^{1-\beta}_{\mathrm{RL},t} u(x,t) := \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,s)}{(t-s)^{1-\beta}} ds$$

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TODAY - Case 2: $T < \infty$, $\Sigma^2 = \infty$, long jumps / Lévy flights

- example: $\lambda(x) \sim A_\alpha \frac{1}{|x|^{1+\alpha}}$ with $\alpha \in (1,2)$,
- Markovian with "diffusion" equation

$$\frac{\partial u}{\partial t} = K_\alpha D^\alpha_{\mathrm{RF},x} u$$

involving the Riemann-Feller fractional operator $D^\alpha_{\mathrm{RF},x}$.

Riesz-Feller Operators

- Schwartz space
  $\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left| x^\rho \frac{\partial^\gamma f}{\partial x^\gamma}(x) \right| < \infty, \ \forall \rho, \gamma \in \mathbb{N}_0 \right\}$
- Fourier transform and Fourier inverse transform
  $\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{+i\xi x} f(x) dx$ and $\mathcal{F}^{-1}f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} f(\xi) d\xi$

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3

Define 2-parameter family of Riesz-Feller operators D^α_θ on $\mathcal{S}(\mathbb{R})$ as

$$\mathcal{F}(D^\alpha_\theta f)(\xi) = \psi^\alpha_\theta(\xi) \mathcal{F}f(\xi) , \qquad \xi \in \mathbb{R} ,$$

with pseudo-differential operator symbol

$$\psi^\alpha_\theta(\xi) = -|\xi|^\alpha \exp\left[i(\mathrm{sgn}(\xi)) \theta \frac{\pi}{2} \right] .$$

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Observe: $$e^{-\psi_\theta^\alpha(\xi)} = e^{|\xi|^\alpha \exp\left[i(\mathrm{sgn}(\xi))\theta\frac{\pi}{2}\right]} = \mathbb{E}\left[e^{i\xi X}\right]$$

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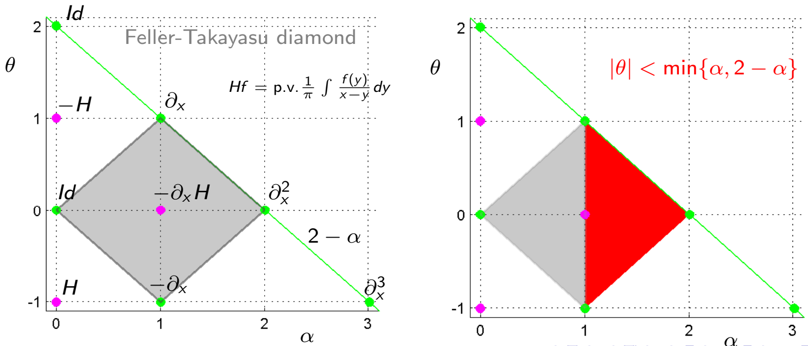

Main Result(s)

Consider the "operator-perturbed" diffusion equation

$$\frac{\partial u}{\partial t} = D_\theta^\alpha u + f(u), \qquad u = u(x,t),\ (x,t) \in \mathbb{R} \times [0,\infty) \qquad (1)$$

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Some results for fractional Laplacian $D_0^\alpha = (\frac{\partial^2}{\partial x^2})^{\alpha/2}$, $\alpha \in (0,2)$:

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Theorem (Achleitner, K., 2013)

Assume $\alpha \in (1,2)$, $|\theta| < \min\{\alpha, 2-\alpha\}$ (and some mild conditions) then a monotone, unique, exponentially stable front exists for (1).

Ingredients of the Proof I

Idea: sub- and super-solutions ("Chen '97, approach").

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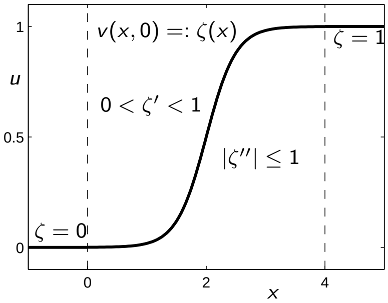

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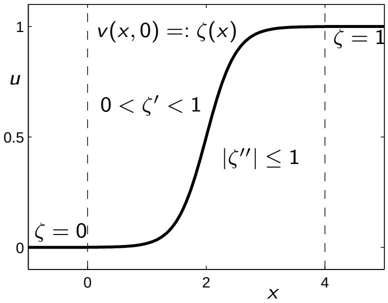

