Exit times of diffusions with incompressible drift

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Exit times of diffusions with incompressible drifts

Transition from long-time Homogenization to strong-flow Freidlin-Wentzel Averaging,
with Gautam Iyer, Tomasz Komorowski, and Lenya Ryzhik.

Do incompressible drifts enhance transport?
with Gautam Iyer, Lenya Ryzhik, Andrej Zlatos.

$$dX_t^x = Au(X_t^x)dt + \sqrt{2}dW_t,\ X_0^x = x,\ \nabla \cdot u(x) = 0.$$

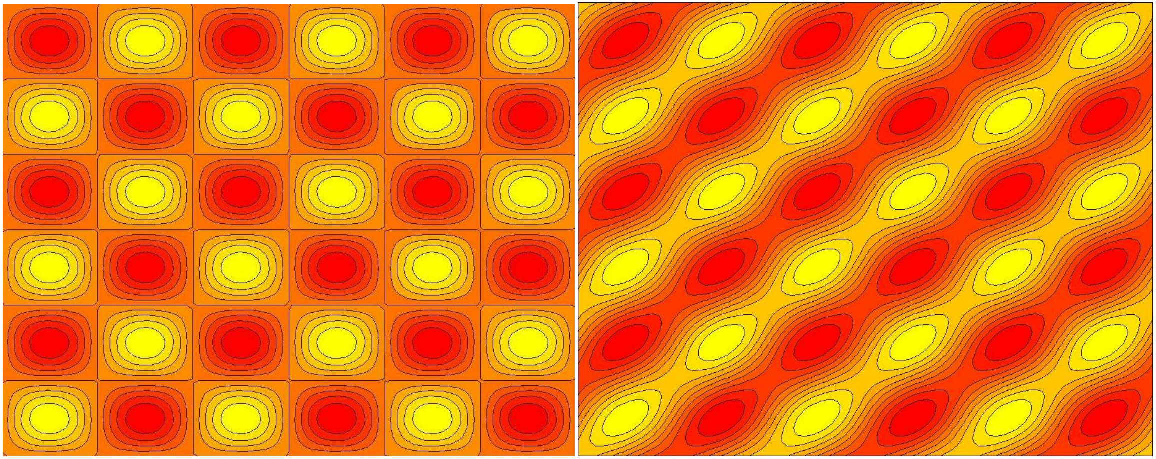
Behavior of X_t^x, when Péclet number $A \gg 1$, and $t \gg 1$.
Behavior of

$$\tau(x) = \mathbb{E}\left(X_{\sigma^x}^x\right),\ \sigma^x = \inf_{t>0}\left(X_t^x \in \partial\Omega\right),$$

with $|\Omega| \to \infty$.

Cellular vs. cat's-eye incompressible flows

$$u(\mathbf{x}) = \nabla^{\perp} H(\mathbf{x}) = \left(-\frac{\partial}{\partial x_2} H(\mathbf{x}), \frac{\partial}{\partial x_1} H(\mathbf{x}) \right), \ \mathbf{x} = (x_1, x_2).$$



Dissipation rate a.k.a. effective diffusivity.

$$dX_t^x = Au(X_t^x)dt + \sqrt{2}dW_t, \ A \text{ is fixed, } t \to \infty.$$

$$\lim_{t\to\infty} \frac{\mathbb{E}\left(X_t^x \times X_t^x\right)}{D^A} = I.$$

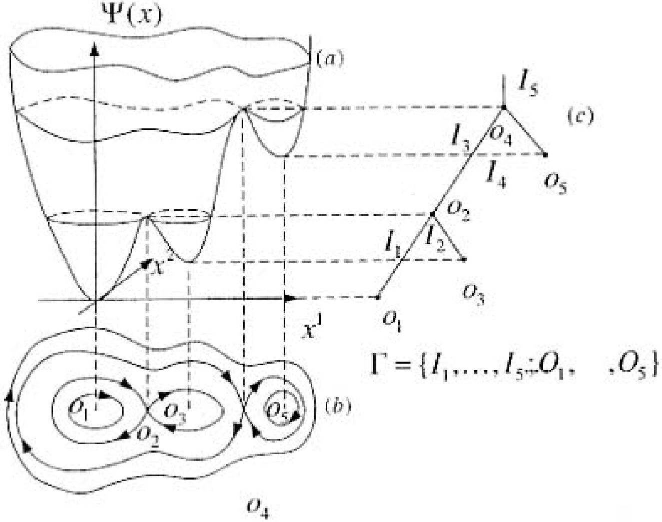
Then $X_t^x \sim Y_t^x$, $dY_t^x = D^A d\tilde{W}_t$.
Let $H = \sin \pi x_1 \sin \pi x_2$. PDE methods:

