

$7^{ ext{th}}$ Workshop on $\overline{ ext{RDS}}$ Bielefeld, December 13, 2014



Front motion

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Stochastic Front Motion & Slow Manifolds

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joint work with:

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Peter Bates (Michigan State)
Alexander Schindler (Augsburg)



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■ General Concept

- Slow Manifold
- Key ideas
- Cahn-Hilliard
 - Construction of the Slow Manifold
 - Motion of interfaces
- Other models
 - Allen-Cahn
 - Motion of Droplets
 - Interface motion in 2D



General Concept



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Setting:

 ∞ -dimensional stochastic system in \mathcal{H}

$$du = \mathcal{L}(u)dt + dW$$
 (SPDE)

- \mathcal{L} nonlinear operator (e.g. $\Delta u + f(u)$)
- small additive noise
- W- Wiener process in \mathcal{H} , covariance operator Q

$$\left(\mathbb{E}\langle u, W(t)\rangle\langle v, W(t)\rangle = tQ\langle u, v\rangle\right)$$



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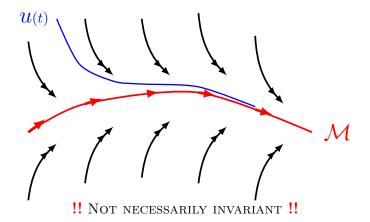
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Deterministic slow manifold $\mathcal{M} \subset \mathcal{H}$

(Dynamics for W = 0)

For example, parametrized by position of interfaces





Some Related Results



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- Otto, Reznikoff [07] Slow manifold gradient systems
- Beck, Wayne [09] Slow manifold Burgers on \mathbb{R}
- Brassesco, Butta,et.al.[98,02,...] single interface motion (stochastic Ginzburg-Landau, Phase Field Equation)
- Fatkullin, Kovacic, Vanden Eijnden [10] gradient systems
- Statistics, numerics, asymptotic expansions vanden Eijnden, Fatkullin[03], Fatkullin[10], Lythe[00,...],
- S. Weber [14], Shardlow [00] multi-kink stoch. Allen-Cahn
- Stannat et al [14,....] travelling waves
- For SDE see Gentz & Berglund

Stochastic Cahn-Hilliard – Brassesco [14]

The motion of a single interface in Cahn-Hilliard on \mathbb{R} is non-Markovian (fractional Brownian motion)





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General Idea of the Approach

(usually hidden in technical details)



Meta-Theorems



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Theorems (for stochastic dynamics)

- \blacksquare Attractivity of the manifold \mathcal{M}
 - only for a neighboorhood
 - \mathcal{M} not necessarily invariant for deterministic dynamics
- \blacksquare Stability of \mathcal{M}
 - exit from a neighboorhood of ${\mathcal M}$
 - here: weaker results than Large Deviation (not optimal)
- Motion along the manifold
 - Deterministic dynamics
 - Wiener process projected to ${\mathcal M}$



Theorem: Stability



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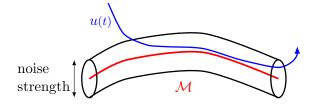
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Metatheorem (Stability)

With high probability, a solution

- \blacksquare stays on the "order of noise-strength" close to ${\mathcal M}$
- **2** until it exits "at the end" of \mathcal{M}





Motion along the manifold



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For some parameter space $\mathcal{P} \subset \mathbb{R}^N$

$$\mathcal{M} = \{ u^{\xi} \in \mathcal{H} : \xi \in \mathcal{P} \} .$$

AIM:

Determine $b: \mathcal{P} \mapsto \mathbb{R}^N$ and $\sigma_j: \mathcal{P} \mapsto \mathcal{H}$ such that

$$u(t) \approx u^{\xi(t)}$$
 with $d\xi_j = b_j(\xi)dt + \langle \sigma_j(\xi), dW \rangle$.