Existence:

1. Start nice profile $v(x,0)$
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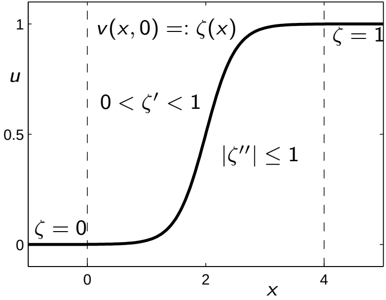

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Sample step: let $w := v + \epsilon e^{Kt}$ and $\cdots \Rightarrow$ supersolution

$$\frac{\partial w}{\partial t} \geq D_\theta^\alpha w + f(w).$$

Ingredients of the Proof II

For uniqueness, stability (and existence) need key lemma:

Lemma ("Two-Fence Lemma")

(U, c) is a front. $\exists\ 0 < \delta_0 \ll 1$, $\sigma \gg 1$ s.t. $\forall \delta \in (0, \delta_0]$ and $\xi_0 \in \mathbb{R}$

$$w^{\pm}(x,t) := U\left(x - ct + \xi_0 \pm \sigma\delta[1 - e^{-\beta t}]\right) \pm \delta e^{-\beta t},$$

are super- and sub-solutions with $\beta := \frac{1}{2}\min(-f'(0), -f'(1))$.

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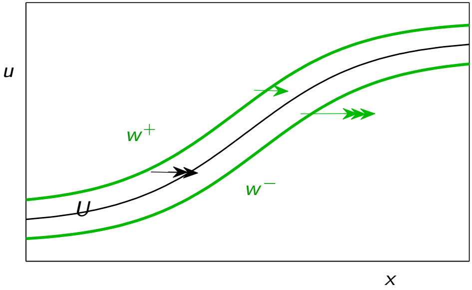

Ingredients of the Proof III

Need further several components:

- Well-definedness of $D_\theta^\alpha g$ for $g \notin \mathcal{S}(\mathbb{R})$.
- Properties of Green's function $G(x,t)$ for $\frac{\partial u}{\partial t} = D_\theta^\alpha u$ e.g.

$$G \geq 0, \quad \|G(\cdot,t)\|_{L^1} = 1, \quad G(x,t) = t^{-1/\alpha} G(xt^{-1/\alpha}, t), \quad \ldots$$

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- A-priori bounds on Riesz-Feller operators

$$\sup_{x\in\mathbb{R}} |D_\theta^\alpha g(x)| \leq \text{const.} \left(\|g''\|_{C_b(\mathbb{R})} \frac{M^{2-\alpha}}{2-\alpha} + \|g'\|_{C_b(\mathbb{R})} \frac{M^{1-\alpha}}{\alpha-1} \right)$$

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Lemma

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Proof.

Infinitesimal generators of Lévy processes (e.g. $\rightarrow$ *Sato*, CUP, 1999)

$$\Rightarrow D^{\alpha}_{\theta} g(x) = c_1 \int_0^{\infty} \frac{g(x+\xi) - g(x) - g'(x)\xi}{\xi^{1+\alpha}} d\xi$$
$$+ c_2 \int_0^{\infty} \frac{g(x-\xi) - g(x) + g'(x)\xi}{\xi^{1+\alpha}} d\xi.$$

Therefore, D^{α}_{θ} is well-defined on $C^2_b(\mathbb{R})$.

Ingredients of the Proof IV

Lemma

$\exists$ *integral representation of* D_θ^α*; from it* $\Rightarrow$ *a-priori bounds.*

Proof.

Infinitesimal generators of Lévy processes (e.g. $\rightarrow$ *Sato*, CUP, 1999)

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$$\int_M^\infty \frac{g(x+\xi) - g(x) - g'(x)\xi}{\xi^{1+\alpha}} d\xi = \int_M^\infty \frac{1}{\xi^{1+\alpha}} \left[\int_0^1 g'(x+s\xi)\xi ds - g'(x)\xi \right] d\xi$$
$$= \int_M^\infty \frac{\xi}{\xi^{1+\alpha}} \underbrace{\left[\int_0^1 g'(x+s\xi) - g'(x) \; ds \right]}_{\text{bounded by } 2\|g'\|_{C_b(\mathbb{R})}} d\xi$$

Topic 2: Critical Transitions for SPDEs

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Geoscience (climate change, climate subsystems, earthquakes)