$$D^A \sim C\sqrt{A}, \text{ S.Childress '79.}$$

$$\lim_{A\to 0} \frac{D^A}{\sqrt{A}} = C > 0, \text{ A.Fannjiang \& G.Papanicolaou'94.}$$

Probabilistic methods: L.Koralov'01.

Averaging. Diffusion on graphs
(after M. Freidlin & A.Wentzel '78)



$$dX^x_t = Au(X^x_t)dt + \sqrt{2}dW_t.$$

Time t is fixed, $A \to \infty$

Small Random Perturbation

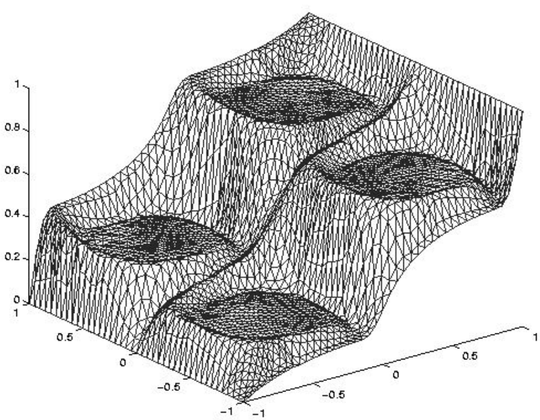
of Hamiltonian System

$$\dot{X}^x_t = Au(X^x_t)$$

Figure taken from M.Freidlin, Reaction-Diffusion in Incompressible Fluid: Asymptotic Problems, 2002.

Boundary layer theory. Cellular flows

Probability to exit through top,
Péclet number is 30.



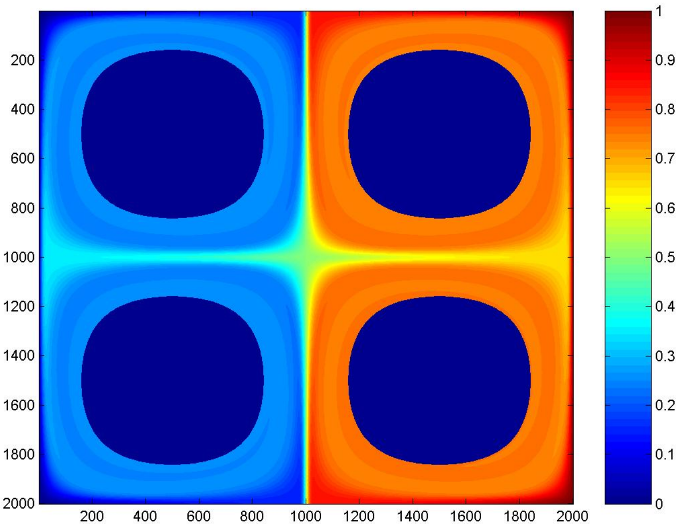
$$\Delta\phi - Au \cdot \nabla\phi = 0,$$

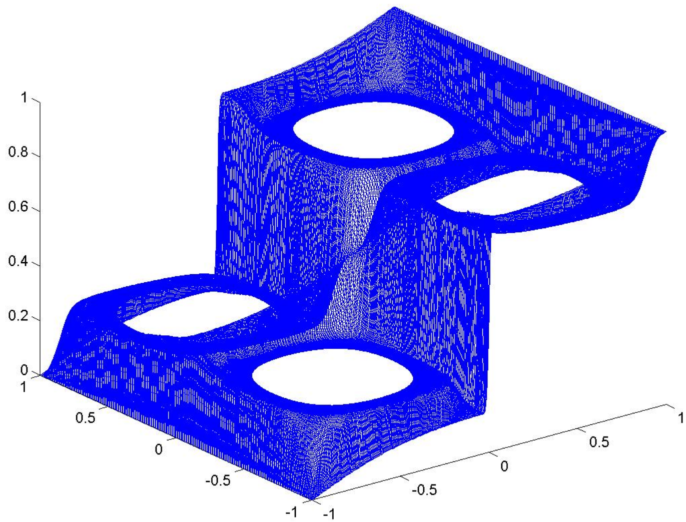
$$u = \nabla^{\perp} H,$$
$$H = \sin \pi x_1 \sin \pi x_2,$$

