The influence of the disorder in the Kuramoto model

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1. Mean-field interacting diffusions

2. $N \to \infty$: Law of Large Numbers

3. The example of the Kuramoto model

4. The symmetric case: fluctuations around the McKean-Vlasov equation

5. Spatially extended neurons

Synchronization of individuals

Emergence of synchrony is widely encountered in complex systems of individuals in interaction (networks of neurons, collective behavior of social insects, chemical interactions between cells, planets orbiting, . . .).

Three main ingredients for interacting individuals:

1. a dynamics for each individual (e.g. FitzHugh-Nagumo or Hodgkin-Huxley for neurons)
2. a network of interactions (possibly heterogeneous and delayed)
3. absence or presence of a thermal noise (possibly correlated).

Under these conditions, for a sufficiently strong interaction between individuals and for a sufficiently large population, synchronization should occur (individuals exhibit similar simultaneous behavior).

Outline

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2. $N \to \infty$: Law of Large Numbers
3. The example of the Kuramoto model
4. The symmetric case: fluctuations around the McKean-Vlasov equation
5. Spatially extended neurons

General framework: mean-field interacting diffusions in $\mathbf{R}^m$

Here, each individual θ is a diffusion in $\mathbf{R}^p$.

For $T > 0$, $N \geqslant 1$, consider $t \in [0, T] \mapsto (\theta_1(t), \dots, \theta_N(t))$ solution to

$$d\theta_i(t) = c(\theta_i)\,dt + \frac{1}{N}\sum_{j=1}^{N}\Gamma(\theta_i, \theta_j)\,dt + \sigma\,dB_i(t), \quad i = 1, \cdots, N,$$

- $c(\cdot)$: local dynamics of one individual
- $\Gamma(\cdot, \cdot)$: interaction kernel
- B_i: i.i.d. Brownian motions (thermal noise).

Exchangeability

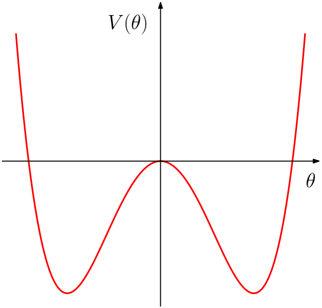
If at $t = 0$, the vector $(\theta_1(0), \dots, \theta_N(0))$ is exchangeable, then, at all time $t > 0$, the law of the vector $(\theta_1(t), \dots, \theta_N(t))$ is also exchangeable.

An example

Interesting examples include the granular media system:

$$d\theta_i(t) = -\nabla V(\theta_i)\,dt - \frac{1}{N}\sum_{j=1}^{N} \nabla W(\theta_i - \theta_j)\,dt + \sigma\, dB_i(t),$$

for V and W having convexity properties.



[✎ Carillo, McCann, Villani, Malrieu, Guillin, Cattiaux, Berglund, Gentz, Tugaut, etc.]

Interacting diffusions in a random environment

Exchangeability may not be a suitable property: one needs to encode the fact that the dynamics may not be the same for each individual (e.g., inhibition or excitation for a neuron).

Idea: set a sequence of i.i.d. random variables (ω_i) encoding the intrinsic behavior of the individual θ_i. For this choice of disorder $(\omega_1, \omega_2, \ldots, \omega_N)$, we modify the dynamics and the interaction in the following way:

$$\mathrm{d}\theta_i(t) = c(\theta_i, \omega_i)\,\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N}\Gamma(\theta_i, \theta_j, \omega_i, \omega_j)\,\mathrm{d}t + \sigma\,\mathrm{d}B_i(t), \quad i = 1, \cdots, N.$$

Question

What is the (quenched) influence of the disorder on the long-time/large N behavior of the system in comparison with the case without disorder?

Disordered mean-field models in neuroscience

Consider FitzHugh-Nagumo dynamics for the spiking activity of one neuron $\theta = (v, W) \in \mathbf{R}^2$ i.e.

$$\begin{cases} \varepsilon \dot{V} & = V - V^3/3 + W + I \\ \dot{W} & = aW + bV, \end{cases}$$

where V is the membrane potential and W is a recovery variable. The disorder $\omega = (a, b)$ encodes the state (inhibited/excited) of one neuron.

$$\mathrm{d}\theta_i(t) = c(\theta_i, \omega_i)\,\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N} \Gamma(\theta_i, \theta_j, \omega_i, \omega_j)\,\mathrm{d}t + \sigma\,\mathrm{d}B_i(t), \quad i = 1, \cdots, N,$$

Here Γ models synaptic connections between neurons.