- Alley et al., *Abrupt climate change*. Science, 2003
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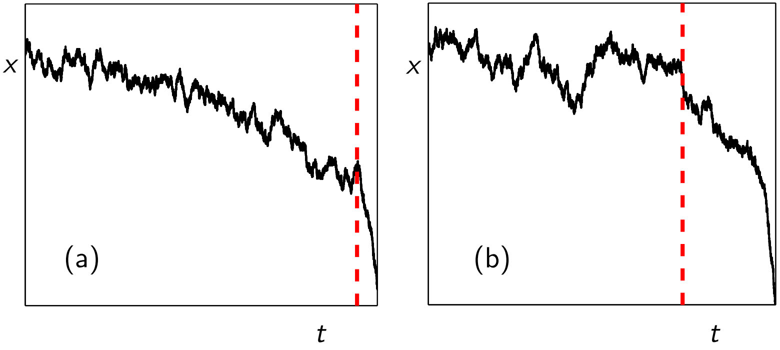

Deterministic Generic Models: Fast-Slow Systems

Fast variables $x \in \mathbb{R}^m$, slow variables $y \in \mathbb{R}^n$, time scale separation $0 < \epsilon \ll 1$.

$$\begin{cases} \frac{dx}{dt} = x' = f(x,y) \\ \frac{dy}{dt} = y' = \epsilon g(x,y) \end{cases} \quad \xleftrightarrow{\epsilon t = s} \quad \begin{cases} \epsilon \frac{dx}{ds} = \epsilon \dot{x} = f(x,y) \\ \frac{dy}{ds} = \dot{y} = g(x,y) \end{cases}$$

$$\downarrow \ \epsilon = 0 \qquad\qquad \downarrow \ \epsilon = 0$$

$$\begin{cases} x' = f(x,y) \\ y' = 0 \end{cases} \qquad\qquad \begin{cases} 0 = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

fast subsystem — slow subsystem

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fast subsystem — slow subsystem

- $C := \{f = 0\}$ = critical manifold = equil. of fast subsystem.
- C is normally hyperbolic if $D_x f$ has no zero-real-part eigenvalues.
- **Fenichel's Theorem:** Normal hyperbolicity $\Rightarrow$ "nice" perturbation.
- Critical transitions at fast subsystem bifurcations possible.

What about Noise and Warning Signs...

(W1) The system recovers slowly from perturbations: slowing down.
(W2) The autocorrelation increases before a transition.
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$$
\begin{aligned}
dx_t &= \tfrac{1}{\epsilon}(-y_t - x_t^2)\ dt + \tfrac{\sigma}{\sqrt{\epsilon}} dW_t, \\
dy_t &= 1\ dt.
\end{aligned}
$$

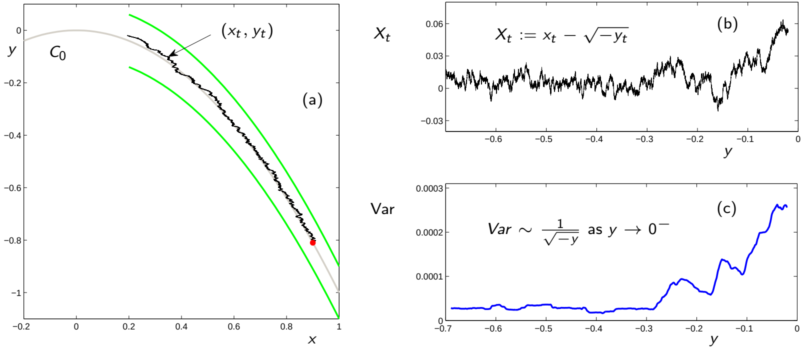

Figure : $(x_0, y_0) = (0.9, -0.9^2)$ [red dot], $\sigma = 0.01$, $\epsilon = 0.01$.

A Classification Result

Theorem (K. 2011/2012)

Classification of generic critical transitions for all fast subsystem bifurcations up to codimension two:

- *Fold, Hopf, (transcritical), (pitchfork)*
- *Cusp, Bautin, Bogdanov-Takens*
- *Gavrilov-Guckenheimer, Hopf-Hopf*

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The main results are:

1. *(Existence:) Conditions on slow flow to get a critical transition.*
2. *(Scaling:) Leading-order covariance scaling $H_\epsilon(y)$ for*
$$Cov(x_s) = \sigma^2[H_\epsilon(y)] + \mathcal{O}(\delta(s,\epsilon)).$$
3. *((ϵ,σ)-expansion:) Higher-order calculations for the fold.*
4. *(Technique:) Covariance estimates without martingales.*

Spatio-Temporal Stochastic Dynamics

- Bounded domain $\rightarrow$ 'finite-dim.' bifurcations, warning signs.
- Unbounded domain $\rightarrow$???