$$\phi(x_1 = -1, x_2) = 0,$$
$$\phi(x_1 = 1, x_2) = 1,$$
$$\frac{\partial}{\partial n}\phi(x_1, x_2 = \pm 1) = 0.$$

- Level-set $H(\mathbf{x}) = 0$ separates fluid motion into 4 eddies.
- Large gradients near separatrices, boundary layers
- Inside cells temperature is constant.

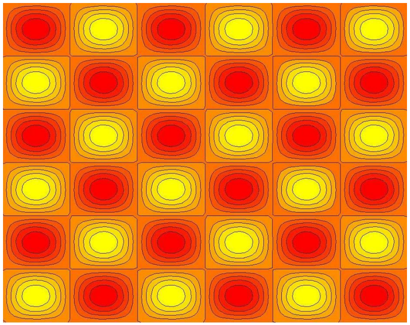
Boundary Layer Approximation





Numerical simulation for cellular flows $H = \sin x_1 \sin x_2$, $A = 10^3$.

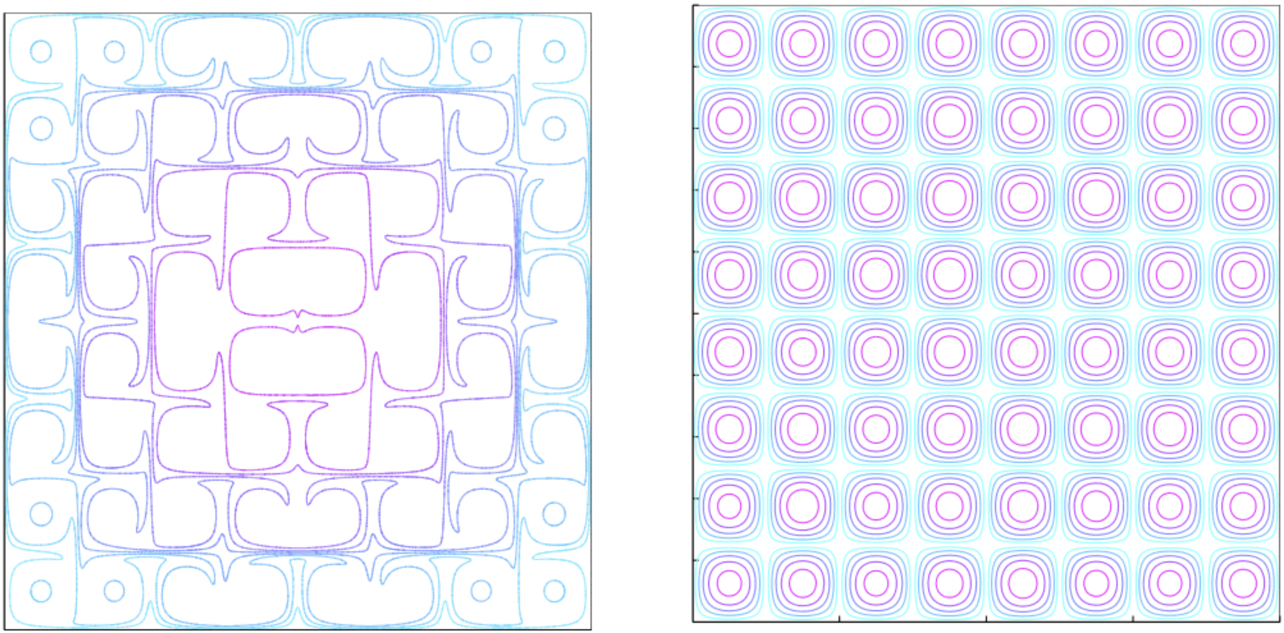
From Homogenization to Averaging in Cellular Flows