[✎ O. Faugeras, J. Touboul et al.: similar systems with delay, two-scale of population, disorder, etc.]

Difficulty: absence of reversibility.

A simpler model: the Kuramoto model

✎ *Kuramoto ['75], Strogatz, Giacomin, Pakdaman, Pellegrin, Poquet, Bertini, L.*

The state space here is the one-dimensional circle: $\mathbf{S} := \mathbf{R}/2\pi$:

$$\mathrm{d}\theta_i(t) = \omega_i\,\mathrm{d}t + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\,\mathrm{d}t + \sigma\,\mathrm{d}B_i(t), \quad i = 1, \ldots, N,$$

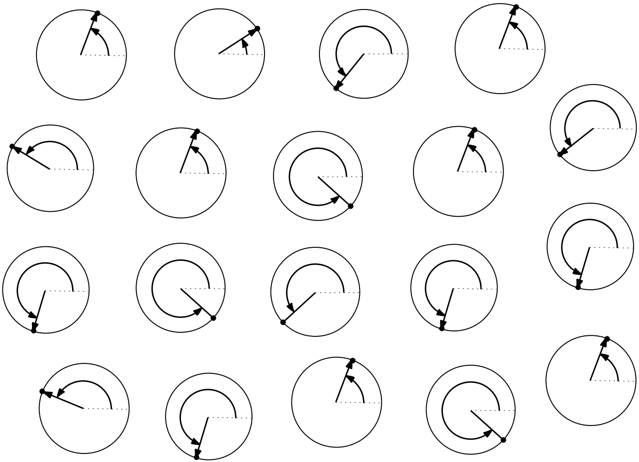
Intuition: competition between

- $\omega_i dt$: random intrinsic speed of rotation for each rotator θ_i,
- $K\sin(\cdot)dt$: synchronizing kernel between rotators.

Absence of disorder = reversibility

If $\forall i = 1, \ldots, N, \omega_i = 0$, the dynamics is reversible.

Example: The disorder is chosen by a toss of coins $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$.



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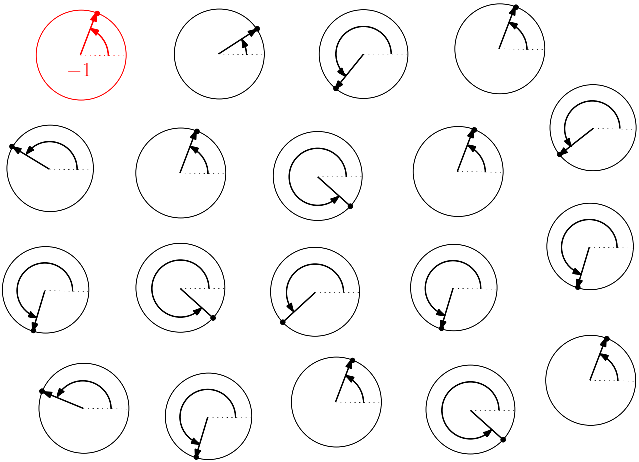

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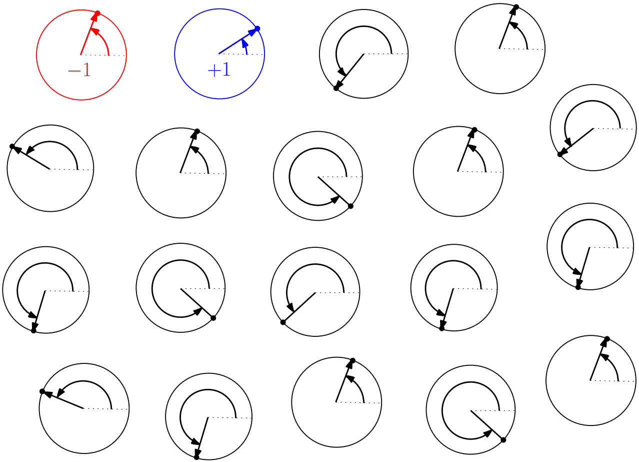