Coordinate System One Possibility



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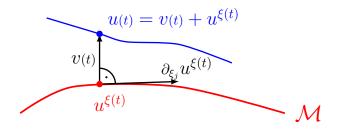
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 $u = u^{\xi} + v$ with $v \perp \partial_{\xi_j} u^{\xi}$ and $\partial_{\xi_j} u^{\xi}$ is tangential



Now:

Differentiate (Itô-formula)

$$u = u^{\xi} + v$$
 and $\langle \partial_{\xi_i} u^{\xi}, v \rangle = 0$,

eliminate dv and derive an equation for $d\xi_i$.



Stability



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For Stability

$$dv = du + du^{\xi}$$

$$= \mathcal{L}(v + u^{\xi})dt + \partial_{\xi}u^{\xi} \cdot d\xi + \text{Itō-Correction} + dW$$

$$= D\mathcal{L}(u^{\xi})v \ dt + N(u^{\xi}, v)dt + \partial_{\xi}u^{\xi} \cdot d\xi + \dots$$

Now take scalar product with v

NEED:

- ... Bounds on linearized operator $\langle v, D\mathcal{L}(u^{\xi})v \rangle$
- ... Control of Nonlinearity $N(u^{\xi}, v)$

To show bounds for v over large times with high probability



Stability



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Show for
$$t \leq \tau^*$$

(τ^{\star} exit time from a neighboorhood of $\mathcal M$ - distance R_{ϵ})

$$d||v||^2 = -a_{\epsilon}||v||^2 dt + \mathcal{O}(K_{\epsilon})dt + \langle \mathcal{O}(||v||), dW \rangle$$

Inductively

$$\mathbb{E}\|v(\tau^{\star})\|^{2p} \le C_p \left(\frac{K_{\epsilon} + \|Q_{\epsilon}\|}{a_{\epsilon}}\right)^p \cdot \mathbb{E}\tau^{\star}$$

Theorem

If $K_{\epsilon} + ||Q_{\epsilon}|| \ll a_{\epsilon} R_{\epsilon}^{2} \epsilon^{\kappa}$, then:

Exiting the neighboorhood of \mathcal{M} before any time of order ϵ^{-q} due to v being large has small probability.





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Results of [Antonopoulu, Karali, DB, 12]

- Motion of interfaces between several "pure" phases
- Approximate slow manifold
 - parametrized by the interface positions
 - Based on Bates & Xun [94,95] + Carr & Pego [89,90]



Cahn-Hilliard model Cahn & Hilliard [58,59], Cook [70]



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Phenomenological model for phase separation

For binary alloy, fluid,....

starting from an initially homogeneous mixture

u(t,x) – concentration of one component

(t > 0 time, x space)

 $u = \pm 1$ (almost) "pure" phases.



Cahn-Hilliard-Cook



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Stochastic Cahn-Hilliard Equation

$$\partial_t u = -\partial_x^2 [\epsilon^2 \partial_x^2 u - f(u)] + \epsilon^\delta \partial_t \partial_x W, \tag{CH}$$

Neumann-type (no flow) boundary conditions

$$\partial_x u(t,0) = \partial_x u(t,1) = 0$$
 $\partial_x^3 u(t,0) = \partial_x^3 u(t,1) = 0$

- F smooth double well, F' = f (E.g. $F(u) = \frac{1}{4}(1 u^2)^2$)
- Noise strength: ϵ^{δ} , $\delta > 9/2$ (technical reason)
- Interaction length/interface width: $0 < \epsilon \ll 1$



Noise



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$\partial_t u = -\partial_x^2 [\epsilon^2 \partial_x^2 u - f(u)] + \epsilon^{\delta} \partial_t \partial_x W$

$$W(t,x) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) e_k(x)$$

- e_k eigenfunctions of Dirichlet Laplacian (i.e., $\sin(\pi kx)$)
- \bullet $\alpha_k \to 0$ for $k \to \infty$, sufficiently fast (for Itō-formula)
- $\{\beta_k\}_{k\in\mathbb{N}}$ i.i.d. standard Brownian motion

Physics: Mass conservative space-time white noise, $\alpha_k = 1$





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Slow Manifold Construction

Carr & Pego [89,90]



Stationary Solution Φ



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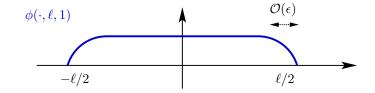
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Stationary solutions of deterministic (CH)!