- Let $H(x) = \dfrac{1}{\pi} \sin(\pi x_1) \sin(\pi x_2)$.
- Let $u(x) = \nabla^{\perp} H = \begin{pmatrix} -\partial_2 H \\ \partial_1 H \end{pmatrix}$.
- Let $\Omega = (0,1)^2 \subset \mathbb{R}^2$.
- $\begin{cases} -\triangle \tau + \dfrac{A}{\varepsilon} v\left(\dfrac{x}{\varepsilon}\right) \cdot \nabla \tau = 1 & \text{in } \Omega, \\ \tau = 0 & \text{on } \partial\Omega \end{cases}$

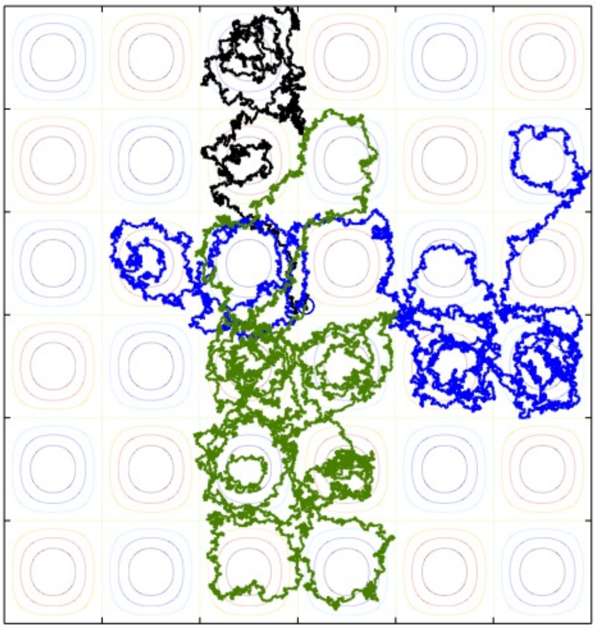
- Here A and $\varepsilon = 1/L$ are two non-dimensional parameters.
  - ▷ A is the strength of stirring (the Péclet number).
  - ▷ ε is the cell size.

Contour plots of τ

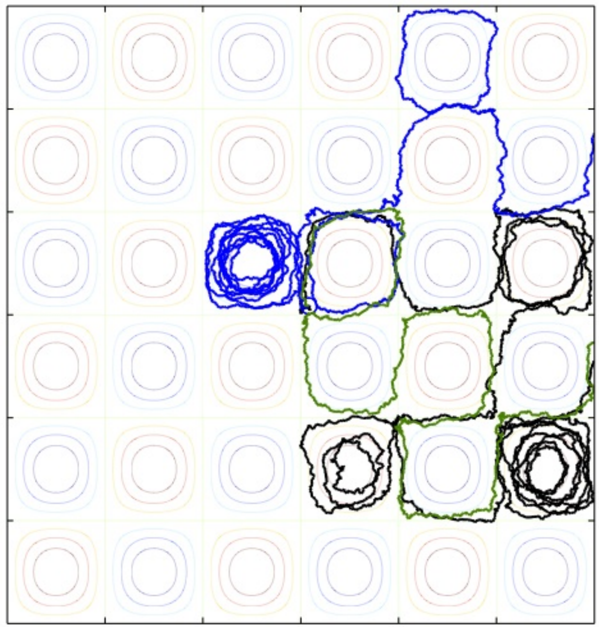


(a) A 'small' compared to $L = 1/\varepsilon$. (b) A 'large' compared to $L = 1/\varepsilon$.

Particle's behaviour $dX_t = u(X_t)\,dt + \sqrt{2}\,dW_t$



(c) A 'small' compared to L.



(d) A 'large' compared to L.

Homogenization vs. Averaging

- **Large amplitude. (Freidlin-Wentzel Averaging)**
  - ▷ For fixed cell size ε, and amplitude $A \longrightarrow \infty$.
  - ▷ τ is nearly constant on stream lines of u.
  - ▷ Forces τ to be almost identical in each cell. Separatrices are highways.
  - ▷ M.Friedlin, A.Wentzell; Yu.Kifer; H.Berestycki, F.Hamel, N.Nadirashvili.
- **Small amplitude. (Homogenization)**
  - ▷ For fixed amplitude A, and cell size $\varepsilon \longrightarrow 0$.
  - ▷ τ converges to the solution of an 'effective' enhanced diffusion equation.
  - ▷ No difference whether you start near or away from separatrices.
  - ▷ S.Childress; A.Fannjiang; G.Papanicolaou; L. Koralov.