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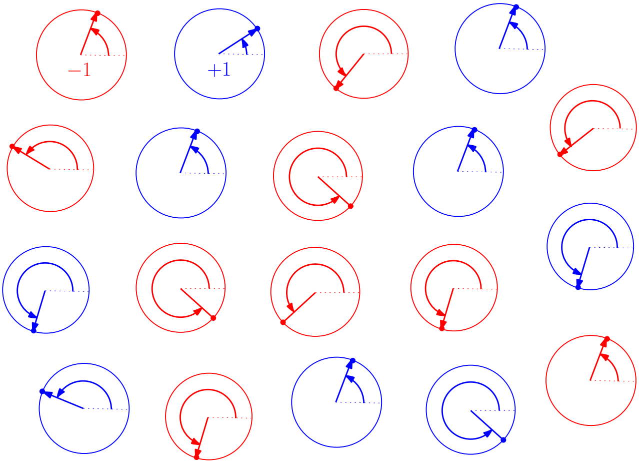
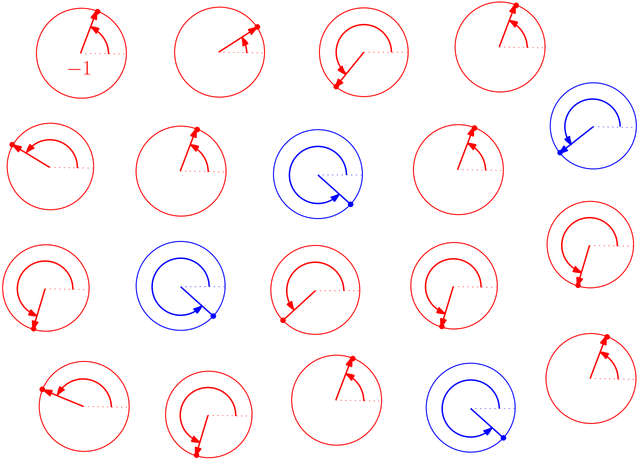


Questions: what is the influence of the disorder on the system? Does it depend only on its law μ (centered, symmetric or not) or on a typical realization $(\omega_1, \ldots, \omega_N)$?

Example: The disorder is chosen by a toss of coins $\mu = p\delta_{-1} + (1-p)\delta_1$.



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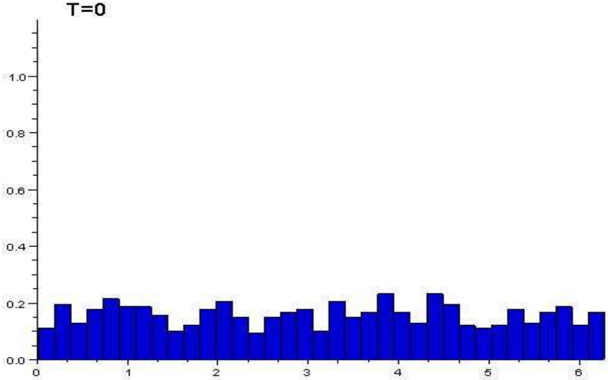
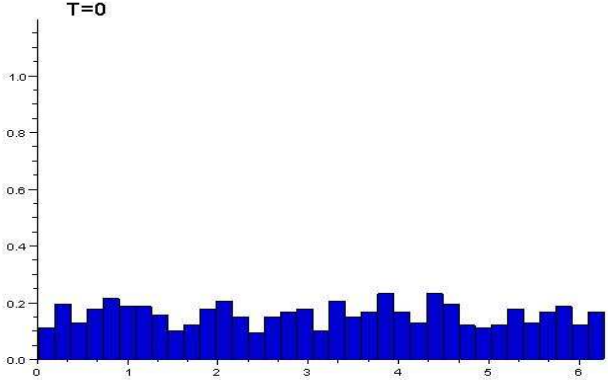
Simulation I: $N = 500$, $K = 3$, $\sigma = 1$, no disorder

$$d\theta_i(t) = \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)\, dt + \sigma\, dB_i(t), \quad i = 1, \ldots, N,$$



Simulation II: $N = 600$, $K = 6$, $\sigma = 1$, $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$

$$d\theta_i(t) = \omega_i\,dt + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)\,dt + \sigma\,dB_i(t), \quad i = 1, \dots, N,$$



Outline

The empirical measure

$$d\theta_i(t) = c(\theta_i, \omega_i)\,dt + \frac{1}{N}\sum_{j=1}^{N} \Gamma(\theta_i, \theta_j, \omega_i, \omega_j)\,dt + \sigma\, dB_i(t), \quad i = 1, \cdots, N.$$

We want to understand the (quenched vs annealed) behavior as $N \to \infty$ of the empirical measure

$$\nu_{N,t} := \frac{1}{N}\sum_{j=1}^{N} \delta_{(\theta_j(t), \omega_j)}.$$

- Law of Large Numbers?
- Does the continuous limit say anything on the particle system?
- Central Limit Theorem? Large Deviations?