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Example: Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

Background - FKPP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

- Model for waves $u = u(x - ct)$ in biology, physics, etc.
- Take $x \in \mathbb{R}$ and localized initial condition $u(x, t = 0)$.

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Basic propagating front(s):

- $u \equiv 0$ and $u \equiv 1$ are stationary.
- Wave connecting the two states:

$$u(\eta) = u(x - ct), \quad \lim_{\eta\to\infty} u(\eta) = 1, \quad \lim_{\eta\to-\infty} u(\eta) = 0.$$

- Propagation into unstable state $u = 0$ since

$$D_u f = D_u[u(1-u)] \quad \Rightarrow D_u f(0) = (1-2u)|_{u=0} > 0.$$

SPDE Version of FKPP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) + \sigma g(u)\,\xi(x,t), \qquad \sigma > 0.$$

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Possible choices for 'noise process' $\xi(x,t)$

- white in time $\xi = \dot{B}$, $\mathbb{E}[\dot{B}(t)\dot{B}(s)] = \delta(t-s)$
- space-time white $\xi = \dot{W}$, $\mathbb{E}[\dot{W}(x,t)\dot{W}(y,s)] = \delta(t-s)\delta(x-y)$
- Q-trace-class noise

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Possible choices for 'noise term' $g(u)$

- $g(u) = u$, ad-hoc (Elworthy, Zhao, Gaines,...)
- $g(u) = \sqrt{2u}$, contact-process (Bramson, Durrett, Müller, Tribe,...)
- $g(u) = \sqrt{u(1-u)}$, capacity (Müller, Sowers,...)

Propagation Failure

FKPP SPDE exhibits propagation failure

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) + \sigma g(u)\, \xi(x,t), \qquad g(0) = 0.$$

i.e. solution may get absorbed into $u \equiv 0$.

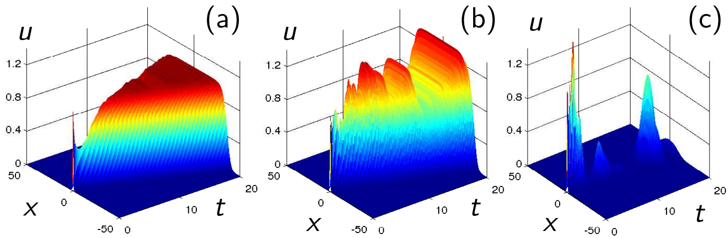

Figure : $g(u) = u$, $\xi = \dot{B}$. (a) $\sigma = 0.02$, (b) $\sigma = 0.3$ and (c) $\sigma = 1.2$.

Scaling near transition: single-point observer statistics:

$$\bar{u} = \frac{1}{T - t_0} \int_{t_0}^{T} u(0, t)\, dt, \quad \Sigma = \left[\frac{1}{T - t_0} \int_{t_0}^{T} (u(0, t) - \bar{u})^2\, dt \right]^{1/2}.$$

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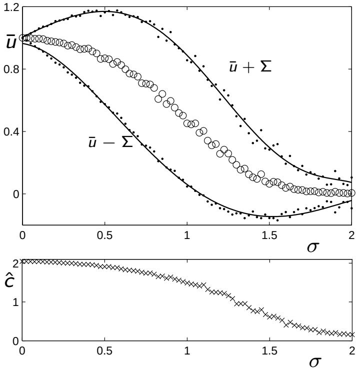

Figure : Average over 200 sample paths; $t \in [10, 20]$.

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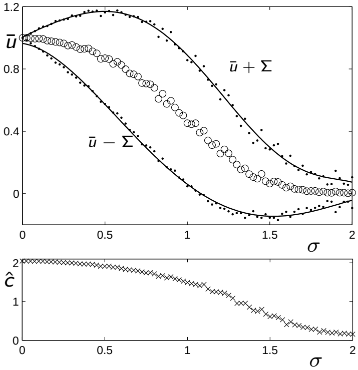

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Challenge: Statistics (of SPDEs) near instability?

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For more references see also:

- http://www.asc.tuwien.ac.at/∼ckuehn/

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