Large amplitude, <u>and</u> a large number of cells.

With T.Komorowski, G.Iyer, L.Ryzhik'13

Send **both** $A \to \infty$, $\varepsilon \to 0$. Let $\tau = \tau_{A,\varepsilon}$ as before.

Theorem. *(Homogenization;* $A \ll 1/\varepsilon^4$*)*

- Suppose $\alpha > 0$, and $A \approx 1/\varepsilon^{4-\alpha}$.
- If $\Omega = B(0,1)$, then $\tau(x) \approx \tau_{eff}(x) = \dfrac{1-|x|^2}{2D_{eff}(A)} \approx \dfrac{1-|x|^2}{c\sqrt{A}}$.
- If $\Omega = (0,1)^2$, only have $\dfrac{1}{c\sqrt{A}} \leqslant \tau(x) \leqslant \dfrac{c}{\sqrt{A}}$ on the *interior* of Ω.

Theorem. *(Averaging;* $A \gg 1/\varepsilon^4$*)*

- Suppose $\lim_{A\to\infty} \dfrac{\varepsilon^2\sqrt{A}}{\log A \, \log(1/\varepsilon)} = \infty$.
- Oscillation of τ along streamlines tends to 0.
- On cell boundaries $\tau(x) \leqslant \dfrac{\log A \, \log(1/\varepsilon)}{\sqrt{A}} \to 0$.

Let $\varphi = \varphi_{\varepsilon,A}$ be the (positive) principal eigenfunction:

$$\begin{cases} -\triangle\varphi + \dfrac{A}{\varepsilon} v\left(\dfrac{x}{\varepsilon}\right) \cdot \nabla\varphi = \lambda\varphi & \text{in } \Omega = (0,1)^2, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem. There exists c_1, c_2 *independent* of L and A such that.

(A) If $A \gg \dfrac{(\log A)^2(\log(1/\varepsilon))^2}{\varepsilon^4}$, then $\lambda \approx \lambda_{avg}$.

(H) If $A \approx \dfrac{1}{\varepsilon^{4-\alpha}}$, then $\lambda \approx \lambda_{eff}$.

- $\lambda_{avg} = \dfrac{c_0}{\varepsilon^2}$, for some explicitly computable c_0.
- $\lambda_{eff} = \lambda_0(\nabla \cdot D_{eff}(A)\nabla) \approx c_1\sqrt{A}$, for explicitly computable c_1.

Asymptotics of the transition.

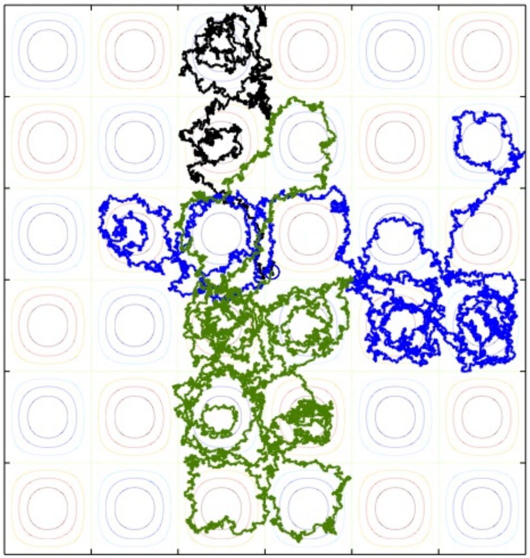
- The transition should occur when $\tau_{avg} \approx \tau_{eff}$.
- Freidlin-Wentzel Averaging $\tau_{avg} = \tau_{\text{one cell}} \sim \varepsilon^2$.
- Homogenization $X_t^x \sim Y_t^x = \sqrt{A}\tilde{W}_t$ $\tau_{eff} \sim 1/\sqrt{A}$.
- Transition should occur for $\sqrt{A} \approx \dfrac{1}{\varepsilon^2}$, or $A \approx \dfrac{1}{\varepsilon^4}$.