Quenched convergence of the empirical measure ν_N

Under Lipschitz regularity on c and Γ, moment condition on μ and convergence of the initial condition $\nu_{N,0}^{(\omega)} \xrightarrow{N\to\infty} \nu_0$,

Proposition (Quenched LLN - L. 2011)

For a.e. $(\omega_i)_{i\geqslant 1}$, the empirical measure $\nu_N^{(\omega)}$ converges in law, as a process to

$$t \mapsto \nu_t(\mathrm{d}\theta, \mathrm{d}\omega)$$

that is the unique weak solution to the following McKean-Vlasov equation:

$$\partial_t \nu_t = \frac{1}{2} div_\theta \left(\sigma\sigma^T \nabla_\theta \nu_t\right) - div_\theta \left[\nu_t \left(c(\theta,\omega) + \int \Gamma(\theta,\omega,\cdot,\cdot)\,\mathrm{d}\nu_t\right)\right].$$

Self-averaging phenomenon

At the level of the LLN, the dependence in the disorder lies in its law, not a typical realization.

Outline

McKean-Vlasov equation in the Kuramoto model

In the limit of an infinite population: $q_t(\theta, \omega)$, density at time t of oscillators with phase θ and frequency ω solves

$$\partial_t q_t(\theta,\omega) = \frac{\sigma^2}{2}\Delta q_t(\theta,\omega) - K\partial_\theta\Big[q_t(\theta,\omega)\Big(\langle \sin * q_t\rangle_\mu(\theta)+\omega\Big)\Big].$$

What makes the Kuramoto tractable is that the nonlinearity is nice (it only concerns the first Fourier coefficients of q). In particular, if there is no disorder, one can show that

- the microscopic system is reversible,
- there exists a Lyapounov functional for the continuous model.

From this, one can derive many things in the case of small disorder, by perturbation arguments.

Non-symmetric disorder: existence of traveling waves

1. The law of the disorder is not centered ($\mathbb{E}_\mu(\omega) \neq 0$): we can go back to the centered case by the change of variables $\tilde{\theta}_i(t) := \theta_i(t) - t\mathbb{E}_\mu(\omega)$ (existence of traveling waves).
2. The law of the disorder is centered ($\mathbb{E}_\mu(\omega) = 0$) but not symmetric:

Theorem (Giacomin, L., Poquet, 2012)

If the disorder is small, there exist solutions to the McKean-Vlasov equation of the following type

$$q^{\psi}(\theta, \omega) := q(\theta - c(\mu)t - \psi)$$

and the family $q^{(\psi)}$ is stable by perturbation.

Question

What if the law of the disorder is centered and symmetric?

Symmetric disorder: synchronization

In this case, the McKean-Vlasov admits stationary solutions that can be explicitly computed:

$$0 = \partial_t q_t(\theta, \omega) = \frac{\sigma^2}{2} \Delta q_t(\theta, \omega) - K \partial_\theta \Big[q_t(\theta, \omega) \Big(\langle \sin * q_t \rangle_\mu(\theta) + \omega \Big) \Big],$$

Theorem (Giacomin, L., Poquet, 2012)

For small disorder, there exists $K_c > 0$ such that

- *if $K \leqslant K_c$, $q_i \equiv \frac{1}{2\pi}$ is the only stationary solution (incoherence),*
- *if $K > K_c$, $\frac{1}{2\pi}$ coexists with a circle (rotation invariance) of nontrivial stationary solutions $\{(\theta, \omega) \mapsto q_s(\theta + \theta_0, \omega);\ \theta_0 \in \mathbf{S}\}$ (synchronization).*

Moreover, such a circle of synchronized solutions is stable under perturbations.

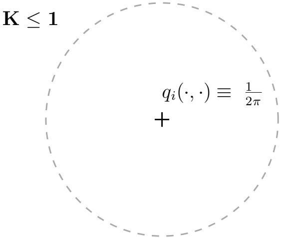
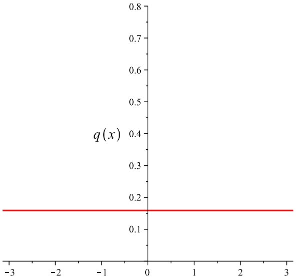
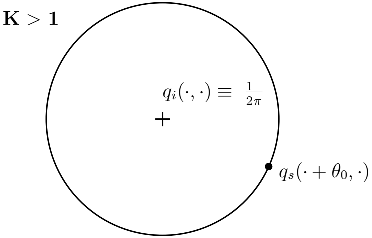