(upto boundary conditions)



 $\Phi(x,\ell,\pm 1)$ solves

$$\epsilon^2 \partial_x^2 \Phi = f(\Phi)$$
, $\Phi(-\ell/2) = 0 = \Phi(\ell/2)$



Approximation u^h



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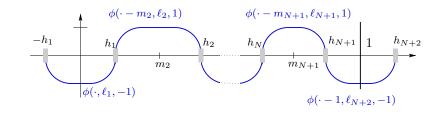
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 u^h indexed by interfaces $h \in \Omega_\rho \subset \mathbb{R}^{N+1}$

Distance:
$$\ell_j = h_{j+1} - h_j > \epsilon/\rho$$
 (later $\rho = \epsilon^{\kappa}$, $0 < \kappa \ll 1$)



Near h_i use cut-off to glue the ϕ 's smoothly.



Slow Manifold \mathcal{M}



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(i.e. $\int_0^1 u(t)dx = \int_0^1 u(0)dx$) (CH) is mass conservative. Fix mass $M = \int_0^1 u(t, x) dx$.

Definition (Slow Manifold)

$$\mathcal{M} = \left\{ u^h : h \in \Omega_\rho, \int_0^1 u^h dx = M \right\}$$

Interface h_{N+1} is a function of $\xi = (h_1, \dots, h_N)$.

Define $u^{\xi} := u^h$



Integrated Cahn-Hilliard Equation



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Technical Point:

If u(t, x) solves (CH), then

$$\tilde{u}(t,x) = \int_0^x u(t,z)dz$$

solves the Integrated Cahn-Hilliard equation (ICH).

Advantage:

The linearized operator (around \tilde{u}^{ξ})

$$L^c = -\epsilon^2 \partial_x^4 + \partial_x f'(u^{\xi}) \partial_x$$

is self-adjoint.

(It depends on ξ !)



Attraction – Deterministic (ICH)



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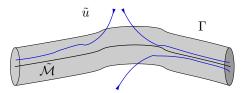
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 \exists exponentially (in ϵ) small neighboorhood Γ of $\widetilde{\mathcal{M}}$ that is locally exponentially attracting with rate $\mathcal{O}(1)$



On \mathcal{M} lies a stationary solution u^* (equidistant interfaces) with its N-dim. unstable manifold $W^u(u^*)$ exponentially close to $\tilde{\mathcal{M}}$.





Full SDE for ξ and \tilde{v}



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Theorem

$$\tilde{u}(0) = \tilde{u}^{\xi(0)} + \tilde{v}(0).$$

As long as $B_{\epsilon}(\tilde{v}) = \epsilon^2 \|\partial_x^2 \tilde{v}\|^2 + \|\partial_x \tilde{v}\|^2 = \mathcal{O}(\epsilon^3)$:

$$\tilde{u} = \tilde{u}^{\xi} + \tilde{v}$$
 is a solution of (ICH)

$$\Leftrightarrow$$

 ξ and \tilde{v} solve (SDE) & (SPDE)

$$d\xi_k = b_k(\xi, \tilde{v})dt + \epsilon^{\delta} \langle \sigma_k(\xi \tilde{v}), dW \rangle$$
 (SDE)

with b and σ stated later.

$$d\tilde{v} = \mathcal{L}(\tilde{u}^{\xi} + \tilde{v})dt + \epsilon^{\delta}dW - d\tilde{u}^{\xi},$$
 (SPDE)

where
$$\mathcal{L}(\tilde{u}) = -\epsilon^2 \partial_x^4 \tilde{u} + \partial_x f(\partial_x \tilde{u}).$$