Three scales of diffusion in cellular flows

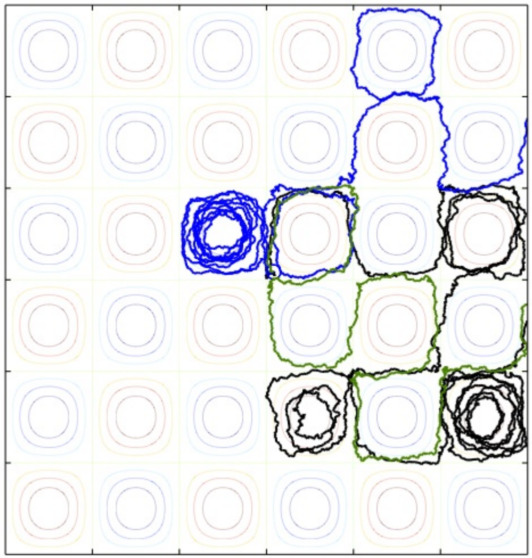
$$dX_t^x = Au(X_t^x)dt + \sqrt{2}dW_t,\ X_0^x = x.$$

- $t_{avg} \ll t_{rw} \ll t_{eff}$
- Freidlin-Wentzell averaging time t_{avg}.
- Random walk on separatrices time t_{rw}.
- Effective diffusion t_{eff}

Three scales of diffusion in cellular flows



(e) A 'small' compared to L.



(f) A 'large' compared to L.

Universality of diffusion in periodic fluid flows

$$dX_t = u(X_t)dt + dV_t,\ \ X_0 = 0.$$

- $t_{avg} \ll t_{rw} \ll t_{eff}$
- Time of V_t.
- Random walk time t_{rw}.
- Effective diffusion t_{eff}

Theorem (T.Komorowski, A.N., L.Ryzhik, '13) If V_t is a fractional Brownian motion with $H < 1/2$, $u(x)$ is shear $u = (u_1(x_2), 0)$ then

$$\varepsilon X_{t/\varepsilon^2} \to W_t,\ \ \text{as } \varepsilon \to 0.$$

Do incompressible flows improve mixing?

Suppose λ^u is the principal eigenvalue of $L^u = -\Delta + u \cdot \nabla$.

If

$$\partial_t \phi + L^u \phi = 0, \ \ \phi|_{\partial\Omega} = 0$$

then $||\phi||_{L^2} \sim e^{-\lambda^u t}$ as $t \to \infty$.

If $\nabla \cdot u = 0$, principal eigenvalue of $L^u = -\Delta + u \cdot \nabla$ (with $\phi|_{\partial\Omega} = 0$) is larger than that of $L^0 = -\Delta$.

Incompressible flows improve mixing in L^2-sense

Suppose $\phi \in H_0^1(\Omega)$ and $L^u\phi = \lambda^u\phi$, with $||\phi||_{L^2}^2 = \int_\Omega \phi^2 = 1$ then

$$\lambda^u \int_\Omega \phi^2 = \int_\Omega \phi L^u \phi = \int_\Omega |\nabla\phi|^2.$$

On the other hand the Raleigh quotient characterizes the principal eigenvalue of L^0:

$$\lambda^0 = \inf_{||\psi||_{L^2}=1} \int_\Omega |\nabla\psi|^2 \leqslant \int_\Omega |\nabla\phi|^2 = \lambda^u.$$

Exit time problem

(with G.Iyer, L.Ryzhik & A.Zlatos'10)

For any incompressible u

$$-\Delta\tau^u + u\cdot\nabla\tau^u = 1,\ \ \tau^u|_{\partial\Omega} = 0.$$

$\tau^u(x) = \mathbb{E}(X^x_{\sigma^x})$ is the expected exit time from Ω of the diffusion:

$$dX^x_t = u(X^x_t)\,dt + \sqrt{2}\,dW_t, \sigma^x = \inf_{t>0}(X^x_t \in \partial\Omega).$$

Theorem 1 Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and Lipschitz domain. Then $u \equiv 0$ maximizes $||\tau^u||_{L^\infty(\Omega)}$ if and only if Ω is a disk.