Figure : The incoherent solution $q_i \equiv \frac{1}{2\pi}$





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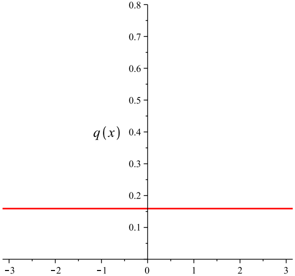
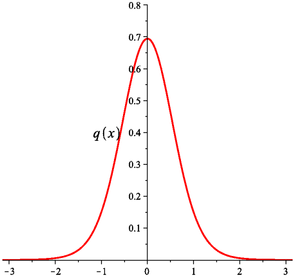


Figure : One synchronized solution q_s



Outline

Fluctuations of ν_N around its McKean-Vlasov limit

For fixed $t \in [0, T]$, fixed disorder (ω), consider the random tempered distribution

$$\eta_{N,t}^{(\omega)} := \sqrt{N}\left(\nu_{N,t}^{(\omega)} - \nu_t\right) \in \mathcal{S}'.$$

Semi-martingale representation of $\eta_N^{(\omega)}$: for all φ regular, $t \leqslant T$:

$$\left\langle \eta_{N,t}^{(\omega)}, \varphi \right\rangle = \left\langle \eta_{N,0}^{(\omega)}, \varphi \right\rangle + \int_0^t \left\langle \eta_{N,s}^{(\omega)}, L_N(\varphi) \right\rangle \mathrm{d}s + M_{N,t}^{(\omega)}(\varphi),$$

where L_N is a linear operator and $M_{N,t}^{(\omega)}(\varphi)$ a martingale.

Some negative answer

Remark

There cannot be any real quenched Central Limit Theorem, in the sense that for fixed disorder (ω), the process $\eta_N^{(\omega)}$ may not converge.

Why? Consider the example of independent Brownian motions with random drifts (i.e. $\Gamma \equiv 0$ and $c(\theta, \omega) = \omega$):

$$\mathrm{d}\theta_i(t) = \omega_i \,\mathrm{d}t + \mathrm{d}B_i(t).$$

In the quenched model, the $(\omega_i)_{i \geqslant 1}$ are fixed. In order to study the fluctuations of this system, one needs to understand the quantity

$$\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} \omega_i - \mathbb{E}(\omega)\right),$$

which does not converge for fixed $(\omega_i)_{i \geqslant 1}$ (but only in law w.r.t. $(\omega_i)_{i \geqslant 1}$).

The correct set-up

Instead of looking at

$$\eta_N^{(\omega)} := \sqrt{N}\left(\nu_N^{(\omega)} - \nu\right) \in \mathcal{C}([0,T], \mathcal{S}'),$$

for fixed (ω), one can always consider the application

$$(\omega) \mapsto H_N(\omega) := \text{law of } \eta_N^{(\omega)} \in \mathcal{M}_1(\mathcal{C}([0,T], \mathcal{S}')).$$

The correct set-up is to say that the sequence of random variables $(H_N)_N$ converges in law in the big space $\mathcal{M}_1(\mathcal{C}([0,T], \mathcal{S}'))$.

Quenched CLT

Hypothesis

- b and c are regular,
- (ω_j) are i.i.d. and $\int_{\mathbf{R}} |\omega|^{4\alpha} \mu(\mathrm{d}\omega) < \infty$ for some $\alpha > 0$.

Theorem (L. 2011)

Let $H_N(\omega)$ be the law of the process $\eta_N^{(\omega)}$. Then $(H_N)_N$ converges in law in $\mathcal{M}_1(\mathcal{C}([0,T], \mathcal{S}'))$ to $\omega \mapsto H(\omega)$ satisfying the following characterization: for all ω, $H(\omega)$ is the law of the solution η^ω of the SPDE:

$$\eta_t^\omega = X(\omega) + \int_0^t L_{q_s} \eta_s^\omega \, \mathrm{d}s + W_t,$$

where, W_t is explicit and for all ω, $X(\omega)$ is a Gaussian process that is not centered. W is independent with X.

Kuramoto: asymptotic behavior of the fluctuation process

Binary disorder: $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$, $\omega_0 > 0$.