Definition of b and σ



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Recall:

 $A_{ri}(\xi, \tilde{v}) \approx 4\ell_{i+1}(\delta_{i,r} + \delta_{r,i-1}) + C_{ri}\epsilon$

 $\sigma_r(\xi) = \sum_i A_{ri}^{-1}(\xi) E_i^{\xi} . \qquad (E_i^{\xi} \approx \partial_{\xi_i} \tilde{u}^{\xi})$

 $b_r(\xi) = \sum_i A_{ri}^{-1}(\xi) \Big\{ \langle \mathcal{L}^c(\tilde{u}^{\xi} + \tilde{v}), E_i^{\xi} \rangle + \sum_i \langle \mathcal{Q}_{\epsilon} E_{ij}^{\xi}, \sigma_j(\xi) \rangle$

Covariance operator of W is $\mathcal{Q}_{\epsilon}e_k = \epsilon^{2\delta}\alpha_k^2 e_k$

Most of the terms in b are Itō-Stratonovic correction.

Indices are derivatives, $E_{ij}^{\xi} = \partial_{\xi_i} E_i^{\xi}$ and $\tilde{u}_{ij}^{\xi} = \partial_{\xi_i} \partial_{\xi_i} \tilde{u}^{\xi}$

 $+ \sum_{i=1}^{k} \frac{1}{2} \Big[\langle \tilde{v}, E_{ilk}^{\xi} \rangle - \langle \tilde{u}_{kl}^{\xi}, E_{i}^{\xi} \rangle - 2 \langle \tilde{u}_{k}^{\xi}, E_{il}^{\xi} \rangle \Big] \langle \mathcal{Q}_{\epsilon} \sigma_{k}(\xi), \sigma_{l}(\xi) \rangle \Big\}$



Attractivity – Stability



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Theorem – Stability

Consider a solution $\tilde{u} = \tilde{u}^{\xi} + \tilde{v}$ of (ICH).

Fix $\kappa > 0$.

If $\mathcal{A}_{\epsilon}(\tilde{v}(0)) < \epsilon^{2\delta}$, then with high probability

$$\mathcal{A}_{\epsilon}(\tilde{v}(t)) = \mathcal{O}(\epsilon^{2\delta - \kappa})$$

for all $t \le \epsilon^{-q}$ for any q > 0 or until interface breaks down.

Def.
$$\mathcal{A}_{\epsilon}(\tilde{v}) = \langle \tilde{v}, L^c \tilde{v} \rangle$$
 with $c\epsilon^3 \|\tilde{v}\|_{H^1}^2 \leq \mathcal{A}_{\epsilon}(\tilde{v}) \leq C \|\tilde{v}\|_{H^2}^2$.

Attraction

Exponential attraction holds for a larger neighbourhood of order $B_{\epsilon}(\tilde{v}) = \mathcal{O}(\epsilon^6)$.





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Is the (SDE) for ξ usefull?



Examples



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Two interfaces (N=1)

Second interface determined by first (mass conservation).

 $d\xi \approx \exp. \text{ small} + \epsilon^{\delta} \|\partial \tilde{u}^{\xi}\|^{-2} \langle \partial \tilde{u}^{\xi}, \circ dW \rangle$

Stratonovic equation for $\epsilon^{\delta}W$ projected to $\tilde{\mathcal{M}}$

Space-time white noise $(Q_{\epsilon} = \epsilon^{2\delta} \cdot Id)$ – (too rough)

 ξ is close to a Brownian motion with variance $\epsilon^{2\delta}/4\ell$.

[B-X-94]: $\|\tilde{u}_1^{\xi}\|^2 \approx 4\ell$ - distance between interfaces.