Theorem 2 Let $D \subset \mathbb{R}^n$ be a ball. Then $||\tau^u||_{L^p(D)} \leqslant ||\tau^0||_{L^p(D)}$ for all incompressible u, and all $1 \leqslant p \leqslant \infty$.

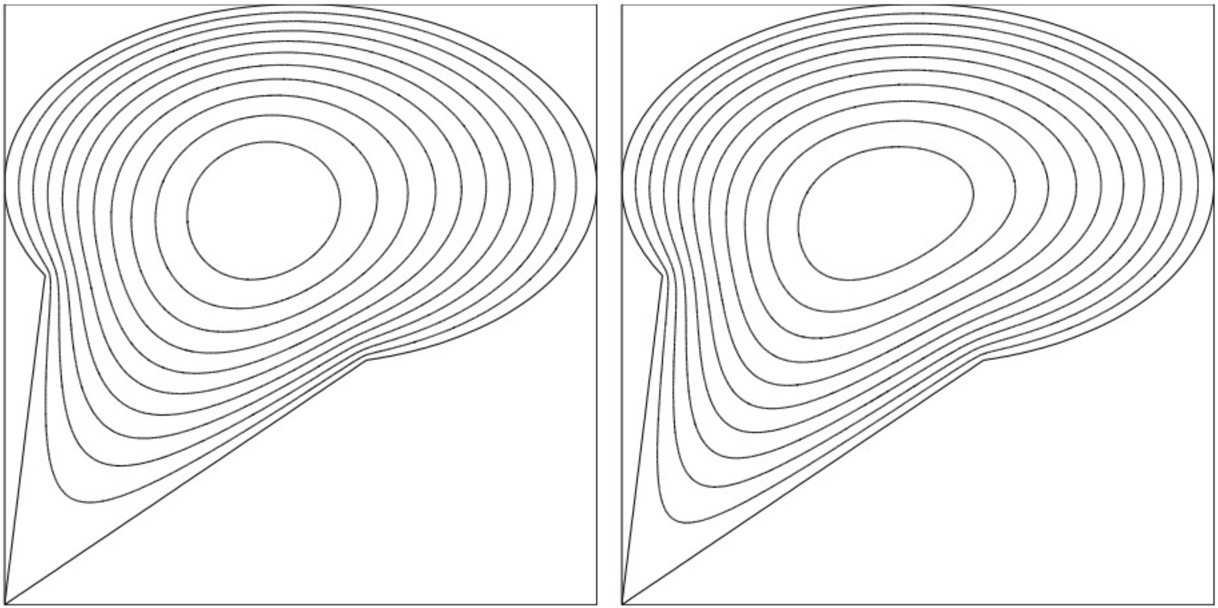
The 2-dimensional case. General domain

The stream function H for the "worst" flow $u = \nabla^{\perp} H$ solves

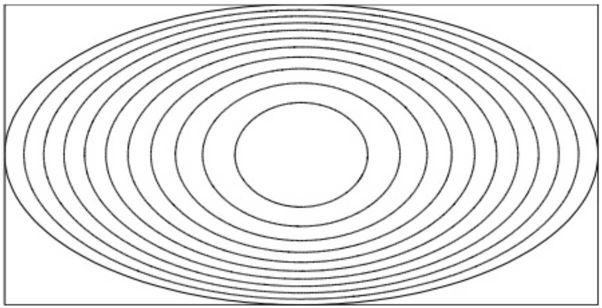
$$-2\Delta H = 1 + |\nabla H|^2 \left(\int_{\partial\Omega_h} |\nabla H| \, d\sigma \right)^{-1} \left(\int_{\partial\Omega_h} \frac{d\sigma}{|\nabla H|} \right),$$

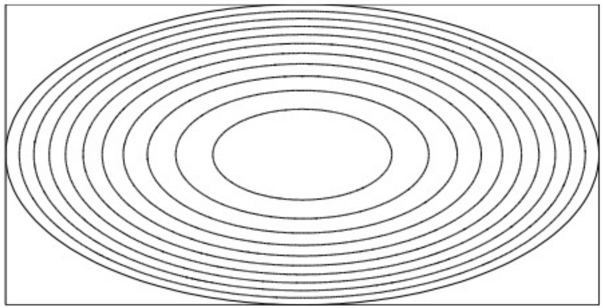
where $\Omega_h = \{x \in \Omega,\ H(x) \geqslant h\}$.