Theorem (L. 2012)

For all $K > 1$, there exists $\omega_0 = \omega_0(K)$ such that η satisfies

$$\forall \omega,\ \frac{\eta_t^\omega}{t} \xrightarrow[t\to\infty]{\text{in law}} v(\omega) q'.$$

Moreover, as a function of ω, $\omega \mapsto v(\omega)$ is a Gaussian random variable with variance

$$\sigma_v^2 := \frac{\omega_0^2}{4}.$$

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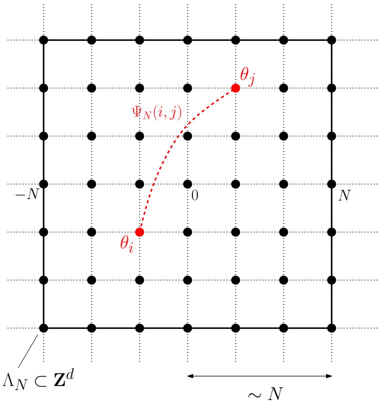
The case with spatial extension

- Joint work with W. Stannat.
- We want to take into account the positions of the particles θ_i: we place one particle at each point of the lattice $\mathbf{Z}^d$ and the interaction between two particles depends on the distance between them.



Spatially extended weakly interacting diffusions

The system becomes

$$d\theta_i(t) = c(\theta_i, \omega_i)\,dt + \frac{1}{|\Lambda_N|} \sum_{j \in \Lambda_N, j \neq i} \Gamma(\theta_i, \theta_j, \omega_i, \omega_j) \cdot \Psi\left(\frac{i}{2N}, \frac{j}{2N}\right) dt + \sigma\, dB_i(t),$$

where

- Λ_N is a box in $\mathbf{Z}^d$ of size $\sim N$, with volume $|\Lambda_N|$,
- $\Psi(\cdot, \cdot)$ is a spatial weight.

Possible choices of weights Ψ are:

- A cut-off function: $\Psi(x, y) \approx \mathbf{1}_{|x-y| \leqslant R}$
- A power-law interaction: $\Psi(x, y) = \frac{1}{|x-y|^{\alpha}}$

The power-law case for $\alpha < d$

$$d\theta_i(t) = c(\theta_i, \omega_i)\,dt + \frac{1}{|\Lambda_N|} \sum_{j \in \Lambda_N, j \neq i} \frac{\Gamma(\theta_i, \theta_j, \omega_i, \omega_j)}{\left|\frac{i-j}{2N}\right|^{\alpha}}\,dt + \sigma\,dB_i(t).$$

Proposition (L. - Stannat, 2013)

The empirical measure

$$\nu_N := \frac{1}{|\Lambda_N|} \sum_{j \in \Lambda_N, j \neq i} \delta_{(\theta_i, \omega_i, \frac{i}{2N})}$$

converges, as a process, to the unique solution $d\nu_t = q_t(\theta, \omega, x)\,d\theta\mu(d\omega)\,dx$ *where*

$$\partial_t q_t = \frac{1}{2} div_\theta \left(\sigma\sigma^T \nabla_\theta q_t\right) - div_\theta \left[q_t \left(c(\theta, \omega) + \int \frac{\Gamma(\theta, \omega, \cdot, \cdot)}{|x - \cdot|^{\alpha}}\,dq_t \right) \right].$$

Precise fluctuations estimates in the case $\alpha < d$

For an appropriate *weighted Wasserstein distance* (for some $p > 1$ and for some adequate domain $\mathcal{D}$),

$$d(\lambda, \nu) := \sup_{f \in \mathcal{D}} \left(\mathbb{E} \left\| \int f \, \mathrm{d}\lambda - \int f \, \mathrm{d}\nu \right\|^p \right)^{1/p}.$$

Theorem (L. - Stannat, 2013)

For any $\gamma < \frac{d}{2}$, there exists a constant $C > 0$ such that:

$$\sup_{0 \leqslant t \leqslant T} d(\nu_{N,t}, \nu_t) \leqslant C \begin{cases} N^{-(\gamma \wedge 1)}, & \text{if } \alpha \in \left[0, \frac{d}{2}\right), \\ (\ln N) \cdot N^{-(\frac{d}{2} \wedge 1)}, & \text{if } \alpha = \frac{d}{2}, \\ N^{-((d-\alpha)\wedge 1)}, & \text{if } \alpha \in \left(\frac{d}{2}, d\right). \end{cases}$$

Perspectives

- Is it possible to prove a quenched LDP in the mean field case?
- What can we say about the phase transition in the spatial case? About the central limit theorem?
- What if the positions of the particles are chosen randomly?
- Can we derive similar McKean-Vlasov equations for more general graphs (small-world graphs, etc.)?

Thank you for your attention!