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1D-Allen-Cahn



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$$du = [\epsilon^2 \partial_x^2 u + u - u^3] dt + dW$$
 & Neumann b.c

- similar dynamics than Cahn-Hilliard
- no mass conservation
- Interface motion similar [S. Weber 14, PhD]



Results of [Weber 14]



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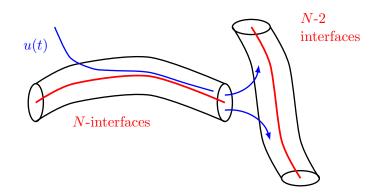
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- based on [X. Chen 04]
- Using maximum principle
- Exit from manifold with N-layers into the domain of attraction for N-2-layers







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 ${\bf Motion\ of\ Droplets}$



Mass conservative Allen-Cahn



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On some sufficiently large domain $D \in \mathbb{R}^2$

$$du = \left[\epsilon^2 \Delta u + u - u^3 - \frac{1}{|D|} \int_D (u - u^3) dx\right] dt + \epsilon^{\delta} dW$$
 & Neumann b.c.



Mass conservative Allen-Cahn



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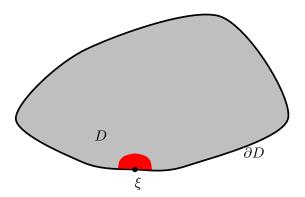
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$$u^{\xi} \approx \begin{cases} 1 & : \text{ on ball of radius } 1 - \epsilon \text{ around } \xi \in \partial D \\ -1 & : \text{ outside ball of radius } 1 + \epsilon \text{ around } \xi \in \partial D \end{cases}$$

$$\mathcal{M} = \{ u^{\xi} : \xi \in \partial D \} \simeq S^1$$





Motion of a droplet [Antonopoulou, Bates, Karali, DB 14]



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Theorem

Assume $\delta < 4$ and W sufficiently smooth in space.

Then:

Solutions stay close to \mathcal{M} up to times of order ϵ^{-q} .

$$d\xi \approx \epsilon^2 c(\xi) dt + \epsilon^\delta \|\partial_\xi u^\xi\|^{-2} \langle \partial_\xi u^\xi, \circ dW \rangle$$

with $c(\epsilon) \approx \mathcal{K}'(\xi)$, where $\mathcal{K}'(\xi)$ is the curvature of ∂D at ξ





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2D Case - Cahn-Hilliard

Results of [Antonopoulou, Karali, DB 15(?)]



2D-Case – Setting



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Rescale time, such that:

$$du_{\epsilon} = \Delta(-\epsilon \Delta u_{\epsilon} + \epsilon^{-1} f'(u_{\epsilon})) dt + \epsilon^{\sigma} d\mathcal{W}$$
 (CH)

with Neumann boundary conditions on domain \mathcal{D} .

Chemical potential v_{ϵ} such that

$$\begin{cases} du_{\epsilon} = -\Delta v_{\epsilon} dt + \epsilon^{\sigma} dW \\ v_{\epsilon} = -\frac{1}{\epsilon} f'(u_{\epsilon}) + \epsilon \Delta u_{\epsilon} \end{cases}$$
 (SYS)



2D-Case — very small noise (based on Alikakos, Bates, Chen 94)



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If $\sigma < \frac{14}{3}$, then the limit (u, v) satisfies free boundary problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{D} \backslash \Gamma, \\ \partial_n v = 0 & \text{on } \partial \mathcal{D} \\ v = \lambda H & \text{on } \Gamma \\ 2V = \partial_n v^+ - \partial_n v^- & \text{on } \Gamma \end{cases}$$

NEED:

 $\Gamma = \Gamma(t)$ is closed hypersurface dividing u = 1 from u = -1 with mean curvature H, velocity V, unit outward normal n, constant $\lambda > 0$



2D-Case – Conjecture



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Conjecture for $\sigma = 1$

The limit (u, v) solves the stochastic Hele-Shaw problem.

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{D} \backslash \Gamma, \\ \partial_n v = 0 & \text{on } \partial \mathcal{D} \\ v = \lambda H + \mathcal{W} & \text{on } \Gamma \\ 2V = \partial_n v^+ - \partial_n v^- & \text{on } \Gamma \end{cases}$$





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Thank you very much for your attention