Streamfunction H for the "worst" flow, and T^0



Streamfunction H for the "worst" flow, and T^0





Remarks

- If $\nabla \cdot u = 0$ is dropped, the problem is trivial (a flow with a sink).
- Recall that presence of incompressible flow always improves mixing in the sense of increasing the first eigenvalue.
- Using fast flows, we can always make T^u arbitrarily small.
- A more general question: $\begin{cases} -\Delta \tau^u + u \cdot \nabla \tau^u = f & \text{in } \Omega \subset \mathbb{R}^n, \\ \tau^u = 0 & \text{on } \partial\Omega \end{cases}$ Then there is an $L^p \to L^\infty$ bound:

$$||\tau^u||_{L^\infty} \leqslant C||f||_{L^p},\ p > n/2,$$

where $C = C(n, p, \Omega)$, but C is independent of u (see Berestycki, Kiselev, Novikov, Ryzhik '09). Find an optimal C.
Theorem (A.N.'13) If Ω is a disk, then optimal C arises when $f = g(|x|, p, n)$, and g is a certain optimal non-increasing function.

Exit times in a ball.

Proposition. Let $\Omega \subset \mathbb{R}^n$ be bounded, simply connected and Lipschitz domain, and u be *any* divergence free vector field which is tangential on $\partial\Omega$. Then

$$||\tau^u||_{L^p(\Omega)} \leqslant ||\tau^{0,D}||_{L^p(D)}$$

where $D \subset \mathbb{R}^n$ is a ball with $|D| = |\Omega|$, and $\tau^{0,D}$ is the expected exit time from D with 0 drift.

Proof of Proposition

- Given any $\tau = \tau^u$, consider its symmetric rearrangement τ^*:
  - D is a ball with $|D| = |\Omega|$, and $\tau^* : D \to \mathbb{R}^+$ is radial.
  - For all h, $|\{\tau > h\}| = |\{\tau^* > h\}|$.
  - $||\tau||_{L^p(\Omega)} = ||\tau^*||_{L^p(D)}$ for all p.
- Denote $\Omega_h = \{\tau > h\}$, $\Omega_h^* = \{\tau^* > h\}$.

Proof of Proposition

$$\int_{\partial\Omega_h^*} |\nabla\tau^*|\, d\sigma \int_{\partial\Omega_h^*} \frac{1}{|\nabla\tau^*|}\, d\sigma = |\partial\Omega_h^*|^2 \leqslant |\partial\Omega_h|^2 \leqslant \int_{\partial\Omega_h} |\nabla\tau|\, d\sigma \int_{\partial\Omega_h} \frac{1}{|\nabla\tau|}\, d\sigma.$$

- Integrating the equation on Ω_h we obtain $\int_{\partial\Omega_h} |\nabla\tau|\, d\sigma = |\Omega_h|$.
- Co-area implies $\int_{\partial\Omega_h} \frac{1}{|\nabla\tau|}\, d\sigma = -\frac{d}{dh}|\Omega_h| = -\frac{d}{dh}|\Omega_h^*| = \int_{\partial\Omega_h^*} \frac{1}{|\nabla\tau^*|}\, d\sigma$
- So $\int_{\partial\Omega_h^*} |\nabla\tau^*|\, d\sigma \leqslant \int_{\partial\Omega_h} |\nabla\tau|\, d\sigma = |\Omega_h| = |\Omega_h^*|$.
- Using τ^* is radial and $\int_{\partial\Omega_h^*} |\nabla\tau^*|\, d\sigma \leqslant |\Omega_h^*|$ we conclude that $\tau^* \leqslant \tau^{0,D}$ point-wise, where $\tau^{0,D}$ is a solution of the exit time problem in the ball D with no flow.

Summary

- Interplay between size of the domain and large convection.
- Open question: Three scales in convection enhanced diffusion.
- Open question: Universality of effective diffusion.
- Does incompressible stirring improve mixing? It depends on your definition of mixing.
- Open question: Relaxation Enhancement and its quantitative characterization. Can (fluid-temperature) coupling may significantly improve mixing?